Homework 8

Abstract Algebra I

Complete the following problems. Note that you should only use results that we've discussed so far this semester.

Problem 1. Suppose G is a finite abelian group. If p is a prime dividing $|G|$, then prove that G contains an element of order p. *Note:* This result is a special case of Cauchy's Theorem, but you are not allowed to use Cauchy's Theorem to prove it. One possible approach to proving this involves using strong induction on the order of G.

Problem 2. Suppose G is an abelian group such that the only normal subgroups of G are the trivial subgroup $\{e\}$ and G itself.[1] Prove that $G \cong \mathbb{Z}_p$ for some prime p. *Hint:* Use the result of the previous problem.

Problem 3. Prove that σ^2 is an even permutation for every permutation $\sigma \in S_n$.

Problem 4. Prove that $S_n = \langle (1,2), (1,2,\ldots,n) \rangle$ for all $n \geq 2$.

Problem 5. Prove that the group of rigid motions of a tetrahedron is isomorphic to A_4.

Problem 6. Prove that the subgroup of order 4 in A_4 is normal and is isomorphic to V_4.

Problem 7. A transitive permutation group G on a set A is called *doubly transitive* if for any (hence all) $a \in A$, the subgroup $\mathrm{Stab}_G(a)$ is transitive on $A \setminus \{a\}$. Prove that S_n is doubly transitive on $\{1, 2, \ldots, n\}$ for all $n \geq 2$.

Problem 8. Exhibit Cayley's Theorem for D_8. That is, find a subgroup of S_8 that is isomorphic to D_8.

Problem 9. Consider the group Q_8.

(a) Find a subgroup of S_8 that is isomorphic to Q_8.

(b) Prove that Q_8 is not isomorphic to a subgroup of S_n for $n \leq 7$. *Hint:* If Q_8 acts on any set A of size less than or equal to 7, show that the stabilizer of any point $a \in A$ must contain the subgroup $\langle -1 \rangle$.

Problem 10. Suppose G is a group of order p^α for some prime p and $\alpha \in \mathbb{Z}^+$.

(a) Prove that every subgroup of index p is normal in G.

(b) Prove that if $\alpha = 2$, then G has a normal subgroup of order p.

Problem 11. Suppose G is a non-abelian group of order 6.

(a) Prove G has a nonnormal subgroup of order 2.

(b) Prove that G is isomorphic to S_3.

[1] Such a group is called a *simple* group.