AN ISOMORPHISM THEOREM FOR QUOTIENT COMMUTATION GRAPHS

By Scott Akin

A Thesis

Submitted in Partial Fulfillment

of the Requirements for the Degree of

Master of Science

in Mathematics

Northern Arizona University

May 2026

Approved:

Dana C. Ernst, Ph.D., Chair

Nándor Sieben, Ph.D.

Michele Torielli, Ph.D.

ABSTRACT

AN ISOMORPHISM THEOREM FOR QUOTIENT COMMUTATION GRAPHS

SCOTT AKIN

Any two reduced expressions for an element of a Coxeter group are related by a sequence of commutation and braid moves. Two reduced expressions are called braid equivalent if they are related by a sequence of only braid moves. Braid equivalence is an equivalence relation, and the corresponding equivalence classes are called braid classes, which are encoded in braid graphs. Similarly, reduced expressions related by only commutation moves form commutation classes. The relationships between commutation classes are encoded in a graph called the quotient commutation graph, where vertices are commutation classes and edges represent braid moves between representatives of adjacent classes.

In this thesis, we investigate the structural relationship between braid graphs and quotient commutation graphs in simply-laced Coxeter systems. We introduce the transversal property: a condition where a single braid class contains exactly one representative from every commutation class of a given element. We prove an isomorphism theorem stating that for certain Coxeter systems, the quotient commutation graph is isomorphic to the braid graph of one of its reduced expressions if and only if the element possesses the transversal property. This work extends the theoretical framework established in recent research on reduced expressions and the combinatorial properties of simply-laced Coxeter systems. We conclude by providing several families of elements that exhibit the transversal property and hence satisfy the isomorphism.

Contents

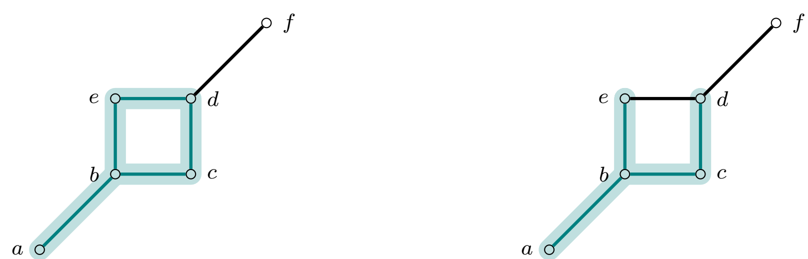
List of Figures

Chapter 1

Graph theory preliminaries

This chapter introduces the necessary terminology and results from graph theory. All graphs discussed throughout this thesis are assumed to be finite, connected, and simple. If G is a graph, let $V(G)$ denote the vertex set of G and let $E(G)$ denote its edge set. If $S \subseteq V(G)$ is any subset of vertices of G, then we define the *subgraph induced by S* to be the graph whose vertex set is S and whose edge set consists of all edges with endpoints in S.

Example 1.1. Consider the subgraphs highlighted in teal in Figure 1.1. The subgraph pictured in Figure 1.1(a) is induced by $S = \{a, b, c, d, e\}$, while the subgraph pictured in Figure 1.1(b) is not induced by S since the edge $\{d, e\}$ is not included.



(a) (b)

Figure 1.1: Example and non-example of an induced subgraph.

A *geodesic* between two vertices u and v of G is a shortest path (sequence of vertices connected by edges) between u and v. In this thesis, we will often use the notion of distance between two vertices in a graph to establish results. A connected graph G can be viewed as a metric space by taking the standard metric. That is, the *distance* between $u, v \in V(G)$ is defined via

$$d_G(u, v) := \text{length of any geodesic between } u \text{ and } v.$$

Note that if two vertices u and v are adjacent, then $d_G(u, v) = 1$.

Let G be a graph and H a subgraph of G. We say that H is an *isometric subgraph* of G if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. If in addition H is a cycle graph, then H is called an *isometric cycle*. Since preserving distance also preserves adjacency, every isometric subgraph is also an induced subgraph. However, as the next example illustrates, the converse is not true.

Example 1.2. Consider the graph G given in Figure 1.2. The subgraph highlighted in teal is induced by $\{a, b, c, d, e\}$, but it is not an isometric subgraph of G, since $d_H(a, e) = 4$ yet $d_G(a, e) = 2$.

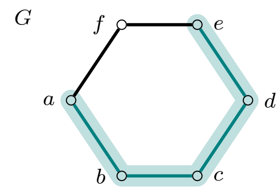


Figure 1.2: Example of an induced subgraph that is not isometric.

Given two graphs G_1 and G_2, the *box product*, denoted $G_1 \square G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$, where there is an edge from (x_1, y_1) to (x_2, y_2) if and only if either

(a) $x_1 = x_2$ and there is an edge from y_1 to y_2 in G_2, or

(b) $y_1 = y_2$ and there is an edge from x_1 to x_2 in G_1.

Up to isomorphism, the operation $\square$ is both associative and commutative.

For any $n \in \mathbb{N} \cup \{0\}$, we define the set of *binary strings* of length n as:

$$\{0, 1\}^n := \{a_1 a_2 \cdots a_n \mid a_k \in \{0, 1\}\}.$$

Note that the empty string is the only string of length $n = 0$. The *hypercube* Q_n of dimension n is the graph with vertex set $V(Q_n) = \{0, 1\}^n$, where there is an edge between two vertices when the corresponding binary strings differ by exactly one digit.

A graph G is called a *partial cube* if it can be isometricallly embedded in some hypercube Q_n. The *isometric dimension* $\dim_I(G)$ of a partial cube G is defined as the minimum dimension of the hypercube into which G can be isometrically embedded. That is,

$$\dim_I(G) := \min\{n \in \mathbb{N} \cup \{0\} \mid \text{there exists an isometric embedding of } G \text{ into } Q_n\}.$$

The following result from [18] states that the box product of two partial cubes is itself also a partial cube.

Proposition 1.3. If G_1 and G_2 are partial cubes, then $G_1 \square G_2$ is a partial cube with $\dim_I(G_1 \square G_2) = \dim_I(G_1) + \dim_I(G_2)$.

The *interval* between vertices u and v of a graph G is the set

$$I(u, v) := \{w \in V(G) \mid d_G(u, v) = d_G(u, w) + d_G(w, v)\}.$$

That is, $I(u, v)$ is the collection of vertices that lie on at least one geodesic between u and v. A graph G is *median* if

$$|I(u, v) \cap I(u, w) \cap I(v, w)| = 1$$

for all $u, v, w \in V(G)$. In other words, G is median if for all triples of vertices u, v, w, there is a unique vertex x that simultaneously lies on a geodesic between u and v, a geodesic between u and w, and a geodesic between v and w.

The next proposition from [19] connects partial cubes and median graphs.

Proposition 1.4. If G is a median graph, then G is a partial cube.

Note that the graph G in Figure 1.2 is a partial cube with isometric dimension three. However, it is not a median graph because $I(a, c) \cap I(c, e) \cap I(a, e) = \emptyset$, and so the converse to the above proposition is not true. But like the class of partial cubes, median graphs are closed under the box product operation. This result is well known; for example, see [5].

