Enumerative Combinatorics

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This book is intended to be a problem sequence for a one-semester graduate enumerative combinatorics course that utilizes an inquiry-based learning (IBL) approach.

Contents

> The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful.
>
> Henri Poincaré, mathematician & physicist

Chapter 1

Introduction

1.1 What is This Course All About?

This course focuses on enumerative combinatorics with an introduction to generating function techniques.

This course will use an inquiry-based learning (IBL) to learning combinatorics. Rather than spending most of class listening to lectures, you will spend substantial time exploring problems, developing arguments, presenting solutions, and discussing the mathematics with your classmates.

Outside of class, you may use whatever resources help you learn—including books, websites, classmates, and AI tools such as ChatGPT. The important distinction is between **finding a solution** and **understanding the mathematics**. When you come to class, you should be prepared to explain, defend, critique, and adapt the ideas you encountered.

Because outside resources are readily available, homework and preparation will be relatively low-stakes. Your understanding will be assessed primarily through your mathematical contributions in class and regular short, individual, no-resource assessments. In short:

Use whatever helps you learn. Be prepared to demonstrate that you learned it.

1.2 Inquiry-Based Learning

Mathematics is not about calculations, but ideas. My goal as a teacher is to provide students with the opportunity to grapple with these ideas and to be immersed in the process of mathematical discovery. Repeatedly engaging in this process hones the mind and develops mental maturity marked by clear and rigorous thinking. Like music and art, mathematics provides an opportunity for enrichment, experiencing beauty, elegance, and aesthetic value. The medium of a painter is color and shape, whereas the medium of a mathematician is abstract thought. The creative aspect of mathematics is what captivates me and fuels my motivation to keep learning and exploring.

While the content we teach our students is important, it is not enough. An education must prepare individuals to ask and explore questions in contexts that do not yet exist and to be able to tackle problems they have never encountered. It is important that we put these issues front and center and place an explicit focus on students producing, rather than consuming, knowledge. If we truly want our students to be independent, inquisitive, and persistent, then we need to provide them with the means to acquire these skills. Their viability as a professional in the modern workforce depends on their ability to embrace this mindset.

When I started teaching, I mimicked the experiences I had as a student. Because it was all I knew, I lectured. By standard metrics, this seemed to work out just fine. Glowing student and peer evaluations, as well as reoccurring teaching awards, indicated that I was effectively doing my job. People consistently told me that I was an excellent teacher. However, two observations made me reconsider how well I was really doing. Namely, many of my students seemed to depend on me to be successful, and second, they retained only some of what I had taught them. In the words of Dylan Retsek:

"Things my students claim that I taught them masterfully, they don't know."

Inspired by a desire to address these concerns, I began transitioning away from direct instruction towards a more student-centered approach. The goals and philosophy behind inquiry-based learning (IBL) resonate deeply with my ideals, which is why I have embraced this paradigm. According to the Academy of Inquiry-Based Learning, IBL is a method of teaching that engages students in sense-making activities. Students are given tasks requiring them to solve problems, conjecture, experiment, explore, create, and communicate—all those wonderful skills and habits of mind that mathematicians engage in regularly.

In an IBL course, instructor and students have joint responsibility for the depth and progress of the course. While effective IBL courses come in a variety of forms, they all possess a few essential ingredients. According to Laursen and Rasmussen (2019), the Four Pillars of IBL are:

- Students engage deeply with coherent and meaningful mathematical tasks.
- Students collaboratively process mathematical ideas.
- Instructors inquire into student thinking.
- Instructors foster equity in their design and facilitation choices.

This book can only address the first pillar while it is the responsibility of your instructor and class to develop a culture that provides an adequate environment for the remaining pillars to take root.

Just like learning to play an instrument or sport, you will have to learn new skills and ideas. Along this journey, you should expect a cycle of victory and defeat, experiencing a full range of emotions. Sometimes you will feel exhilarated, other times you might be seemingly paralyzed by extreme confusion. You will experience struggle and failure before you experience understanding. This is part of the normal learning process. If you are doing things well, you should be confused on a regular basis. Productive struggle and mistakes provide opportunities for growth. As the author of this text, I am here to guide and challenge you, but I cannot do the learning for you, just as a music teacher cannot move your fingers and your heart for you.

