Your Name:

Names of Any Collaborators:

Instructions

This portion of Exam 1 is worth a total of 32 points and is due at the beginning of class on **Wednesday, October 12**. Your total combined score on the in-class portion and take-home portion is worth 20% of your overall grade.

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any results that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Proposition xyz, then you should say so.
2. Unless you prove them, you cannot use any results that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

Complete any **FOUR** of following problems. Each problem is worth 8 points. Write your solutions on your own paper and please put the problems in order.

1. Consider the set $S_n(123)$.
   (a) Show that every $w \in S_n(123)$ is the "interweaving" of two decreasing sequences $a_1, a_2, \dots, a_k$ ($a_m > a_{m+1}$) and $b_1, b_2, \dots, b_{n-k}$ ($b_m > b_{m+1}$) (where we allow one of the sequences to be empty). That is, if $w = w_1 \cdots w_n \in S_n(123)$ with $w_i = a_m$ and $w_j = a_{m+1}$ (respectively, $w_i = b_m$ and $w_j = b_{m+1}$), then $i < j$.
   (b) Let $xyz \in \{123, 132, 213, 231, 312, 321\}$ and let $w = w_1 w_2 \cdots w_n \in S_n(xyz)$. Choose a decreasing subsequence $a_1, \dots, a_k$ from $w_1 w_2 \cdots w_n$ in the following way. First, choose $a_1 = w_1$. Next, if possible, choose the smallest i such that $a_1 > w_i$ and define $a_2 = w_i$. Again, if possible, choose the smallest i such that $a_2 > w_i$ and define $a_3 = w_i$. Continue this way until you can no longer find a letter in w to the right of the previous choice that is smaller. Let $B = \{w_1, \dots, w_n\} \setminus \{a_1, \dots, a_k\}$. Now, define $\widetilde{w}$ be the permutation obtained from w by fixing the subsequence $a_1, \dots, a_k$ in their current positions of $w_1 \cdots w_n$ but rearranging (if necessary) the letters from B in the remaining positions of $w_1 \cdots w_n$ in a decreasing manner. For which xyz is this process a bijection from $S_n(xyz)$ to $S_n(123)$? When will the bijection(s) preserve the number of descents?

2. Find an explicit bijection between $S_n(132)$ and $NC(n)$. *Note:* We know that both sets are counted by the Catalan numbers, but this problem is asking you to avoid using this fact. If you want, you may compose bijections between other sets, but if you do this, you must explicitly describe each map. However, a "better" solution would not involve a composition.

3. For $w = w_1 \cdots w_n \in S_n$ define the *trace* of w, denoted $\mathrm{tr}(w)$, to be the sum of the positions of the descents of w. An *inversion* of w is a pair (i, j) such that $i < j$ and $w_i > w_j$. Define $\mathrm{inv}(w)$ to be the number of inversions of w. Prove that
$$\sum_{w \in S_n} t^{\mathrm{tr}(w)} = \sum_{w \in S_n} t^{\mathrm{inv}(w)}.$$

4. The *wicked awesome length* (WAL) of a permutation $w = w_1 \cdots w_n \in S_n$ is the number of adjacent transpositions required to unscramble w into the identity permutation $12 \cdots n$. For example, 31542 has WAL at most 5 since the permutation can be unscrambled in five moves as follows: $31542 \to 13542 \to 13524 \to 13254 \to 12354 \to 12345$. In fact, the WAL of 31542 is exactly 5.
   (a) Show that applying an adjacent transposition to a permutation either increases the number of inversions by one, or decreases the number of inversions by one (see Problem 3).
   (b) Describe an unscrambling algorithm (using adjacent transpositions) that decreases the number of inversions after each swap.
   (c) Prove that the WAL of a permutation w is equal to $\mathrm{inv}(w)$ (see Problem 3).

5. Let $p(n, k)$ equal the number of ways of partitioning $[n]$ into k blocks. Prove that $p(n, k) = p(n-1, k-1) + k \cdot p(n-1, k)$ and determine the values of n and k for which this makes sense.

6. Consider a circle with $2n$ fixed points on the circle. Determine the number of ways of drawing n nonintersecting chords (where each point is connected to exactly 1 chord).

7. Determine the number of ways we can stack coins in the plane such that the bottom row consists of n consecutive coins. You should think of this as the two-dimensional version of stacking apples. For example, here are all the ways to stack coins so that the bottom row has 3 coins.