Proposition 1.5. If G_1 and G_2 are median graphs, then $G_1 \square G_2$ is also median.

Chapter 2

Introduction to Coxeter systems and braid graphs

We now turn our attention to the necessary information concerning Coxeter systems.

A *Coxeter matrix* is a finite dimensional $n \times n$ symmetric matrix $M = (m_{ij})$ with entries $m_{ij} \in \{1, 2, \ldots, \infty\}$ such that $m_{ii} = 1$ for all $1 \leq i \leq n$ and $m_{ij} \geq 2$ for $i \neq j$. A *Coxeter system* is a pair (W, S) consisting of a finite set $S = \{s_1, s_2, \ldots, s_n\}$ and a group W, called a *Coxeter group*, with presentation

$$W = \langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m(s_i, s_j)} = e \rangle,$$

where $m(s_i, s_j)$ is m_{ij} for some $n \times n$ Coxeter matrix $M = (m_{ij})$. For $s, t \in S$, the conditions $m(s, t) = \infty$ means that there is no relation imposed between s and t. In this thesis, all $m(s, t)$ will be finite. It turns out the elements of S are distinct as group elements and $m(s, t)$ is the order of st [14]. Since elements of S have order two, the relation $(st)^{m(s,t)} = e$ can be written as

$$\underbrace{sts \cdots}_{m(s,t)} = \underbrace{tst \cdots}_{m(s,t)}$$

with $m(s, t) \geq 2$ generators. When $m(s, t) = 2$, $st = ts$ is called a *commutation relation*, and when $m(s, t) \geq 3$, the corresponding relation is called a *braid relation*. The replacement

$$\underbrace{sts \cdots}_{m(s,t)} \longmapsto \underbrace{tst \cdots}_{m(s,t)}$$

is called a *commutation move* if $m(s, t) = 2$ and a *braid move* if $m(s, t) \geq 3$.

A Coxeter system (W, S) can be encoded by a unique *Coxeter graph* Γ with vertex set S and edges $\{s, t\}$ for each $m(s, t) \geq 3$. Each edge is labeled with the corresponding $m(s, t)$, and labels of 3 are often omitted because they are most common. In this case, we say that (W, S), or just W, is of type Γ, and we may denote the Coxeter group as $W(\Gamma)$ and the generating set as $S(\Gamma)$ for emphasis. If $m(s, t) \leq 3$ for all $s, t \in S$, we say that a Coxeter system is *simply laced*. Additionally, if a Coxeter graph has no three-cycles, we say the Coxeter system is *triangle-free*. Following [2], a Coxeter system (W, S) (or the group W) is of type Λ if (W, S) is both simply laced and triangle-free. The primary focus of this thesis will be on simply-laced Coxeter systems, often of type Λ.

Example 2.1. The Coxeter graphs shown in Figure 2.1 correspond to four common Coxeter systems. We summarize the defining relations for the Coxeter systems determined by the graphs in Figures 2.1(a) and 2.1(d) below.

(a) The Coxeter system of type A_n is given by the Coxeter graph in Figure 2.1(a). In this case, $W(A_n)$ is generated by $S(A_n) = \{s_1, s_2, \ldots, s_n\}$ and has the defining relations

1. $s_i^2 = e$ for all i,
2. $s_i s_j = s_j s_i$ when $|i - j| > 1$, and
3. $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group S_{n+1} under the mapping that sends s_i to the adjacent transposition $(i, i + 1)$.

(b) The Coxeter system of type D_n is given by the graph in Figure 2.1(d). The Coxeter group $W(D_n)$ is generated by $S(D_n) = \{s_1, s_2, \ldots, s_n\}$ and has the defining relations

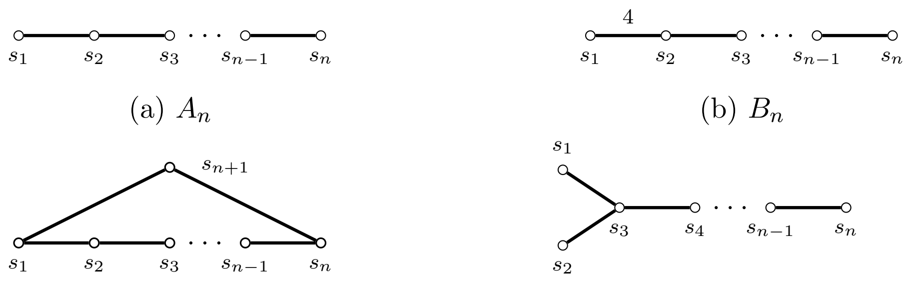
1. $s_i^2 = e$ for all i,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$ and $i, j \neq 1$,
3. $s_i s_j = s_j s_i$ if $i = 1$ and $j \neq 3$, and
4. $s_1 s_3 s_1 = s_3 s_1 s_3$ and $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$.

The Coxeter group $W(D_n)$ is isomorphic to the index-two subgroup of the group of signed permutations on n letters having an even number of sign changes.

Each of the Coxeter graphs displayed in Figure 2.1 represents a simply-laced Coxeter system, except for the one of type B_n. Each also represents a triangle-free Coxeter system, except for the one of type $\tilde{A}_2$.

Given a Coxeter System (W, S), let S^* denote the free monoid on the alphabet S. An element $\alpha = s_{x_1} s_{x_2} \cdots s_{x_m}$ in S^* is called a *word*. A *factor* of α is a word of the form $s_{x_i} s_{x_{i+1}} \cdots s_{x_{j-1}} s_{x_j}$ for $1 \leq i \leq j \leq m$. The word $\alpha = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is an *expression* for $w \in W$ if α is equal to w when considered as an element of the group W. If m is minimal among all possible expressions for w, we say that α is a *reduced expression* for w, and m is the *length* of w, denoted $\ell(w)$. Note that any proper factor of a reduced expression is also a reduced expression, though it is one for a different group element. We will denote the set of all reduced expressions for $w \in W$ by $\mathcal{R}(w)$.

The relationship between reduced expressions for a given group element is characterized by the following proposition, called Matsumoto's Theorem [9].



(a) A_n (b) B_n (c) $\widetilde{A}_n$ (d) D_n

Figure 2.1: Coxeter graphs for common Coxeter systems.

Proposition 2.2. In a Coxeter system (W, S), any two reduced expressions for the same group element differ by a sequence of commutation and braid moves.

In light of Matsumoto's Theorem, we can define a graph on the set of reduced expressions of a given element in a Coxeter group. For $w \in W$, define the *Matsumoto graph* $\mathcal{G}(w)$ to be the graph having vertex set equal to $\mathcal{R}(w)$, where two reduced expressions α and β are connected by an edge if and only if α and β are related via a single commutation or braid move. We will distinguish commutation and braid moves by coloring an edge orange if it corresponds to a commutation move, and blue if it corresponds to a braid move. Matsumoto's Theorem implies that $\mathcal{G}(w)$ is connected. In [12], the authors prove that every cycle in a Matsumoto graph has even length. This fact yields the following result.

Proposition 2.3. If (W, S) is a Coxeter system and $w \in W$, then $\mathcal{G}(w)$ is bipartite.

For the remainder of this thesis, if we are considering a particular labeling of a Coxeter graph, we will often replace s_i with i for brevity.

