Counting generators in type B Temperley–Lieb diagrams

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Definition

A Coxeter group consists of a group W together with a generating set S consisting of elements of order 2 with presentation

$$W = \langle S : s^2 = e, (st)^{m(s,t)} = e \rangle,$$

where $m(s,t) \geq 2$ for $s \neq t$.

Comment

Since s and t are elements of order 2, the relation $(st)^{m(s,t)} = e$ can be rewritten as

$$\left.\begin{array}{lcr} m(s,t)=2 & \Longrightarrow & st = ts \end{array}\right\} \quad \text{short braid relations}$$

$$\left.\begin{array}{lcr} m(s,t)=3 & \Longrightarrow & sts = tst \\ m(s,t)=4 & \Longrightarrow & stst = tsts \\ & \vdots & \end{array}\right\} \quad \text{long braid relations}$$

Comment

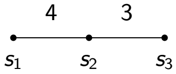
Given a Coxeter group W, we can encode the defining relations in a Coxeter graph.

Example

The Coxeter group of type B_3 is defined to be the group generated by s_1, s_2, s_3 subject to:

- $s_i^2 = e$ for $i = 1, 2, 3$
- $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$
- $s_2 s_3 s_2 = s_3 s_2 s_3$
- $s_1 s_3 = s_3 s_1$

The corresponding Coxeter graph is:

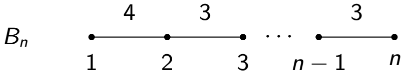


Comment

For brevity, we will no longer use s_i, but instead we will just use i. For example, we will write $1212 = 2121$ in place of $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$.

Definition

The Coxeter group of type B_n is determined by the following graph:



Then $W(B_n)$ is subject to:

- $1212 = 2121$,
- $iji = jij$, if $|i-j| = 1$ and $i, j \geq 2$,
- $ij = ji$, if $|i-j| \geq 2$,
- $i^2 = e$.

Definition

A "word" $s_{x_1} s_{x_2} \cdots s_{x_m}$ is called an expression for $w \in W$ if it is equal to w when considered as a group element.

If m is minimal, it is a reduced expression.

Example

Let 132121 be an expression for $w \in W(B_3)$. We see that

$$132121 = 131212 = 311212 = 3212,$$

showing that the original expression is *not* reduced. However, the last expression on the right is reduced.

Theorem (Matsumoto)

Any two reduced expressions for $w \in W$ differ by a sequence of braid relations.

Definition (Stembridge 1996)

Let $w \in W$. Then w is fully commutative (FC) iff every reduced expression "avoids long braid relations."

Examples

- Let 12142 be an expression for w in B_4. We see that

  $$12142 = 12124.$$

  The element clearly contains a long braid and therefore is not FC.
- Let 121342 be a reduced expression for w in B_4. Since

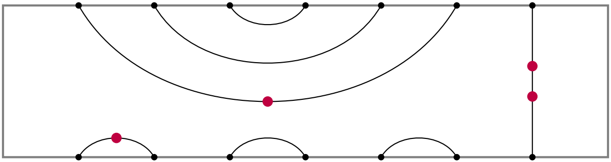
  $$121342, 121324, 123142, 123124, 123412$$

  is an exhaustive list of all possible reduced expressions and none of these contain long braids, w is FC.

We will introduce the collection of Temperley–Lieb diagrams of type B_n by way of example.

Example

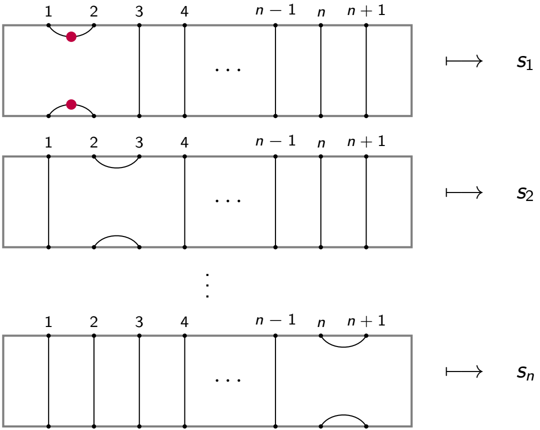
The following figure depicts a Temperley–Lieb diagram of type B_6.



Comments

- There is a one-to-one correspondence between the FC elements of type B_n and a certain collection of decorated $(n+1)$-diagrams.
- There is a combinatorial description of this collection of diagrams which we will not elaborate on.

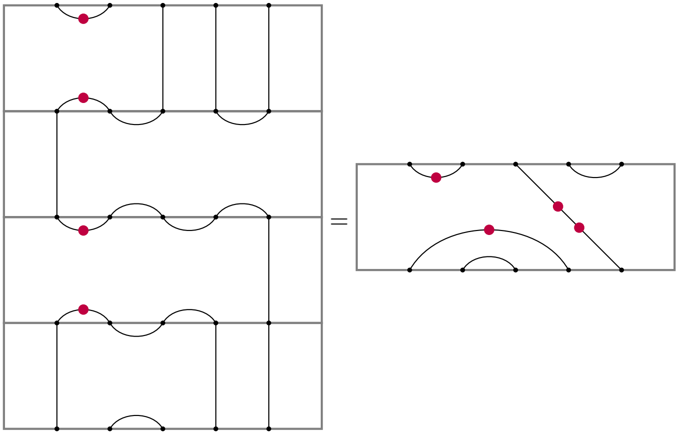
Consider the following Temperley–Lieb diagrams of type B_n.



Under this correspondence, we can create the diagrams that correspond to the FC elements by concatenating the diagrams and "pulling the strings tight."

Example

Here is an example of a product of several "generator" diagrams that corresponds to the FC element $124132 \in W(B_4)$.



One of our current results deals with the problem of counting the number of occurrences of a given generator in a diagram corresponding to an unknown FC element in a Coxeter group of type B_n.

Given a diagram d, embed d in the plane so that the lower left hand corner is at the origin and node i in the south face sits at $(i, 0)$. Also, let ℓ_i be the vertical line $x = i + \frac{1}{2}$.

Theorem (Ernst, Otis, Rivanis)

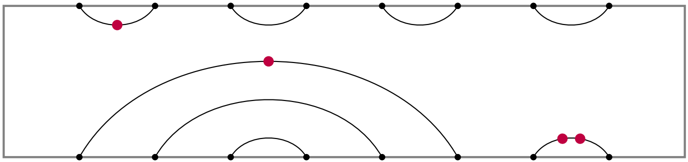
Let d be a diagram corresponding to an unknown FC element w in a Coxeter group of type B_n. Then the number of occurrences of s_i, $\#(s_i)$, in any reduced expression for w is given by

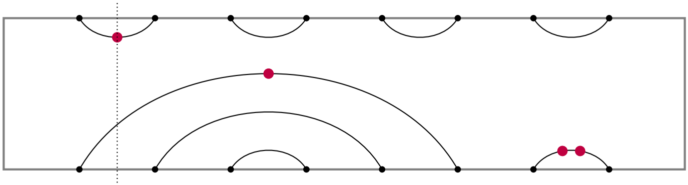
$$\frac{(\#\ \textit{intersections with}\ \ell_i) + (\#\ \textit{beads on non-intersected edges to the right of}\ \ell_i)}{2}.$$

Let's see an example.

Example

Consider the following diagram that corresponds to some FC element in type B_7.





Example (continued)

It turns out that the diagram on the previous slide corresponds to the FC element 1357246135243 in a Coxeter group of type B_7.