You could view this book as mountaineering guidebook. I have provided a list of mountains to summit, sometimes indicating which trailhead to start at or which trail to follow. There will always be multiple routes to top, some more challenging than others. Some summits you will attain quickly and easily, others might require a multi-day expedition. Oftentimes, your journey will be laced with false summits. Some summits will be obscured by clouds. Sometimes you will have to wait out a storm, perhaps turning around and attempting another route, or even attempting to summit on a different day after the weather has cleared. The strength, fitness, and endurance you gain along the way will allow you to take on more and more challenging, and often beautiful, terrain. Do not forget to take in the view from the top! The joy you feel from overcoming obstacles and reaching each summit under your own will and power has the potential to be life changing. But make no mistake, the journey is vastly more important than the destinations.

> Don't fear failure. Not failure, but low aim, is the crime. In great attempts it is glorious even to fail.
>
> Attributed to Bruce Lee, martial artist & actor

Chapter 2

Preliminaries

2.1 Combinatorial Structures

One of our objectives is to study combinatorial structures that arise in mathematics, with an eye towards enumeration and classification. We will begin by investigating several sets of discrete mathematical objects and exploring the connections among them. In no particular order, here are some combinatorial species[1] we may encounter throughout the semester:

Binary String	Pixel Strip	Lattice Path
Starfolks Path	Compositions	Pebble Piles
Domino Stacking	Domino Tiling	Permutation
Nonattacking Rooks	Sparse Strip	Odd Composition
Binary Tree	Increasing Tree	Triangulated Polygon
Ballot Path	Noncrossing Partition	Parking Function
Unlabeled Tree	Set Partition	Integer Partition
Dodger Rooks	Cayley Permutation	Fubini Arrangement
Fraenkel Tree	Knuth Rooks	Penrose Doodle

Note that I invented some of the names mentioned above. Figure 2.1 provides a few examples of individuals one can find among the species mentioned above.
Below are some questions we should ponder when exploring a new species:

- What patterns do we see?
- How do they "grow"?
- What is a sensible definition of the species? How does your definition tie in to the **index** n and the **rank** r?
- Can you provide a generic example with index $n = 10$? For your example, what is r?

[1] In this context, I am using the word "species" in a colloquial sense. Later in the semester, this word will have a more precise mathematical meaning.

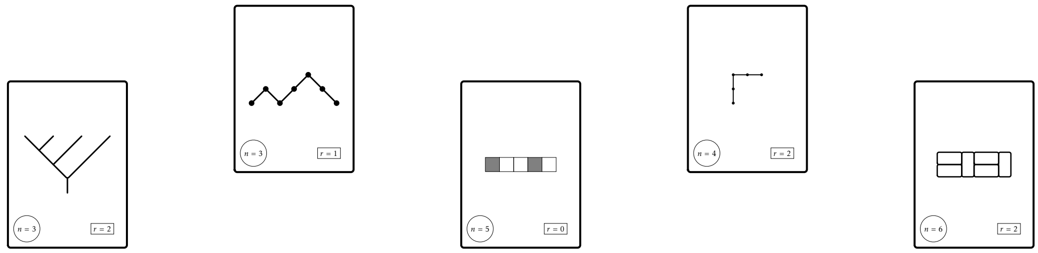

Figure 2.1: Examples of combinatorial species.

- How many objects are there of index n? Is there a nice formula?
- How many objects are there of index n and rank r? Is there a nice formula?
- For a fixed index n how do you generate all the possibilities? What is the simplest/most efficient way to generate them?
- How does this species relate to other species we have studied?

You may not be able to answer some of these questions immediately, and that's okay. The goals here are to have a variety of examples at our finger tips and to start a taxonomy of combinatorial species.

Problem 2.1. For each of the species you encountered or were assigned, complete the following.

(a) List the name of the species.

(b) Provide a working definition for the species. Your definition should include mention of the index n and the rank r, including the domain for n (and possibly r). For example, your definition might start as, "A *dinglehopper* of index $n \in \mathbb{N} \cup \{0\}$ and rank r is..."

(c) Provide a generic example for index $n = 10$. For your example, specify the rank r. Your example should illustrate most or all of the key features and not be too "boring."

