Homework 3

Abstract Algebra II

Complete the following problems. Note that you should only use results that we've discussed so far this semester or last semester.

For Problems 3–7, assume that F is a field.

Problem 1. Let R be a commutative ring with 1. Prove that a polynomial ring in more than one variable over R is not a PID.

Problem 2. Consider the polynomial ring $\mathbb{Q}[x,y]$.

(a) Prove that the ideals (x) and (x,y) are prime in $\mathbb{Q}[x,y]$.

(b) Prove that (x,y) is a maximal ideal but (x) is not maximal.

(c) Prove that (x,y) is not a principal ideal.

Problem 3. Prove that the rings $F[x,y]/(y^2-x)$ and $F[x,y]/(y^2-x^2)$ are not isomorphic for any field F.

Problem 4. Let $f(x) \in F[x]$ such that $\deg(f(x)) = n \geq 1$. Prove that for each $\overline{g(x)} \in F[x]/(f(x))$ there is a unique $g_0(x) \in F[x]$ with $\deg(g_0(x)) \leq n-1$ such that $\overline{g(x)} = \overline{g_0(x)}$. *Note:* $\overline{g(x)}$ denotes passage to the quotient $F[x]/(f(x))$.

Problem 5. Let $f(x) \in F[x]$. Prove that $F[x]/(f(x))$ is a field iff $f(x)$ is irreducible.

Problem 6. Let F be a finite field. Prove that $F[x]$ contains infinitely many primes. *Hint:* Mimic one of the well-known proofs that there are infinitely many primes in the natural numbers.

Problem 7. Prove that the set R of polynomials in $F[x]$ whose coefficient of x is equal to 0 is a subring of $F[x]$ and that R is not a UFD. *Hint:* One approach is to find two distinct factorizations of x^6 into irreducibles.