Example 2.4. Consider the so-called "longest element" w with reduced expression $\alpha = 121321$ in the Coxeter system of type A_3. It turns out that there are 16 reduced expressions in $\mathcal{R}(w)$. The corresponding Matsumoto graph is given in Figure 2.2. The 16 reduced expressions are the vertices of $\mathcal{G}(w)$ and the edges show how pairs of reduced expressions are related via commutation or braid moves. In each reduced expression, we use underlines or overlines to indicate the opportunity to apply a braid move.

Matsumoto's Theorem allows us to define two equivalence relations on the set of reduced expressions for a given element of a Coxeter group. Let (W, S) be a Coxeter system and let $w \in W$. For $\alpha, \beta \in \mathcal{R}(w)$, define $\alpha \sim_c \beta$ if we can obtain α from β by applying a single commutation move of the form $st \mapsto ts$. We define the equivalence relation $\approx_c$ by taking the reflexive and transitive closure of $\sim_c$. Each equivalence class under $\approx_c$ is called a *commutation class*, denoted $[\alpha]_c$ for $\alpha \in \mathcal{R}(w)$. Two reduced expressions are said to be *commutation equivalent* if they are in the same commutation class. We denote the collection

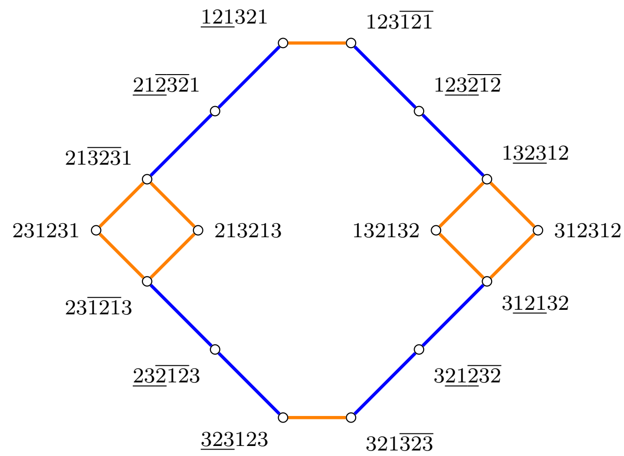


Figure 2.2: The Matsumoto graph in type A_3 from Example 2.4.

of all braid classes for w by $\mathrm{Comm}(w)$. Commutation classes have been studied extensively. For Coxeter systems of type A_n, Elnitsky [7] showed that the set of commutation class for a given w is in one-to-one correspondence with the set of rhombic tilings of a certain polygon determined by w; Bedard [3] developed recursive formulas to find the number of reduced expressions in each commutation class; and Meng [17] studied enumerations of commutation classes and relationships between them via braid moves.

Similarly, we define $\alpha \sim_b \beta$ if we can obtain α from β by applying a single braid move. In simply-laced Coxeter systems, this braid move is always of the form $sts \mapsto tst$, where $m(s,t) = 3$. The equivalence relation $\approx_b$ is defined by taking the reflexive and transitive closure of $\sim_b$. Each equivalence class under $\approx_b$ is called a *braid class*, denoted $[\alpha]_b$ for $\alpha \in \mathcal{R}(w)$. Two reduced expressions are said to be *braid equivalent* if they are in the same braid class. We denote the collection of all braid classes for w by $\mathrm{Braid}(w)$. Although not as widely studied as commutation classes, braid classes are discussed in the work of those such as Bergeron, et al. [4]. For Coxeter systems of type A_n, Fishel et al. [8] provided upper and lower bounds on the number of reduced expressions for a fixed permutation by studying commutation classes together with braid classes. Zollinger [22] provided formulas for the cardinality of braid classes, again in Coxeter systems of type A_n. More recently, braid classes have been the focus of [1] and [2].

Example 2.5. Recall the longest element w in the Coxeter system of type A_3 from Example 2.4. The set $\mathcal{R}(w)$ consisting of 16 reduced expressions is partitioned into eight commutation classes and eight braid classes:

$$
\begin{aligned}
&[121321]_c = \{121321, 123121\} && [121321]_b = \{121321, 212321, 213231\}\\
&[212321]_c = \{212321\} && [231231]_b = \{231231\}\\
&[213231]_c = \{213231, 231231, 213213, 231213\} && [213213]_b = \{213213\}\\
&[232123]_c = \{232123\} && [323123]_b = \{323123, 232123, 231213\}\\
&[323123]_c = \{323123, 321323\} && [321323]_b = \{321323, 321232, 312132\}\\
&[321232]_c = \{321232\} && [132132]_b = \{132132\}\\
&[312132]_c = \{312132, 132132, 312312, 132312\} && [312312]_b = \{312312\}\\
&[123212]_c = \{123212\} && [123121]_b = \{123121, 123212, 132312\}
\end{aligned}
$$

Notice that the commutation classes of size 2 and 4 correspond to the vertices in the orange connected components of the Matsumoto graph given in Figure 2.2, whereas the singleton commutation classes correspond to the four vertices that are not incident to any orange edges. A similar structure holds for the braid classes. The number of commutation classes for a fixed group element generally does not equal the number of braid classes. For example, consider the group element in type A_4 with reduced expression $\beta = 121343$, which has four commutation classes and fourteen braid classes.

We can see the relationship among reduced expression in a fixed braid or commutation class by looking at the maximal blue or orange connected components of the underlying Matsumoto graph. This leads to the following definitions. Let α be a reduced expression for $w \in W$. We define the *commutation graph* of α, denoted $\mathcal{C}(\alpha)$, to be the graph with vertex set $[\alpha]_c$, where $\alpha, \beta \in [\alpha]_c$ are connected by an edge if and only if α and β are related via a single commutation move. Similarly, the *braid graph* of α, denoted $\mathcal{B}(\alpha)$, is defined as the graph with vertex set $[\alpha]_b$, where $\alpha, \beta \in [\alpha]_b$ are connected by an edge if and only if α and β are related via a single braid move. We conjecture that for any Coxeter system, commutation and braid graphs are isometric subgraphs of the underlying Matsumoto graph.

Note that braid graphs are defined with respect to a fixed reduced expression as opposed to the corresponding group element. Moreover, if α and β are braid equivalent, then $\mathcal{B}(\alpha) = \mathcal{B}(\beta)$. A similar line of reasoning follows for commutation graphs.

Example 2.6. In the following examples, we present three distinct braid classes and their corresponding braid graphs.