(d) Provide a table of values that enumerates the number of structues of index n for small n. Your table should start at either $n = 0$ or $n = 1$ depending on the species and it should look something like the following:

n	0	1	2	3	4	5
count	1	1	2	3	5	8

(e) If you can, write down a formula (either a closed form or recursive) that counts the number of structures of index n. Justify your answer. Please do not look this up or ask AI. If you cannot do it at this time, I would prefer we leave it blank!

(f) Now, provide a table of values that enumerates the number of structures of index n and rank r for small values of n and r. Your table should start at either $n = 0$ or $n = 1$ and $r = 0$ or $r = 1$ depending on the species, with n labeling the rows and r labeling the columns. Table 2.1 illustrates the requested format.

$n \backslash r$	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

Table 2.1: Sample counts of structures organized by index n and rank r.

(g) Ignoring the row labels, what happens if you sum the enteries in a row of your table in part (f)?

(h) If you can, write down a formula (either a closed form or recursive) that counts the number of structures of index n and rank r. Justify your answer. Please do not look this up or ask AI. If you cannot do it at this time, I would prefer we leave it blank!

(i) Are there any other species you have already encountered that seems related to this species?

2.2 First Principles

Let's establish some notation and make sure we are on the same page regarding some basic counting principles.

In this book, the set of **natural numbers** is given by $\mathbb{N} := \{1, 2, 3, \ldots\}$. Sometimes mathematicians include 0 in the natural numbers, but that will not be our convention. For $n \in \mathbb{N} \cup \{0\}$, the following notation is useful:

$$[n] := \begin{cases} \{1, 2, \ldots, n\}, & \text{if } n \in \mathbb{N} \\ \varnothing, & \text{if } n = 0. \end{cases}$$

The **cardinality** of a set A, denoted by $|A|$, is the number of elements in A. If A and B are sets, the **Cartesian product** of A and B, denoted $A \times B$ (read as "A times B" or "A cross B"), is the set of all **ordered pairs** where the first component is from A and the second component is from B. In set-builder notation, we have

$$A \times B := \{(a, b) \mid a \in A, b \in B\}.$$

Cartesian products are named after French philosopher and mathematician René Descartes (1596–1650).

Problem 2.2. If A is a set, then what is $A \times \varnothing$ equal to?

Problem 2.3. Under what circumstances is $A \times B = B \times A$?

The Cartesian product of sets $A_1, \dots, A_n$ is the collection of n-**tuples** given by

$$A_1 \times \cdots \times A_n := \{(a_1, \dots, a_n) \mid a_j \in A_j \text{ for all } 1 \leq j \leq n\},$$

where A_i is referred to as the ith **factor** of the Cartesian product. As a special case, the set

$$\underbrace{A \times \cdots \times A}_{n \text{ factors}}$$

is often abbreviated as A^n.

Theorem 2.4 (Product Principle)**.** Each of the following are referred to as the **Product Principle**.

(a) If $A_1, \dots, A_k$ are finite sets, then

$$|A_1 \times \cdots \times A_k| = |A_1| \cdots |A_k|.$$

(b) If $\mathcal{O}$ is the set of outcomes for a k-step process, where for $1 \leq i \leq k$, there are n_i choices for step i, no matter what earlier choices were made, then

$$|\mathcal{O}| = n_1 n_2 \cdots n_k.$$

The key difference between the two versions of the Product Principle is that the second version does not assume that the set of choices for step i is independent of the previous choices.

Problem 2.5. Is your answer for Problem 2.2 compatible with the Product Principle?

For $n \in \mathbb{N} \cup \{0\}$, a **string** of objects of **length** n is a linearly ordered arrangements of n objects. The objects in a string may be repeated unless specified otherwise. A string of length 0 is referred to as the **empty string**. If a string consists of letters, then it may be called a **word** (we do not require a word to be in the dictionary!). For example, $NNENEEE$ is a word of length 7 consisting of N's and E's.

For $n \in \mathbb{N} \cup \{0\}$, a **binary string** is a string consisting of 0's and 1's. Each occurrence of a 0 or 1 in a binary string is called a **bit**. The strings 010 and 100 are two different binary strings of length 3.

Problem 2.6. For $n \in \mathbb{N} \cup \{0\}$, how many binary strings of length n are there?

