Section 12.2: Series (part 1)

Goal

In this section, we introduce the concept of an infinite series, which is a sum of an infinite sequence.

Definition of a series and basic examples

Recall 1. A sequence is of the form

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\}.$$

Intuitively, a sequence is a infinite list of objects (usually numbers) that have a specified order, so that it makes sense to refer to the nth term, which in this case is denoted by a_n.

If we add up all the terms of a sequence of numbers, we obtain a *series*. More specifically, we have the following definition.

Definition 2. Let $\{a_n\}$ be a sequence of numbers. Then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots$$

is called the (*infinite*) *series* of $\{a_n\}$. If it is clear from the context that the sum is from $n = 1$ to ∞, then we may denote the series by

$$\sum a_n.$$

Important Note 3. A sequence is list of objects separated by commas and a series is a list of numbers separated by ____________.

Certainly, we understand how to add up a finite number of numbers (no matter how many), but what does it even mean to add up an infinite number of numbers? Perhaps surprisingly, sometimes the sum of an infinite series may actually finite. This is reminiscent of when we encountered infinitely long regions with finite area. Let's explore a couple of examples.

Example 4.

(a) Consider the sequence $\{n^2\}_{n=1}^{\infty} = \{$__________________________$\}$. What is $\sum_{n=1}^{\infty} n^2$? Well, let's see what happens as we add more and more terms. Let s_n denote the sum of the first n terms. Then

$$s_1 = \sum_{i=1}^{1} i^2 = \_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_$$

$$s_2 = \sum_{i=1}^{2} i^2 = \_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_$$

$$s_3 = \sum_{i=1}^{3} i^2 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_$$

$$s_4 = \sum_{i=1}^{4} i^2 = \underline{\hspace{6cm}} = \underline{\hspace{3cm}}$$

$\vdots$

$$s_n = \sum_{i=1}^{n} i^2 = \underline{\hspace{6cm}}$$

You may not remember this, but the last line above is a sum that we encountered in Calculus I when dealing with Riemann sums. It turns out that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \text{(see page 303).}$$

Therefore, the sum of the first n terms of the series is equal to $\dfrac{n(n+1)(2n+1)}{6}$. This implies that

$$\sum_{i=1}^{\infty} i^2 = \lim_{n\to\infty} \sum_{i=1}^{n} i^2$$

$$= \underline{\hspace{6cm}}$$

$$= \underline{\hspace{3cm}}.$$

We probably made this problem look unnecessarily difficult since I don't think anyone is surprised that this sum marched off to infinity. However, our method of attack can be applied to other problems, so it was worth the effort.

(b) Now, consider the sequence $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty} = \{\underline{\hspace{6cm}}\}$. What is $\sum_{n=1}^{\infty} \frac{1}{2^n}$?

$$s_1 = \sum_{i=1}^{1} \frac{1}{2^i} = \underline{\hspace{3cm}} = \underline{\hspace{3cm}}$$

$$s_2 = \sum_{i=1}^{2} \frac{1}{2^i} = \underline{\hspace{3cm}} = \underline{\hspace{3cm}}$$

$$s_3 = \sum_{i=1}^{3} \frac{1}{2^i} = \underline{\hspace{6cm}} = \underline{\hspace{3cm}}$$

$\vdots$

$$s_n = \sum_{i=1}^{n} \frac{1}{2^i} = \underline{\hspace{6cm}} = \underline{\hspace{3cm}}$$

Since $\lim\limits_{n\to\infty} \underline{\hspace{3cm}} = \underline{\hspace{3cm}}$, it must be the case that

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \underline{\hspace{3cm}}.$$

We can capture an intuitive understanding of this second example by thinking of it in the following way.

Imagine you are 1 mile away from home. Walk halfway home. How far have you traveled? Now, walk half of the remaining distance. How far have you traveled in total now? Again walk half of the remaining distance. Continue this process ad infinitum. How far will you have walked in total?

This particular example is known as *Zeno's Dichotomy Paradox* and one that philosophers and mathematicians have thought about quite a bit. According to Aristotle, Zeno of Elea said,

> "The first asserts the non-existence of motion on the ground that that which is in locomotion must arrive at the half-way stage before it arrives at the goal." (Aristotle *Physics*, 239b11)

Zeno was arguing that motion is impossible. For more information, do a search for "Zeno's paradox" or see

http://plato.stanford.edu/entries/paradox-zeno/

Definition 5. Given a series $\sum a_n$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_i + \cdots a_{n-1} + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called *convergent* and we write

$$\sum_{n=1}^{\infty} a_n = s,$$

where the number s is called the *sum* of the series. If a series does not converge, it ____________.

Whoa! Now wait a second...so there is a sequence $\{a_n\}$, which is a infinite list separated by commas, a series $\sum a_n$, which is a infinite summation, and another sequence $\{s_n\}$, which is the sequence formed by adding the first term, the first two terms, the first three terms, etc. of $\{a_n\}$? Yep, that's right.

We definitely need to take a look at some more examples.

Telescoping Series

Example 6. Find the sum of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. (There is room to finish this example on the next page.)

Geometric Series

Definition 7. A *geometric series* is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} \quad \text{or} \quad \sum_{n=0}^{\infty} ar^{n},$$

where a and r are constant real numbers (r is sometimes called the *ratio*).

For example, $\sum_{n=1}^{\infty} 5\left(\frac{-1}{3}\right)^{n-1}$ is a geometric series with $a = 5$ and $r = -1/3$.

Theorem 8.

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \textit{diverges}, & \textit{if } |r| \geq 1 \\ \frac{a}{1-r}, & \textit{if } |r| < 1 \end{cases}$$

For a proof, see page 725.

Example 9. Find sum.

(a) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$

(b) $\displaystyle\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$

(c) $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$