(a) In the Coxeter system of type A_4, the expression $\alpha_1 = 12132$ is reduced. Its braid class consists of the following reduced expressions:

$$\alpha_1 = \underline{121}32,\ \alpha_2 = \underline{21}\overline{232},\ \alpha_3 = 21\overline{323}.$$

(b) In the Coxeter system of type A_5, the expression $\beta_1 = 12132454$ is reduced. Notice that α_1 from the previous example is a factor of β_1. The braid class of β_1 consists of the following reduced expressions:

$$\beta_1 = \underline{121}32\underline{454},\ \beta_2 = \underline{21}\overline{\underline{2}32}\underline{454},\ \beta_3 = 21\overline{323}\underline{454},$$
$$\beta_4 = \underline{121}32\underline{545},\ \beta_5 = \underline{21}\overline{\underline{2}32}\underline{545},\ \beta_6 = 21\overline{323}\underline{545}.$$

(c) The expression $\gamma_1 = 1312434$ is reduced in the Coxeter system of type D_4. Its braid class consists of the following reduced expressions:

$$\gamma_1 = \underline{131}2\underline{434},\ \gamma_2 = \underline{313}2\underline{434},\ \gamma_3 = \underline{131}2\underline{343},\ \gamma_4 = \underline{31}\overline{\underline{3}2\underline{3}}\underline{43},\ \gamma_5 = 31\overline{232}43.$$

The braid graphs $\mathcal{B}(\alpha_1)$, $\mathcal{B}(\alpha_1)$, and $\mathcal{B}(\alpha_1)$ are depicted in Figure 2.3.

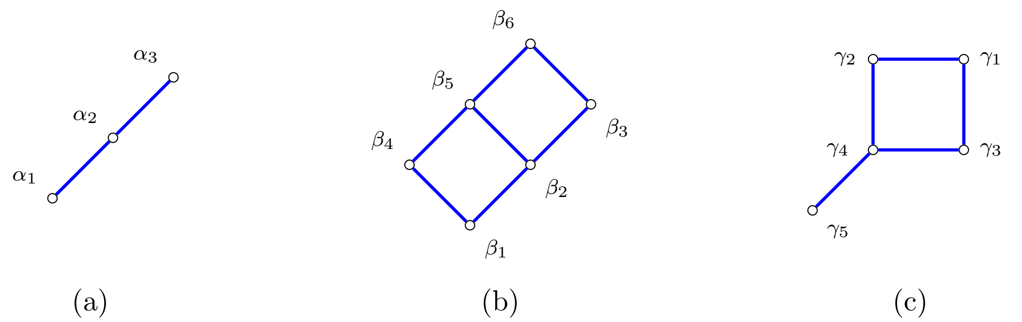


Figure 2.3: The braid graphs generated by the reduced expressions in Example 2.6.

We now discuss more terminology that will allow us to introduce the notions of braid shadow and link. Throughout the rest of this chapter, we assume that (W, S) is a simply-laced Coxeter system. This condition is necessary for many of the following results, but they likely generalize with appropriate modifications.

If $i, j \in \mathbb{N}$ with $i < j$, then we define the interval $[\![i, j]\!] := \{i, i+1, \ldots, j-1, j\}$ and let the degenerate interval $[\![i, i]\!]$ be the singleton set $\{i\}$. We will use the intervals $[\![i, j]\!]$ to represent positions in a reduced expression. If $\alpha = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression for $w \in W$, we define the *support* of α over $[\![i, j]\!]$ to be

$$\mathrm{supp}_{[\![i,j]\!]}(\alpha) := \{s_{x_k} \mid k \in [\![i, j]\!]\}.$$

In other words, $\mathrm{supp}_{[\![i,j]\!]}(\alpha)$ is the set containing the generators that appear in positions $i, i+1, \ldots, j$ of α. We will also let $\alpha_{[\![i,j]\!]}$ denote the subword $s_{x_i} s_{x_{i+1}} \cdots s_{x_{j-1}} s_{x_j}$ of α.

Following [1], if $\alpha = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression for $w \in W$, then the interval $[\![i, i+2]\!]$ is a *braid shadow* for α if $s_{x_i} = s_{x_{i+2}}$ and $m(s_{x_i}, s_{x_{i+1}}) = 3$. That is, $\alpha_{[\![i,i+2]\!]} = sts$ for some s and t with $m(s, t) = 3$. The collection of braid shadows for α is denoted by $\mathcal{S}(\alpha)$ and the set of braid shadows for the braid class $[\alpha]_b$ is given by

$$\mathcal{S}([\alpha]_b) := \bigcup_{\beta \in [\alpha]_b} \mathcal{S}(\beta).$$

The *rank* of α is defined as $\text{rank}(\alpha) := |\mathcal{S}([\alpha]_b)|$.

In summary, a braid shadow for a reduced expression α refers to a location where we can apply a braid move in α. A reduced expression may have one or more braid shadows, or none at all. The set $\mathcal{S}(\alpha)$ is the collection of the braid shadows for a given α, while $\mathcal{S}([\alpha]_b)$ is the collection of braid shadows from every reduced expression that is braid equivalent to α. If $[\![i, i+2]\!]$ is a braid shadow in $\mathcal{S}([\alpha]_b)$, then we will refer to the position $i+1$ in any reduced expression $\beta \in [\alpha]_b$ as the *center* of the braid shadow for β. We refer to position $i+1$ as the center of a braid shadow regardless of whether $[\![i, i+2]\!]$ is in $\mathcal{S}(\beta)$.

Example 2.7. Consider the reduced expressions given in Example 2.6. Here, we see that

(a) $\mathcal{S}(\alpha_1) = \{[\![1,3]\!]\}$ and $\mathcal{S}([\alpha_1]_b) = \{[\![1,3]\!], [\![3,5]\!]\}$,

(b) $\mathcal{S}(\beta_1) = \{[\![1,3]\!]\}$ and $\mathcal{S}([\beta_1]_b) = \{[\![1,3]\!], [\![3,5]\!], [\![6,8]\!]\}$, and

(c) $\mathcal{S}(\gamma_1) = \{[\![1,3]\!], [\![5,7]\!]\}$ and $\mathcal{S}([\gamma_1]_b) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!]\}$.

If α is a reduced expression for $w \in W$, then two braid shadows in $\mathcal{S}([\alpha]_b)$ are either disjoint or overlap by a single position, as stated in the following proposition from [1].

Proposition 2.8. Suppose (W, S) is a simply-laced Coxeter system. If α is a reduced expression for $w \in W$ with $I, J \in \mathcal{S}([\alpha_b])$ such that $I \neq J$, then $|I \cap J| \leq 1$.

Reduced expressions with the property that consecutive braid shadows have no gaps serve an important role as "building blocks" for larger reduced expressions. The previous result motivates the next definition. If $\alpha = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression for $w \in W$ in a simply-laced Coxeter system with $m \geq 1$. We define α to be a *link* if either $m = 1$ or m is odd and

$$\mathcal{S}([\alpha]_b) = \{[\![1,3]\!], [\![3,5]\!], \ldots, [\![m-4, m-2]\!], [\![m-2, m]\!]\}.$$

Loosely speaking, α is a link if there is a sequence of overlapping braid moves that "cover" the positions $1, 2, \ldots, m$. Note that if $m = 1$, then $\text{rank}(\alpha) = 0$. Otherwise, if $m = 2k+1$, then $\text{rank}(\alpha) = k$. For simply-laced Coxeter systems, the center of every braid shadow across the braid class for a link has an even index.

Example 2.9. Consider the reduced expressions from Example 2.6. Recall that $\mathcal{S}([\alpha_1]_b) = \{[\![1,3]\!], [\![3,5]\!]\}$, so α_1 is a link of rank two. However, since $\mathcal{S}([\beta_1]_b) = \{[\![1,3]\!], [\![3,5]\!], [\![6,8]\!]\}$, β_1 is not a link, though the factors 12132 and 454 of β_1 are themselves links. Lastly, since $\mathcal{S}([\gamma_1]_b) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!]\}$, γ_1 is a link of rank three.