Problem 2.7. For $n \in \mathbb{N} \cup \{0\}$, how many subsets does a set with n elements have? You should be able to carefully justify your answer by cleverly utilizing the Product Principle.

For $n \in \mathbb{N}$, a **pebble pile configuration** of index n and rank r consists of n indistinguishable dots arranged into r nonempty vertical piles. Each pile is bottom-justified along a common horizontal baseline, with the dots in a pile stacked vertically. The piles are arranged from left to right, and their order matters. Figure 2.2 depicts a pebble pile configuration with 10 pebbles and 4 piles.

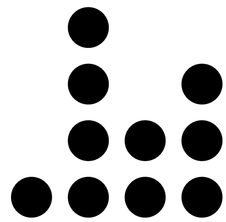

Figure 2.2: A pebble pile configuration consisting of 10 pebbles and 4 piles.

Problem 2.8. For $n \in \mathbb{N}$, how many pebble pile configurations are there consisting of n pebbles?

For $n \in \mathbb{N}$ and $1 \leq r \leq n$, a **composition** of n with r parts is an ordered list of r positive integers whose sum is n. For example, $(1, 2, 1, 3)$ is a composition of 7 with 4 parts. An arbitrary composition with r parts may be denoted as $\alpha = (\alpha_1, \dots, \alpha_r)$.

Problem 2.9. For $n \in \mathbb{N}$, how many compositions of n are there? Is there an obvious connection between pebble pile configurations and compositions? Can you explicitly describe this connection?

Problem 2.10. Imagine we have $n \geq 1$ distinct balls labeled $1, 2, \dots, n$ and $k \geq 1$ distinct buckets labeled $1, 2, \dots, k$. How many ways can we distribute the n distinct balls into the k distinct buckets?

Problem 2.11. Let $n, k \in \mathbb{N}$.

(a) How many functions are there from $[n]$ to $[k]$?

(b) If $k \leq n$, how many injective functions are there from $[k]$ to $[n]$?

(c) How many bijective functions are there from $[n]$ to $[n]$?

The game of chess is played on an 8×8 grid. We will consider chessboards with arbitrary dimensions, say $n \times k$ (n rows, k columns) with $1 \leq k \leq n$. A rook is a castle-shaped piece that can move horizontally or vertically any number of squares. In a typical chess game, there are two black rooks and two white rooks, but for our purposes we will assume we have k rooks, where all the rooks are the same color. We say that an arrangement of rooks on an $n \times k$ board is **nonattacking** if no two rooks lie in the same row or column. Figure 2.3 shows one possible nonattacking rook arrangement on a 4×4 board.

Problem 2.12. For $n \in \mathbb{N}$ and $1 \leq k \leq n$, how many nonattacking rook arrangements are there on an $n \times k$ board? What about an $n \times n$ board?

Theorem 2.13 (Sum Principle)**.** If $A_1, \dots, A_n$ are pairwise disjoint sets, then

$$|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|.$$

The Sum Principle is especially useful when a collection of objects can be partitioned into disjoint cases: count the objects in each case, then add the resulting counts.

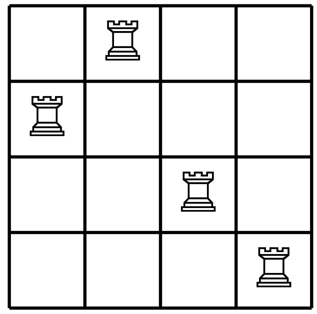

Figure 2.3: A nonattacking rook arrangement on a 4×4 board.

Loosely speaking, a sequence of numbers is defined recursively if the nth term of the sequence is defined in terms of "earlier" terms of the sequence. One famous example of a sequence being defined recursively is the **Fibonacci sequence** (f_n), which we define by $f_0 = 1$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. The equation $f_n = f_{n-1} + f_{n-2}$ is the **recurrence relation** while $f_0 = 1$ and $f_1 = 1$ are the **initial conditions**. The first few terms of the Fibonacci sequence are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

It is important to emphasize that we cannot define the Fibonacci numbers using only the recurrence relation since otherwise, we would not be able to "get started" with the recurrence.

Problem 2.14. How many compositions α of n have the following properties?

