Plymouth State UNIVERSITY

MA 2550: Calculus I

Section 5.3: The Fundamental Theorem of Calculus

Here's a summary of the important stuff in Section 5.3.

So far, we have discussed two types of integrals:

1. Indefinite Integral: Let f be a function. Then

$$\boxed{\int f(x)\ dx = F(x) + C,}$$

where $F'(x) = f(x)$ and C is any constant. We call F the antiderivative of f. It is important to note that the indefinite integral is an entire family of functions (one for every possible constant).

2. Definite Integral: Let f be a function. Then using the standard notation,

$$\boxed{\int_a^b f(x)\ dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x,}$$

provided this limit exists, where $\boxed{\Delta x = \frac{b-a}{n}}$. Now, if f is continuous (or even integrable) on $[a, b]$, we can use the right endpoints of each subinterval of the partition. That is, if f is continuous,

$$\boxed{\int_a^b f(x)\ dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x,}$$

where $\boxed{x_i = a + i\Delta x}$. In this case, we can think of the definite integral as representing net signed area under the graph of f on the interval $[a, b]$.

To summarize, an indefinite integral spits out a family of functions and a definite integral spits out a number! The most important point in this section is that these two seemingly unrelated concepts are actually related in very important way.

The Fundamental Theorem of Calculus, Part 1 (FTC1): Let f be a *continuous* function on the interval $[a, b]$. We define a new function in terms of an integral, which we'll call g, via

$$g(x) = \int_a^x f(t)\ dt.$$

Whoa! What is this function? OK, let's examine it piece by piece. First, what's with the t's? Well, we already used x and we didn't want to use the same variable to mean two different things, so we used a new variable. The upper limit of this integral is the variable x. This is the independent variable of the function g. Imagine that x is a point somewhere in the interval $[a, b]$ on the t-axis (it doesn't have to be, but this is easier to grasp). The integral

$$\int_a^x f(t)\ dt$$

equals the area under the graph of f from a to x. I call g the "painting function." I think of $g(x)$ as equaling the amount of paint I need to paint the region under the graph of f from a to x. Notice that as x gets closer to b, we'll have a "larger" region to paint. (I put larger in quotes because if the graph of f dips below the t-axis, we'll pick up negative signed area.) I think that it's time for an example.

Example: Let $f(x) = x^2$. Define

$$g(x) = \int_0^x f(t)\ dt.$$

But since $f(x) = x^2$, $f(t) = t^2$, and so

$$g(x) = \int_0^x t^2\ dt.$$

Then $g(x)$ equals the area under the graph of $f(t) = t^2$ from 0 to x. You should try to draw a picture of this; there is one similar in the book. For example,

$$g(3) = \int_0^3 t^2\ dt,$$

which equals the area under the graph of f from 0 to 3.

In general, it turns out that the derivative of the "painting" function is equal to f. That is,

$$\boxed{\frac{d}{dx}\int_a^x f(t)\ dt = f(x).}$$

This is FTC1. I'll skip the details on why this is true, but you should notice two things:

1. Informally, the derivative and the integral are inverses of each other since they cancelled each other out and gave us back f.
2. Tackling problems involving FTC1 are pretty easy because all you have to do is drop the integral and replace all the t's with x's.

Example: If we return to our previous example, we see that

$$g'(x) = \frac{d}{dx}\int_0^x t^2\ dt = x^2$$

by FTC1. Yep, it's that easy (even if you don't know what just happened).

Note: If the upper limit is more complicated than just x, then you need to combine FTC1 with the chain rule. The bottom line is that you will replace all occurrences of t's with whatever the upper limit of integration is and then multiply by the derivative of that expression. For example,

$$\frac{d}{dx}\int_0^{\sin x} t^2\ dt = (\sin x)^2 \cos x.$$

The Fundamental Theorem of Calculus, Part 2 (FTC2): This is the more important part of the Fundamental Theorem of Calculus, mostly because we will use it so much. It says that if f is continuous on the interval $[a, b]$, then

$$\boxed{\int_a^b f(x)\ dx = F(x)\Big|_a^b = F(b) - F(a),}$$

where $F' = f$. Holy crap; this is amazing! If you look back to the first page, you'll see that the definite integral of f from a to b was equal to some really complicated limit of a Riemann sum and you know that computing definite integrals that way is pretty dang hard. FTC2 tells us that if f is continuous, there is a much, much, much easier way. All we have to do is figure out what the antiderivative of f is, plug in b, plug in a, and subtract what we get in the correct order.

Example: Consider the function $f(x) = x^2 - x$ on the interval $[0, 2]$. If you look back in your notes from Section 5.2, you will see that I computed the definite integral of f on the interval $[0, 2]$ using the limit of a Riemann sum. It was pretty hard. But since f is continuous, we can just use FTC2 and get

$$\begin{aligned}\int_0^2 f(x)\ dx &= \int_0^2 x^2 - x\ dx \\ &= \frac{x^3}{3} - \frac{x^2}{2}\Big|_0^2 \\ &= \left(\frac{2^3}{3} - \frac{2^2}{2}\right) - \left(\frac{0^3}{3} - \frac{0^2}{2}\right) \\ &= \left(\frac{8}{3} - 2\right) - 0 \\ &= \frac{2}{3}.\end{aligned}$$

(Note: I have some vague memory of getting a negative value when we did this example the hard way in class. If so, I bet we/I made a sign error on the very last step when we subtracted.)

Note: I have two final comments:

1. FTC2 shows us how antiderivatives are related to net signed area. In other words, FTC2 ties together the concepts of indefinite integrals (antiderivatives) with definite integrals (net signed area).
2. The journey from limits of Riemann sums to the much more friendly FTC2, is analogous to how we approached slopes of tangent lines. We started by approximating the slope of a tangent line by first finding slopes of secant lines. Then we took limits of the slopes of secant lines to obtain a complicated formula for the slope of the tangent line (i.e., the derivative). It didn't take us long to realize that working with definition of the derivative was cumbersome. So, we developed shortcuts (power rule, product rule, quotient rule, chain rule, etc.). FTC2 is our shortcut to the cumbersome limit of Riemann sums.