If α is a reduced expression for $w \in W$ with $\ell(w) \geq 1$, then we say that β is a *link factor* of α provided that

(a) β is a factor of α,

(b) β is a link, and

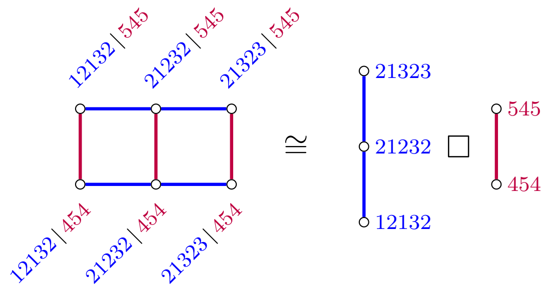


Figure 2.4: The braid graph for the reduced expression in type A_5 from Example 2.6(b) and its decomposition into a box product for the corresponding link factors.

(c) for every factor γ of α, if β is a factor of γ and γ is a link, then $\beta = \gamma$.

It immediately follows that every reduced expression α for a nonidentity group element can be written as a unique product of link factors, say $\alpha_1\alpha_2\cdots\alpha_k$, where each α_i is a link factor of α. This product will be referred to as the *link factorization* of α. Following [1], we may denote the link factorization as $\alpha = \alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_k$ for emphasis.

Originally appearing in [1], the following proposition is an immediate consequence of previous definitions. It says that, using the link factorization of a reduced expression, every braid graph for a reduced expression can be written as a box product of the braid graphs of the corresponding link factors in the link factorization. The decomposition is unique if the ordering of the link factors is respected.

Proposition 2.10. Suppose (W, S) is a simply-laced Coxeter system. If α is a reduced expression for $w \in W$ with link factorization $\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_k$, then

(a) $[\alpha]_b = \{\beta_1 \mid \beta_2 \mid \cdots \mid \beta_k : \beta_i \in [\alpha_i] \text{ for } 1 \leq i \leq k\}$,

(b) $|[\alpha]_b| = \prod_{i=1}^{k} |[\alpha_i]_b|$,

(c) $\operatorname{rank}(\alpha) = \sum_{i=1}^{k} \operatorname{rank}(\alpha_i)$, and

(d) $\mathcal{B}(\alpha) \cong \mathcal{B}(\alpha_1)\Box\mathcal{B}(\alpha_2)\Box\cdots\Box\mathcal{B}(\alpha_k)$.

Example 2.11. Consider the reduced expression β_1 defined in Example 2.6(b). The link factorization for β_1 is 12132|454. The decomposition $\mathcal{B}(\beta_1) \cong \mathcal{B}(12132)\Box\mathcal{B}(454)$ guaranteed by Proposition 2.10 is shown in Figure 2.4. Note that we have utilized colors to help distinguish link factors.

The next result summarizes two of the main results in [1] and [2]. In [1], the authors proved that braid graphs of reduced expressions in Coxeter systems of type Λ are partial cubes, while in [2], the authors extended this result and proved that braid graphs in type

Λ Coxeter systems are median. In both cases, the authors first established the results for links, then utilized Propositions 1.3 and 1.5, together with Proposition 2.10, to generalize to arbitrary reduced expressions.

Proposition 2.12. If (W, S) is type Λ and α is a reduced expression, then $\mathcal{B}(\alpha)$ is a median graph and a partial cube with isometric dimension $\text{rank}(\alpha)$.

Note that not every median graph is a braid graph. A counterexample is provided in [2]. We immediately have the following result, which appeared as Corollary 5.18 in [1].

Proposition 2.13. If (W, S) is type Λ and α is a reduced expression, then $|[\alpha]_b| \leq 2^{\text{rank}(\alpha)}$.

Chapter 3

Geodesics in braid graphs

Adding to our knowledge of braid classes, we now focus precisely on Coxeter systems of type Λ. The next several propositions summarize key results from [1] and [2] concerning the structure of links in Coxeter systems of type Λ. We begin with the following result, which blends Proposition 3.15 and Remark 3.18 from [1].

Proposition 3.1. Suppose (W, S) is type Λ and let α be a link of rank $r \geq 1$. For each $1 \leq i \leq r$, there exists unique $s, t \in S$ with $m(s,t) = 3$ such that if $\beta \in [\alpha]_b$ with $[\![2i-1, 2i+1]\!] \in \mathcal{S}(\beta)$, then $\beta_{[\![2i-1,2i+1]\!]} \in \{sts, tst\}$ and for all $\gamma \in [\alpha]_b$, $\gamma_{[\![2i]\!]} \in \{s, t\}$.

The previous proposition states that in Coxeter systems of type Λ, the support of braid shadows for links is "stable" and determines which generators appear at their corresponding centers across the entire braid class. In particular, in type Λ Coxeter systems, the center of each braid shadow may take on one of two possible values.

The next example illustrates the need for the triangle-free hypothesis in Proposition 3.1.

Example 3.2. Consider the link $\alpha_1 = 1231321$ in the Coxeter system of type $\widetilde{A}_2$, which is determined by the Coxeter graph in Figure 2.1(c). The braid class for α_1 consists of the following links:

$$\alpha_1 = 12\overline{313}21,\ \alpha_2 = \underline{12\overline{1}}\overline{3}\underline{\overline{1}21},\ \alpha_3 = \underline{212}3\underline{121},$$

$$\alpha_4 = \underline{121}3\underline{212},\ \alpha_5 = \underline{21\overline{2}}\overline{3}\underline{\overline{2}12},\ \alpha_6 = 21\overline{323}12.$$

The braid graph $\mathcal{B}(\alpha_1)$ is depicted in Figure 3.1, where we have colored each edge according to its corresponding braid shadow. That is, violet, cyan, and blue correspond to the braid shadows $[\![1,3]\!]$, $[\![3,5]\!]$, and $[\![5,7]\!]$, respectively. Looking across $[\alpha_1]_b$, the generators 1, 2, and 3 each appear in the center of the second braid shadow, and hence Proposition 3.1 may fail if the Coxeter system is not triangle-free.

If α is a link of rank $r \geq 1$, then for each $1 \leq i \leq r$, define the i^{th} *signature* of α via

$$\operatorname{sig}_i(\alpha) := \alpha_{[\![2i]\!]}.$$

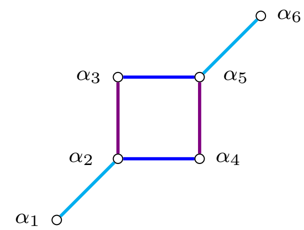


Figure 3.1: The braid graph for the link in type $\tilde{A}_2$ from Example 3.2.