(a) α has parts of size 1 and 2 only.

(b) α has only odd parts.

Problem 2.15. For $n \in \mathbb{N} \cup \{0\}$, how many binary strings of length n never have two consecutive 1's?

Problem 2.16. For $n \in \mathbb{N}$, consider a $2 \times n$ grid of squares. Using n dominoes with dimensions 1×2 squares, how many ways can we tile (i.e., perfectly cover with no overlap) the grid?

If a finite collection of sets is not pairwise disjoint, then we can still enumerate their union using the **Principle of Inclusion and Exclusion**.

Theorem 2.17 (Principal of Inclusion and Exclusion)**.** The number of elements in the union of finite sets $A_1, A_2, \ldots, A_n$ is

$$|A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cdots \cap A_n|.$$

Problem 2.18. To ensure you understand the formula in the previous theorem, write down formulas for computing $|A \cup B|$ and $|A \cup B \cup C|$.

The next result, sometimes referred to as the **Subtraction Principle**, is simply a special case of Sum Principle.

Theorem 2.19 (Subtraction Principle)**.** If $A \subseteq U$, then $|A| = |U| - |A^c|$.

There are two use cases for the Subtraction Principle. Sometimes it might be easier to count the opposite of what you are looking to count and then to subtract from the size of the universe of discourse. The second approach involves "inflating" the universe of discourse to include additional elements, counting the larger set, and then subtracting off the extra stuff. The two approaches are really just two different perspectives of the same idea.

The **Division Principle** provides a method for ignoring "unimportant" differences when counting things. Loosely speaking, the general idea is to count a larger set of distinct objects and then use the Division Principle to "merge" the ones that are not significantly different from each other.

Theorem 2.20 (Division Principle)**.** If a finite set A is the union of n pairwise disjoint subsets each with d elements, then $|A| = n \cdot d$. Equivalently, $n = |A|/d$.

Figure 2.4 illustrates the Division Principle, where the set A contains 12 elements that are partitioned into $n = 3$ pairwise disjoint subsets, each of size $d = 4$. In this case, we see that $|A|/d = 12/4 = 3 = n$.

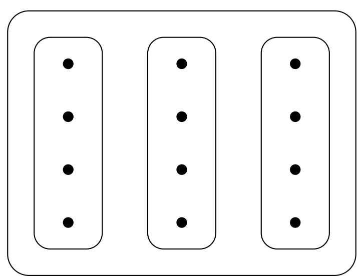

Figure 2.4: Visualization of the Division Principle with $|A| = 12$, $n = 3$, and $d = 4$.

Problem 2.21. How many ways can the letters of the word FLAGSTAFF be arranged?

Problem 2.22. Six friends sit around a circlular table to play a game. Rotations of the group do not constitute different seating orders.

(a) How many circular seating arrangements are there?

(b) How many circular seating arrangements are there if Sally and Maria always sit next to each other?

For $n \in \mathbb{N}$, a **set partition** of $[n]$ is a collection of nonempty pairwise disjoint subsets of $[n]$ whose union is $[n]$. Each subset in a set partition is called a **block**. For example, there are 7 set partitions of [4] with 2 blocks, namely:

$$\begin{array}{ll} \{\{1,2,3\},\{4\}\} & \{\{1,2,4\},\{3\}\} \\ \{\{1,3,4\},\{2\}\} & \{\{2,3,4\},\{1\}\} \\ \{\{1,2\},\{3,4\}\} & \{\{1,3\},\{2,4\}\} \\ \{\{1,4\},\{2,3\}\} & \end{array}$$

We define the **Stirling numbers** (of the second kind) via

$$\left\{ n \atop k \right\} := \text{number of set partitions of } [n] \text{ with } k \text{ blocks.}$$

I usually pronounce this as "n Stirling k". Based on the information above, we know $\left\{ 4 \atop 2 \right\} = 7$. For $n \in \mathbb{N}$, we define the nth **Bell number** B_n to be the total number of set partitions of $[n]$ into any number of blocks. We also set $B_0 = 1$.

The next problem is quite involved but provides an opportunity to put all of the counting principles to use.

Problem 2.23. Answer each of the following.

