Homework 5

Abstract Algebra I

Complete the following problems. Note that you should only use results that we've discussed so far this semester.

Problem 1. Let G be a group and let $x \in G$ such that $|x| = n$. Prove that $x^m = e$ iff n divides m.[1]

Problem 2. Let G be a group acting on a set A. Prove one of the following.

(a) The set $\{g \in G \mid g \cdot a = a \text{ for all } a \in A\}$ is a subgroup of G. This set is called the *kernel* of the action of G.

(b) Fix $b \in A$. The set $\{g \in G \mid g \cdot b = b\}$ is a subgroup of G. This set is called the *stabilizer* of b in G.

Problem 3. Prove that the kernel of an action of a group G on a set A is the same as the kernel of the corresponding permutation representation $G \to S_A$.

Problem 4. Prove that a group G acts faithfully on a set A iff the kernel of the action is trivial.

Problem 5. Find the kernel of the left regular action of a group G on itself.

Problem 6. Let G be a group. For all $g, a \in G$, define $g \cdot a = gag^{-1}$. Prove that this defines a left action of G on itself (called *conjugation*).

Problem 7. Let G be a group and fix $g \in G$. Prove that the function determined by left conjugation by g, i.e., $x \mapsto gxg^{-1}$, is an automorphism of G. Quickly deduce that $|x| = |gxg^{-1}|$ for all $x \in G$ and that for any subset A of G, $|A| = |gAg^{-1}|$, where $gAg^{-1} = \{gag^{-1} \mid a \in A\}$.

Problem 8. Let H be a group acting on a set A. Prove that the relation $\sim$ on A defined via $a \sim b$ iff $a = hb$ for some $h \in H$ is an equivalence relation. For each $x \in A$, the equivalence class of x under $\sim$ is called the *orbit* of x under the action of H. It follows immediately from $\sim$ being an equivalence relation that the orbits form a partition of A.

Problem 9. Let H be a subgroup of the finite group G and let H act on G by left multiplication. Let $x \in G$ and let $\mathcal{O}_x$ be the orbit of x under the action of H. Prove that the map $H \to \mathcal{O}_x$ defined via $h \mapsto hx$ is a bijection.

Problem 10. Prove that if G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. This is called *Lagrange's Theorem*.

Problem 11. Show that the group of rigid motions of a cube is isomorphic to S_4. *Hint:* Consider the action of the group of rigid motions on the set of four long diagonals that join pairs of opposite corners of the cube.

Problem 12. Explain why the action of the group of rigid motions of a cube on the set of three pairs of opposite faces is not faithful. Find the kernel of this action.

[1] This problem doesn't have anything to do with group actions. Several of you have been implicitly using it on previous homework assignments, so I think we should make it an official tool.