That is, the i^{th} signature of a link α is the generator appearing at the center of the i^{th} braid shadow in $\mathcal{S}([\alpha]_b)$. In light of Proposition 3.1, in Coxeter systems of type Λ, each $\text{sig}_i(\alpha)$ takes on one of the two values from the support of the corresponding center. Note that Example 3.2 shows this may fail in Coxeter systems that are not of type Λ. Now, define the *signature* of α as

$$\text{sig}(\alpha) := (\text{sig}_1(\alpha), \ldots, \text{sig}_r(\alpha)).$$

Simply put, $\text{sig}(\alpha)$ is the ordered list of generators in α that appear at the centers of the braid shadows in $\mathcal{S}([\alpha]_b)$. For convenience, we define the signature of a link of rank 0 to be the empty list. One can extend the definition of signature to arbitrary reduced expressions by concatenating the signature of the corresponding link factors.

For braid-equivalent links α and β of rank at least one, we define $\Delta(\text{sig}(\alpha), \text{sig}(\beta))$ to be the number of entries that differ between the signatures of α and β. In Corollary 3.6, we will see that $\Delta(\text{sig}(\alpha), \text{sig}(\beta))$ is the length of any geodesic between α and β.

Example 3.3. Consider the braid class containing $\gamma_1 = 1312434$ and $\gamma_4 = 3132343$ in the Coxeter system of type D_4 from Example 2.6(c). We see that $\text{sig}(\gamma_1) = (3, 2, 3)$ and $\text{sig}(\gamma_4) = (1, 2, 4)$, and so $\Delta(\text{sig}(\gamma_1), \text{sig}(\gamma_4)) = 2$. Moreover, observe that $d_{\mathcal{B}(\alpha)}(\gamma_1, \gamma_2) = 2$.

If α and β are links that are related by a single braid move that occurs in the j^{th} braid shadow (i.e., only the j^{th} entry of the signature differs between α and β), we may accordingly label the edge in $\mathcal{B}(\alpha)$ connecting α and β with j.

Mimicking [2], if α and β are braid-equivalent reduced expressions in a simply-laced Coxeter system, let $b_1^{j_1}, b_2^{j_2}, \ldots, b_k^{j_k}$ denote a minimal sequence of braid moves from α to β, where $b_i^{j_i}$ is the i^{th} braid move in the sequence that occurs in the j_i^{th} shadow of $\mathcal{S}([\alpha]_b)$. A minimal braid sequence from α to β corresponds to a geodesic in $\mathcal{B}(\alpha)$ from α to β consisting of edges consecutively labeled $j_1, j_2, \ldots, j_k$. Informally, the index i tracks "when" a braid move appears in the sequence, and the superscript j_i tracks which shadow supports the braid move, i.e., the "where." Graphically, the next proposition from [2] states that each geodesic between two links in a braid graph utilizes the same set of edge labels, and each label appears once.

Proposition 3.4. Suppose (W, S) is type Λ and let α and β be two braid equivalent links of rank at least one. A braid sequence $b_1^{j_1}, b_2^{j_2}, ..., b_k^{j_k}$ from α to β is minimal if and only if each

j_i appears exactly once. Moreover, if $b_1^{j_1}, b_2^{j_2}, \ldots, b_k^{j_k}$ and $b_1^{l_1}, b_2^{l_2}, \ldots, b_k^{l_k}$ are minimal braid sequences from α to β, then $\{j_1, ..., j_k\} = \{l_1, ..., l_k\}$.

Example 3.5. Figure 3.2 depicts the braid graph for the link $\beta_1 = 313234313$ of a group element w in the Coxeter system of type D_4. Note that β_1 is a link. The blue, cyan, green, and violet edges correspond to the braid shadows $[\![1, 3]\!]$, $[\![3, 5]\!]$, $[\![5, 7]\!]$, and $[\![7, 9]\!]$, respectively. One can verify that each geodesic between any pair of vertices utilizes the same set of colors, with each color appearing exactly once. However, for the braid graph in Figure 3.1 (see Example 3.2), we see that each geodesic between α_1 and α_6 repeats the (cyan) braid shadow $[\![3, 5]\!]$. Example 3.2 shows that Proposition 3.4 requires the type Λ hypothesis.

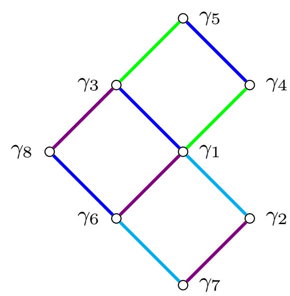


Figure 3.2: The braid graph for the link in type D_4 from Example 3.5, where edges are colored according to their corresponding braid shadow.

The following fact is an immediate consequence of Proposition 3.4 and appears in [2].

Corollary 3.6. If (W, S) is type Λ and α and β are braid-equivalent links, then $d_{\mathcal{B}(\alpha)}(\alpha, \beta) = \Delta(\text{sig}(\alpha), \text{sig}(\beta))$.

We now extend the first part of the previous proposition to arbitrary reduced expressions.

Proposition 3.7. Suppose (W, S) is type Λ and let α and β be two braid-equivalent reduced expressions. A braid sequence $b_1^{j_1}, b_2^{j_2}, \ldots, b_m^{j_m}$ from α to β is minimal if and only if each j_i appears exactly once.

Proof. For the forward implication, assume $b_1^{j_1}, b_2^{j_2}, \ldots, b_m^{j_m}$ is minimal. Note that two terms $b_x^{j_x}$ and $b_y^{j_y}$ in the sequence commute if the braid shadows corresponding to j_x and j_y appear in different link factors. Let $\alpha = \ell_1 | \ell_2 | \cdots | \ell_k$ be the link factorization of α. Sorting by link factor, we may rearrange the sequence $b_1^{j_1}, b_2^{j_2}, \ldots, b_m^{j_m}$ to construct $B_{\ell_1}, B_{\ell_2}, \ldots, B_{\ell_k}$, another minimal braid sequence from α to β in which each B_{ℓ_n} is a sequence of the braid moves acting on its corresponding link factor ℓ_n. By Proposition 3.4, since each B_{ℓ_n} is a minimal braid (sub)sequence, each j_i appears at most once in each B_{ℓ_n}. Thus, each j_i appears exactly once in the complete braid sequence from α to β.

For the reverse implication, assume $b_1^{j_1}, b_2^{j_2}, \ldots, b_k^{j_k}$ is a braid sequence from α to β where each j_i appears exactly once. Because there is no repeated j_i, $k = \Delta\text{sig}(\alpha, \beta)$. By Corollary 3.6, $d_{\mathcal{B}(\alpha)}(\alpha, \beta) = \Delta\text{sig}(\alpha, \beta)$. Hence, $k = d_{\mathcal{B}(\alpha)}(\alpha, \beta)$, so k is minimal. □

Chapter 4

Notions concerning adjacent commutation classes

This chapter focuses on local relationships between commutation classes and establishes the foundational results that lead to our main result (see Theorem 5.6). Although both braid and commutation classes of a group element w are formally subsets of $\mathcal{R}(w)$, we develop the results of this chapter by considering their representations in $\mathcal{G}(w)$.

For a reduced expression α and pair of generators s, t such that $m(s,t) = 3$, let the *st-subexpression* of α, denoted $\alpha(s,t)$, be defined as the (possibly empty) subexpression obtained from α by restricting to the letters s and t. Note that $\alpha(s,t)$ may no longer be reduced. The following proposition is apparent, and we certainly need not restrict ourselves to the simply-laced context.