(a) Explain why $\left\{ n \atop 1 \right\} = 1 = \left\{ n \atop n \right\}$ for every $n \in \mathbb{N}$.

(b) Explain why $\left\{ n \atop 2 \right\} = 2^{n-1} - 1$ for every $n \in \mathbb{N}$.

(c) Explain why the Stirling numbers satisfy the following recurrence for $1 \leq k \leq n$:

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\}.$$

(d) Compute the Bell number B_4 by hand by writing down all of the appropriate set partitions.

(e) For $n \geq 1$, explain why

$$B_n = \sum_{k=1}^{n} \left\{ n \atop k \right\}.$$

Problem 2.24. Notice that back in Problem 2.11 we avoided counting surjective functions. We now have the tools to at least describe the count. Let $n, k \in \mathbb{N}$ wih $k \leq n$. How many surjective functions are there from $[n]$ to $[k]$?

Problem 2.25. Imagine we have $n \in \mathbb{N}$ people competing in a contest where ties are allowed.

(a) Explain why the number of final rankings is given by

$$\sum_{k=1}^{n} k! \left\{ {n \atop k} \right\}.$$

(b) Now, imagine that the people are numbered 1 through n. How many final rankings are possible under the proviso that person i's ranking is less than or equal to person $(i+1)$'s ranking? Such rankings are called **weakly increasing**.

2.3 Bijections

The next theorem states an important counting technique, which we refer to as the **Bijection Principle**.

Theorem 2.26 (Bijection Principle)**.** If there exists a bijection between finite sets A and B, then $|A| = |B|$.

For $n \in \mathbb{N} \cup \{0\}$, a **lattice path** of length n is a path in the integer lattice that begins at the origin and consists of n steps, each of which moves one unit north or one unit east. A lattice path of length n with r north steps is said to have rank r. Figure 2.5 provides an example of a lattice path of length 5 with rank 2.

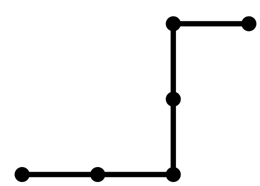

Figure 2.5: A lattice path of length 5 with rank 2.

For $n \in \mathbb{N}$ and $0 \leq r \leq n-1$, a **sticks and stones configuration** with n stones and r sticks is a string of n $\circ$'s and r $|$'s such that every $|$ is framed by two $\circ$'s. For example, $\circ \mid \circ\circ\circ \mid \circ\circ$ is a sticks and stones configuration with 6 stones and 2 sticks.

For $n \in \mathbb{N} \cup \{0\}$, a **sparce strip** of length n is a $1 \times n$ grid such that each square in the grid is colored white or gray with the proviso that two gray squares are never adjacent. A sparce strip with r gray squares is said to have rank r. Figure 2.6 provides an example of a sparce strip of length 5 with rank 2.

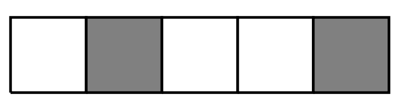

Figure 2.6: A sparce strip of length 5 with rank 2.

Problem 2.27. Describe bijections between the following pairs of collections.

(a) For $n \in \mathbb{N} \cup \{0\}$, the collection of subsets of $[n]$ and the collection of binary strings of length n.

(b) For $n \in \mathbb{N} \cup \{0\}$, the collection of subsets of $[n]$ and the collection of lattice paths of length n.

(c) For $n \in \mathbb{N}$, the collection of compositions of n and the collection of sticks and stones configurations with n stones.

(d) For $n \in \mathbb{N}$, the collection of pebbles piles with n pebbles and the collection of sticks and stones configurations with n stones.

(e) For $n \in \mathbb{N} \cup \{0\}$, the collection of pebble piles with $n+1$ pebbles and the collection of subsets of $[n]$.

(f) For $n \in \mathbb{N}$, the collection of nonattacking rook arrangements on an $n \times n$ board and the collection of bijections from $[n]$ to $[n]$.

(g) For $n \in \mathbb{N}$, the collection of outcomes of a contest with n people with ties allowed (see part (a) of Problem 2.25) and the collection of surjective functions with domain $[n]$.

(h) For $n \in \mathbb{N}$, the collection of outcomes of a contest with n people as described in part (b) of Problem 2.25 and the collection of sticks and stones configurations with n stones.

