Homework 6

Abstract Algebra II

Complete the following problems. Note that you should only use results that we've discussed so far this semester or last semester.

Problem 1. Let F be a finite field of characteristic p. Prove that $|F| = p^n$ for some positive integer n.

Problem 2. Find the minimal polynomial of $1+i$ over $\mathbb{Q}$.

Problem 3. Prove that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$ and find an irreducible polynomial having $\sqrt{2}+\sqrt{3}$ as a root.

Problem 4. Suppose the degree of the extension K/F is a prime p. Prove that any subfield E of K containing F is either K or F.

Problem 5. Suppose $F = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$ where $\alpha^2 \in \mathbb{Q}$ for $i = 1, \dots, n$. Prove that $\sqrt[3]{2} \notin F$.

Problem 6. For **three** of the following polynomials, determine the splitting field and its degree over $\mathbb{Q}$.

(a) $x^4 - 2$

(b) $x^4 + 2$

(c) $x^4 + x^2 + 1$

(d) $x^6 - 4$

(e) $x^4 - 5x^2 + 6$

Problem 7. Let K be a finite extension of F. Prove that K is a splitting field over F iff every irreducible polynomial in $F[x]$ that has a root in K splits completely in $K[x]$.

Problem 8. Suppose F is a field with the property that every polynomial $f(x) \in F[x]$ splits completely over F. Prove that F has no proper finite-degree extensions. *Note:* An extension K of F is proper if $[K : F] > 1$.