Proposition 4.1. If α is a reduced expression and $m(s,t) = 3$, then $\alpha(s,t) = \beta(s,t)$ for all $\beta \in [\alpha]_c$.

Recall that for a group element w, braid and commutation classes each partition $\mathcal{R}(w)$. A helpful use of the st-subexpression is showing that any braid class intersects any commutation class by at most one element. Extending a result from [8], who originally assumed Coxeter systems of type A_n, we modify their proof to show the result holds for all simply-laced Coxeter systems. It likely generalizes to arbitrary Coxeter systems.

Proposition 4.2. Let (W, S) be a simply-laced Coxeter system and $w \in W$. If $B \in \mathrm{Braid}(w)$ and $C \in \mathrm{Comm}(w)$, then $|B \cap C| \leq 1$.

Proof. Suppose $\alpha, \beta \in \mathcal{R}(w)$ are distinct braid-equivalent reduced expressions with link factorizations $\alpha = \alpha_1 \mid \cdots \mid \alpha_r$ and $\beta = \beta_1 \mid \cdots \mid \beta_r$. By Proposition 2.10, there is a smallest k such that $\alpha_k \neq \beta_k$, where the rank of each of these link factors is at least one. Moreover, there is a leftmost entry i such that $\mathrm{sig}_i(\alpha_k) \neq \mathrm{sig}(\beta_k)$. Without loss of generality, let $\mathrm{sig}_i(\alpha_k) = s$ and $\mathrm{sig}_i(\beta_k) = t$ where $m(s,t) = 3$. Thus $\alpha(s,t) \neq \beta(s,t)$, and since the st-subexpression is invariant over commutation classes (Proposition 4.1), α and β are not in the same commutation class. $\square$

We now establish the following framework for understanding braid moves between reduced expressions. For the st-subexpression of α, we call $(i, i+1, i+2)_{sts}$ an *sts-braid admissible triple* if there exists $\beta \in [\alpha]_c$ such that the content of $\beta(s,t) = \alpha(s,t)$ at those indices is sts and the corresponding letters in β are all adjacent. Note that an sts-braid admissible triple identifies adjacent generators in an st-subexpression, which may be distinct from the generators appearing at the same positions of the corresponding reduced expression. If such a β exists, then an sts-braid admissible triple is *present* in α, and if the corresponding generators are all adjacent in α (i.e., α satisfies the conditions of β as described above), the braid admissible triple is *active* in α. That is, if a braid admissible triple is present in α, it is active if the triple corresponds to a braid shadow in $\mathcal{S}(\alpha)$. To our knowledge, the concept of a braid admissible triple is new.

Example 4.3. Consider the following reduced expressions and their st-subexpressions.

(a) For $\alpha = 32313$ in type D_4, the 13-subexpression of α is $\alpha(1,3) = 3313$, and there exists an active 313-braid admissible triple in α denoted by $(2,3,4)_{313}$. Note that the corresponding generators appear over the interval $[\![3,5]\!]$ in α.

(b) For $\alpha = 1231$ in type A_3, the 12-subexpression of α is $\alpha(1,2) = 121$, and the 121-braid admissible triple $(1,2,3)_{121}$ is present in α. This triple is not active in α since the corresponding generators appear in nonadjacent positions 1, 3, and 4 in α. In $\beta = 1213$, however, which is obtained from α via a single commutation move, the triple is active.

(c) For $\alpha = 2132$ in type A_3, the 12-subexpression of α is $\alpha(1,2) = 212$, but there is no 212-braid admissible triple present in α since there is no reduced expression in $[\alpha]_c$ for which the corresponding generators are all adjacent.

Since the st-subexpression is invariant over commutation classes (Proposition 4.1), we immediately have that if $(i, i+1, i+2)_{sts}$ is present in α, then $(i, i+1, i+2)_{sts}$ is present in β for all $\beta \in [\alpha]_c$.

To clarify how we will perform braid moves on a reduced expression α within the context of braid admissible triples, we will use $sts \xmapsto{(i,i+1,i+2)} tst$ to denote a braid move on α supported by the braid shadow corresponding to the active braid admissible triple $(i, i+1, i+2)_{sts}$. We also observe that every braid admissible triple for a reduced expression has a corresponding *twin*: if $(i, i+1, i+2)_{sts}$ is active in α, and if β is the reduced expression obtained by performing the corresponding braid move $sts \xmapsto{(i,i+1,i+2)} tst$, then the twin of $(i, i+1, i+2)_{sts}$ is $(i, i+1, i+2)_{tst}$, an active braid admissible triple in β. On occasion, we may seek to consider both twins at the same time. Let

$$(i, i+1, i+2)_{sts}^{tst} := \{(i, i+1, i+2)_{sts}, (i, i+1, i+2)_{tst}\}$$

be called a *twin pair of braid admissible triples.*

Example 4.4. Consider the reduced expression $\alpha = 32313$ from Example 4.3 for a group element w in the Coxeter system of type D_4. Here, the braid admissible triple $(2,3,4)_{313}$ is active in α. Performing the corresponding braid move $313 \xmapsto{(2,3,4)} 131$ on α obtains $\beta = 32131$, for which the twin $(2,3,4)_{131}$ is an active braid admissible triple.

We can label edges in $\mathcal{G}(w)$ that represent braid moves based on the twin pair of braid admissible triples that relate their incident vertices. In light of this and Proposition 4.2, for a group element w, define two commutation classes $C_1, C_2 \in \text{Comm}(w)$ to be *adjacent* if there exist $\beta_1 \in C_1$ and $\beta_2 \in C_2$ such that $\beta_1 \sim_b \beta_2$. More precisely, for simply-laced Coxeter systems, we say that C_1 and C_2 are *adjacent via* $(i, i+1, i+2)^{tst}_{sts}$ if there exist $\beta_1 \in C_1$ and $\beta_2 \in C_2$ such that $\beta_1 \sim_b \beta_2$, where the twins in $(i, i+1, i+2)^{tst}_{sts}$ are the active braid admissible triples in β_1 and β_2 that determine their relation. We will show later in this chapter that the braid edges connecting a fixed pair of adjacent commutation classes all have the same label. To illustrate this point, consider the following example.

Example 4.5. Consider the Matsumoto graph depicted in Figure 4.1 for a group element w with reduced expression $\alpha = 3132343$ in type D_4. Each edge representing a braid move between two reduced expressions (colored in blue, cyan, teal, or violet) corresponds to a unique twin pair of braid admissible triples. For example, consider the braid move between $\alpha_1 = 1321\overline{343}$ and $\alpha_2 = 1321\overline{434}$ supported by generators 3 and 4. In $\alpha_1(3,4) = 3343$ and $\alpha_2(3,4) = 3434$, we observe a reversal in orientation of the generators in positions 2, 3, and 4. So, the twin pair $(2,3,4)^{434}_{343}$ precisely describes the braid move that occurs between α_1 and α_2. In fact, this twin pair describes all braid moves that relate a pair of representatives from the adjacent commutation classes $[\alpha_1]_c$ and $[\alpha_2]_c$.