(i) For $n \in \mathbb{N}$, the collection of compositions of n with parts of size 1 and 2 only (see Problem 2.14) and the collection of domino tilings of a $2 \times n$ grid (see Problem 2.16).

(j) For $n \in \mathbb{N}$, the collection of domino tilings of a $2 \times n$ grid and the collection of sparce strips of length $n-1$.

In part (a) of Problem 2.27 you likely described a bijection between the collection of subsets of $[n]$ and the collection of binary strings of length n that also respected some additional structure. In particular, your bijection probably mapped k-subsets to binary strings with k ones. This is an example of a "nice" bijection, i.e., a set "isomorphism" that preserves combinatorial structure. In combinatorics, we really like these kinds of bijections.

Problem 2.28. Using the Bijection Principle together with the previous problem and problems that appeared earlier, enumerate the following collections.

(a) For $n \in \mathbb{N}$, lattice paths of length n.

(b) For $n \in \mathbb{N}$, sticks and stones configurations with n stones.

(c) For $n \in \mathbb{N}$, the collection of surjective functions with domain $[n]$.

(d) For $n \in \mathbb{N} \cup \{0\}$, sparce strips of length n.

2.4 Combinations and Permutations

If A is a set and $B \subseteq A$ with $|B| = k$, we refer to B as a k-**subset** of A. The collection of all k-subsets of A is defined via

$$\binom{A}{k} := \{B \subseteq A \mid |B| = k\}.$$

For $n, k \in \mathbb{Z}$, the **binomial coefficients** are defined via

$$\binom{n}{k} := \begin{cases} \text{number of } k\text{-subsets of an } n\text{-element set}, & \text{if } 0 \leq k \leq n \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if $|A| = n$, then $|\binom{A}{k}| = \binom{n}{k}$. We read "$\binom{n}{k}$" as "$n$ choose k". Alternate notations for binomial coefficients include $C(n, k)$ and ${}_nC_k$. We will see later why $\binom{n}{k}$ is referred to as a binomial coefficient.

Example 2.29. If $A = \{a, b, c, d\}$, then

$$\binom{A}{2} = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\},$$

which implies that $\binom{4}{2} = 6$.

Problem 2.30. Let A be a finite set with $|A| = n \in \mathbb{N} \cup \{0\}$.

(a) What is $\binom{A}{0}$?

(b) What is $\binom{n}{0}$? Where do these values appear in Pascal's Triangle?

(c) What is $\binom{A}{n}$?

(d) What is $\binom{n}{n}$? Where do these values appear in Pascal's Triangle?

If we let n and k vary, we can organize the (nonzero) binomial coefficients in a triangular array, often referred to as **Pascal's Triangle**. See Table 2.2.

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Table 2.2: Pascal's Triangle of binomial coefficients.

Problem 2.31. Explain why the row sums in Pascal's Triangle are powers of two? In particular, for $n \geq 0$, explain why

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

We say that the binomial coefficients $\binom{n}{k}$ are a **refinement** of 2^k. This illustrates a general phenomenon we will witness over and over again.

Problem 2.32. Using the definition in terms of k-subsets (as opposed to some formula you might know involving factorials) together with a bijection, prove that for $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}.$$

This result is sometimes referred to as the **symmetry identity for binomial coefficients** (or simply **binomial symmetry**). Can you explain why it has this name?

The next result is called **Pascal's Identity** (or **Pascal's Recurrence**).

Problem 2.33 (Pascal's Identity)**.** Using the definition in terms of k-subsets, prove that for $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

The next problem illustrates a powerful idea: If we count the same collection in two different ways, then the two answers must agree.

Problem 2.34. Explain why

$$1 + 2 + \cdots + n = \binom{n+1}{2}$$

by counting the same collection in two different ways.

For $n \in \mathbb{N}$, the number defined by $t_n := 1 + 2 + \cdots + n$ is called the nth **triangular number** (due to the shape we get by representing each number in the sum by a vertical stack of balls).

Problem 2.35. For $n \in \mathbb{N}$ and $1 \leq k \leq n$, prove that the number of compositions of n with k parts is $\binom{n-1}{k-1}$.

