Homework 10

Abstract Algebra II

Complete the following problems. Note that you should only use results that we've discussed so far this semester or last semester.

Problem 1. Determine all the subfields of the splitting field of $x^8 - 2$ that are Galois over $\mathbb{Q}$. *Note:* You are welcome to consult the example involving the splitting field of $x^8 - 2$, which appears at the end of Section 14.2 of Dummit and Foote.

Problem 2. Suppose K/F is Galois such that $[K : F] = p^n$ for some prime p and $n \geq 1$. Prove that there are Galois extensions of F contained in K of degrees p and p^{n-1}.

Problem 3. Give an example of fields F_1, F_2, F_3 with $\mathbb{Q} \subset F_1 \subset F_2 \subset F_3$, $[F_3 : \mathbb{Q}] = 8$, and each field is Galois over all its subfields with the exception of that F_2 is not Galois over $\mathbb{Q}$.

Problem 4. Consider the extension $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$.

(a) Prove that $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ is a Galois extension of degree 4

(b) Exhibit the Galois correspondence of the subfields of $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ containing $\mathbb{Q}$ with the subgroups of the Galois group of $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$.

(c) Determine which subfields of $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ are Galois over $\mathbb{Q}$.

Problem 5. Consider the separable polynomial $f(x) = x^4 - 12x^2 + 35$ over $\mathbb{Q}$

(a) Determine the Galois group over $\mathbb{Q}$ of $f(x)$.

(b) Exhibit the Galois correspondence of the subfields of the splitting field of $f(x)$ containing $\mathbb{Q}$ with the subgroups of the Galois group of $f(x)$.

(c) Determine which subfields of the splitting field of $f(x)$ are Galois over $\mathbb{Q}$.

Problem 6. Consider the separable polynomial $g(x) = x^4 - 2$ over $\mathbb{Q}$

(a) Determine the Galois group over $\mathbb{Q}$ of $g(x)$.

(b) Exhibit the Galois correspondence of the subfields of the splitting field of $g(x)$ containing $\mathbb{Q}$ with the subgroups of the Galois group of $g(x)$.

(c) Determine which subfields of the splitting field of $g(x)$ are Galois over $\mathbb{Q}$.