Homework 4

Abstract Algebra II

Complete the following problems. Note that you should only use results that we've discussed so far this semester or last semester.

Problem 1. Determine all ideals of the ring $\mathbb{Z}[x]/(2, x^3+1)$.

Problem 2. Prove that if $f(x), g(x) \in \mathbb{Q}[x]$ such that $f(x)g(x) \in \mathbb{Z}[x]$, then the product of any coefficient of $f(x)$ with any coefficient of $g(x)$ is an integer.

Problem 3. Determine whether each of the following polynomials is irreducible in the given ring. Justify your answers. If a polynomial is reducible, write it as a product of irreducibles.

(a) x^4+1 in $\mathbb{Z}/5\mathbb{Z}[x]$.

(b) x^4+10x^2+1 in $\mathbb{Z}[x]$.

(c) x^4-4x^3+6 in $\mathbb{Z}[x]$.

(d) $x^6+30x^5-15x^3+6x-120$ in $\mathbb{Z}[x]$.

Problem 4. Prove that the polynomial $(x-1)(x-2)\cdots(x-n)-1$ is irreducible over $\mathbb{Z}$ for all $n \geq 1$.

Problem 5. Prove that the ring $\mathbb{R}[x]/(x^2+1)$ is isomorphic to the field $\mathbb{C}$.

Problem 6. Prove that the polynomial $x^2-\sqrt{2}$ is irreducible over $\mathbb{Z}[\sqrt{2}]$. *Note:* You may assume that $\mathbb{Z}[\sqrt{2}]$ is a UFD.

Problem 7. Prove that x^2+y^2-1 is irreducible over $\mathbb{Q}[x,y]$.