Rings of Fractions

This section of notes roughly follows Section 7.5 in Dummit and Foote.

Throughout this whole section, we assume that R is a commutative ring.

Note 47. We recall a few relevant facts.

(1) Theorem 11 (Cancellation Law) tells us that if $ab = ac$ and a is neither 0 nor a zero divisor, then $b = c$.

(2) Zero divisors are never units.

One upshot of the above is that ring elements that are not zero divisors possess some of the behavior of units. The goal of this section is to prove that every commutative ring R is aways a subring of a larger ring Q in which every nonzero element of R that is not a zero divisor is a unit in Q. In particular, we can apply this to integral domains, in which case Q will be a field. This generalizes the construction of $\mathbb{Q}$ from $\mathbb{Z}$.

Note 48. Recall that in $\mathbb{Q}$, the fraction $\dfrac{a}{b}$ is the equivalence class of order pairs (a, b) of integers with $b \neq 0$ under the equivalence relation:

$$(a, b) \sim (c, d) \text{ iff } \frac{a}{b} = \frac{c}{d} \text{ iff } ad = bc.$$

Also, every nonzero rational number $\dfrac{a}{b}$ has multiplicative inverse $\dfrac{b}{a}$. That is, every nonzero rational number is a unit, making $\mathbb{Q}$ a field. The integers $\mathbb{Z}$ are a subring of $\mathbb{Q}$. But $\mathbb{Z}$ is an integral domain, not a field.

Theorem 49. Let R be a commutative ring. Let D be any nonempty subset of R that does not contain 0, does not contain any zero divisors, and is closed under multiplication. Then there exists a commutative ring Q with 1 such that Q contains R as a subring and every element of D is a unit in Q.

Theorem 50. Let R, D, and Q be as in Theorem 49. Then every element of Q is of the form rd^{-1} for some $r \in R$ and $d \in D$. In particular, if $D = R \setminus \{0\}$, then Q is a field.

Theorem 51. Let R, D, and Q be as in Theorem 49. Then Q is the smallest ring containing R in which all elements of D become units, in the following sense. Let S be any commutative ring with 1 and let $\phi : R \to S$ be any injective ring homomorphism such that $\phi(d)$ is a unit in S for every $d \in D$. Then there is an injective homomorphism $\Phi : Q \to S$ such that $\Phi|_R = \phi$.

Definition 52. Let R, D, and Q be as in Theorem 49.

(1) The ring Q is called the **ring of fractions** of D with respect to R and is denoted $D^{-1}R$.

(2) If R is an integral domain and $D = R \setminus \{0\}$, then Q is called the **field of fractions** (or **quotient field**) of R.

Corollary 53. Let R be an integral domain and let Q be the field of fractions of R. If a field F contains a subring R' isomorphic to R, then the subfield of F generated by R' (i.e., the intersection of all the subfields of F containing R') is isomorphic to Q.

Example 54. Here are a few quick examples.

(1) If R is a field, then its field of fractions is R itself.

(2) The field of fractions of $\mathbb{Z}$ is $\mathbb{Q}$. The field of fractions of $2\mathbb{Z}$ is also $\mathbb{Q}$.

(3) Consider the polynomial ring $\mathbb{Z}[x]$. Since $\mathbb{Z}$ is an integral domain, so is $\mathbb{Z}[x]$. Then the field of fractions of $\mathbb{Z}[x]$ is the set of rational functions (i.e., functions of the form $p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials with integer coefficients and $q(x)$ is not the zero polynomial). Notice that this field contains the field of fractions of $\mathbb{Z}$, namely $\mathbb{Q}$. However, it is interesting to point out that the field of fractions of $\mathbb{Q}[x]$ is the same as the field of fractions of $\mathbb{Z}[x]$.