For $k \in \mathbb{N}$ and a nonempty set A, a k-**permutation** of A is an injective function $w : [k] \to A$. The set of all k-permutations of A is denoted by $S_{A,k}$. If A happens to be the set $[n]$, we use the notation $S_{n,k}$. And if $n = k$, we write $S_n := S_{n,n}$ and refer to each n-permutation in S_n as a **permutation**. It's not coincidence that this notation coincides with the notation for the symmetric group that one encounters in abstract algebra. Define

$$P(n,k) := |S_{n,k}|.$$

By convention, we set $P(n,0) := 1$, including the case when $n = 0$.

Problem 2.36. For $1 \leq k \leq n$, describe a bijection between $S_{n,k}$ and the nonattacking rook arrangments on a $n \times k$ board.

In light of Problems 2.12 and 2.36, we can conclude that

$$P(n,k) = n \cdot (n-1) \cdots (n+1-k) = \frac{n!}{(n-k)!}.$$

Problem 2.37. For $0 \leq k \leq n$, prove that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Problem 2.38. I'm in the mood for a cup of coffee! Let's say that the nearest place to get a cup of coffee is four blocks East and three blocks North. Assuming I only walk East or North, how many different routes can I take to get there?

Problem 2.39. How many binary strings of length 6 have an odd number of 0's?

Problem 2.40. Show that the Fibonacci numbers satisfy the following identity for $n \geq 0$:

$$f_n = \sum_{k \geq 0} \binom{n-k}{k}.$$

Hint: There are a few natural approaches. One elegant approach would be to utilize a combinatorial argument with one of the collections of compositions in Problem 2.14.

What is the previous result saying in terms of Pascal's Triangle?

Problem 2.41. Recall the definition of the Bell numbers (where $B_0 = 1$). Prove that for $n \geq 1$, we have

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k.$$

Hint: There is a block containing 1. Consider cases based on how many elements this block contains. Let k be the number of elements that are not in the block containing 1.

2.5 The Binomial Theorem

Recall that $\binom{n}{k}$ counts the number of subsets of size k taken from a set of size n. Why are these numbers called binomial coefficients? In general a binomial is just a polynomial with two terms. Let's explore a bit. We see that

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= xx + xy + yx + yy\\ &= x^2 + 2xy + y^2.\end{aligned}$$

The coefficients in this expansion are $1, 2, 1$. Hey, we saw those in row $n = 2$ of Pascal's Triangle! Let's try a larger example. We see that

$$\begin{aligned}(x+y)^3 &= (x+y)^2(x+y)\\ &= (xx+xy+yx+yy)(x+y)\\ &= xxx+xyx+yxx+yyx+xxy+xyy+yxy+yyy\\ &= x^3+3x^2y+3xy^2+y^3.\end{aligned}$$

This time the coefficients are $1, 3, 3, 1$, which is the next row of Pascal's Triangle. What's going on here?

Consider the expansion of $(x+y)^n$. The key observation is that *before* commuting factors and collecting like terms, the terms in the expansion consist of *all possible* ordered products (i.e., words) where we choose either x or y from each factor. Each such term will consist of n letters. In particular, if there are k copies of x in a term, there will be $n-k$ copies of y. Moreover, every possible configuration of k copies of x (i.e., location of the x's in the term *before* doing any commuting with x and y) will be represented. This means there will be precisely $\binom{n}{k}$ many terms with k copies of x (and $n-k$ copies of y). Thus, when we commute and then collect like terms, the coefficient on $x^k y^{n-k}$ will indeed be $\binom{n}{k}$. This leads us to the following remarkable fact, known as the **Binomial Theorem**.

Theorem 2.42 (Binomial Theorem)**.** The goal is to prove that for $n \in \mathbb{N} \cup \{0\}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Problem 2.43. Reprove that claim in Problem 2.31 using the Binomial Theorem.

It follows from the Binomial Theorem that

$$(1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k = \binom{n}{0} t^0 + \binom{n}{1} t^1 + \cdots + \binom{n}{n} t^n.$$

That is, the expression $(1+t)^n$ encodes the entries in the nth row of Pascal's Triangle. In particular, $\binom{n}{k}$ is the coefficient on t^k in the expansion of $(1+t)^n$. This is our first example of what is called a **generating function**.

More coming soon…