Viewing braid moves through the lens of braid admissible triples allows us to apply known results about braid classes to our understanding of commutation classes. In the remainder of this chapter, we seek to show that if two representatives from adjacent commutation classes are adjacent via a certain twin pair braid admissible triple, then *any* two braid-equivalent representatives from these classes are related by the same twin pair. As a specific case, which is shown in the proof of the next proposition, distinct braid edges in a Matsumoto graph connecting a pair of commutation classes correspond to the "same" braid move, up to commutations.

Consider the setup depicted in Figure 4.2. Here, the adjacent commutation classes C_1 and C_2 have respective braid-related representatives β_1 and β_2, where $\beta_1 \sim_b \beta_2$. The edge connecting β_1 and β_2 is identified by a corresponding twin pair of braid admissible triples $(i, i+1, i+2)^{tst}_{sts}$. Suppose $(i, i+1, i+2)_{sts}$ is the braid admissible triple that is active in β_1. The forward implication of Proposition 4.6 shows that if $(i, i+1, i+2)_{sts}$ is active in some commutation-equivalent $\alpha_1 \in C_1$, then performing the corresponding braid move results in a reduced expression in C_2, as anticipated. The reverse implication states that if an edge representing a braid move existed between $\alpha_1 \in C_1$ and $\alpha_2 \in C_2$, then it is labeled by $(i, i+1, i+2)^{tst}_{sts}$, just like the edge between β_1 and β_2. It follows that performing any

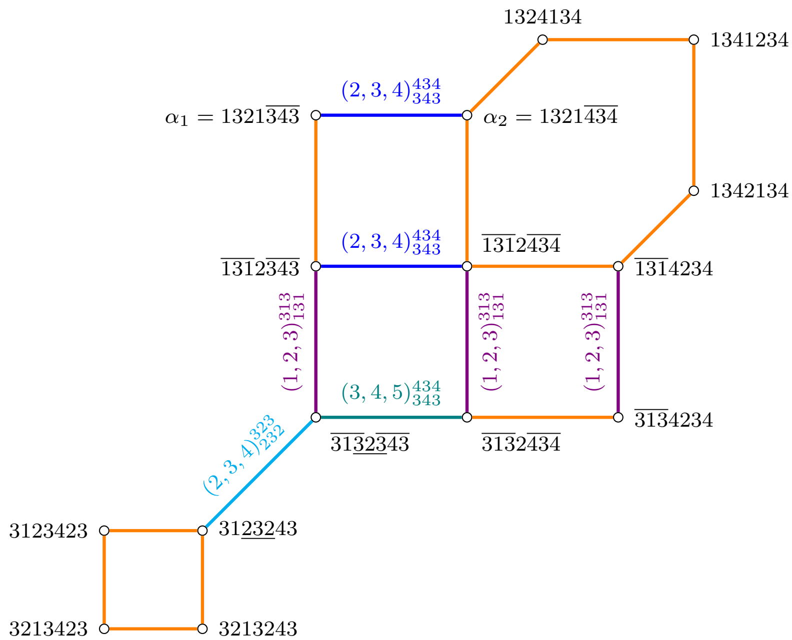


Figure 4.1: The Matsumoto graph in type D_4 from Example 4.5, where the edges are labeled and colored by the corresponding twin pair of braid admissible triples.

"other" braid move on α_1 results in a reduced expression belonging to some commutation class other than C_1 and C_2.

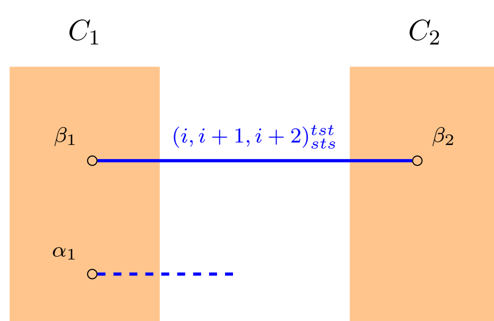


Figure 4.2: An abstraction of two commutation classes connected by a braid edge.

Proposition 4.6. Suppose (W, S) is simply laced and let $w \in W$. For adjacent commutation classes $C_1, C_2 \in \text{Comm}(w)$, assume $\beta_1 \in C_1$ and $\beta_2 \in C_2$ are braid equivalent by applying the braid move $sts \xmapsto{(i,i+1,i+2)} tst$ to β_1. Let $\alpha_1 \in C_1$ and $\alpha_1 \sim_b \alpha_2$. Then α_1 and α_2 are related by applying the braid move $sts \xmapsto{(i,i+1,i+2)} tst$ to α_1 if and only if $\alpha_2 \in C_2$.

Proof. Let α_2 be the reduced expression obtained by performing the braid move $sts \xmapsto{(i,i+1,i+2)} tst$ on $\alpha_1 \in C_1$. We will show that α_2 exists in the same commutation class as $\beta_2 \in C_2$. Consider a sequence of commutation moves that obtains β_1 from α_1. By assumption, the braid admissible triple $(i, i+1, i+2)_{sts}$ is active in β_1, so this sequence cannot end by placing a generator $u \notin \{s,t\}$ somewhere between the positions corresponding to $(i, i+1, i+2)_{sts}$ in β_1, i.e., sts remains a factor of β_1. After applying the braid move $sts \xmapsto{(i,i+1,i+2)} tst$ from β_1 to β_2, performing the "same" sequence of commutation moves in reverse obtains α_2 from β_2, so $\alpha_2 \in C_2$.

Now, if we let $\alpha_2 \in C_2$, then $\beta_1(s,t) = \alpha_1(s,t)$ and $\beta_2(s,t) = \alpha_2(s,t)$ by Proposition 4.1. Furthermore, since β_1 and β_2 are braid equivalent via $sts \xmapsto{(i,i+1,i+2)} tst$, we know that $\beta_1(s,t)$ and $\beta_2(s,t)$—and so $\alpha_1(s,t)$ and $\alpha_2(s,t)$—differ only at positions $i, i+1$, and $i+2$. Hence,

$$\begin{aligned} \alpha_1(s,t) = \beta_1(s,t) = \mu \cdot \underline{sts} \cdot \nu \\ \mu \cdot \underline{tst} \cdot \nu = \beta_2(s,t) = \alpha_2(s,t), \end{aligned}$$

where μ and ν are respectively the left and right-hand factors of the st-subexpression that remain unchanged. Since $\alpha_1 \sim_b \alpha_2$ and $\alpha_2(s,t)$ differs from $\alpha_1(s,t)$ precisely as shown above, α_1 and α_2 must be braid equivalent by applying the braid move $sts \xmapsto{(i,i+1,i+2)} tst$ to α_1. Otherwise, if α_1 and α_2 were braid equivalent by any other braid move, then $\alpha_2(s,t) \neq \beta_2(s,t)$, and we have a contradiction. □

Note that while all braid edges connecting C_1 and C_2 are labeled by some $(i, i+1, i+2)_{sts}^{tst}$, each label is not necessarily unique across pairs of commutation classes. For example, as shown with the violet edges in Figure 4.1, we see that the commutation classes $[1314234]_c$ and $[3134234]_c$ are adjacent via two braid edges labeled by $(1,2,3)_{343}^{434}$, which is the same label for the braid edge between the distinct pair of commutation classes $[3132343]_c$ and $[1312343]_c$.

When examining the braid edges connecting a pair of commutation classes, specifically in a Matsumoto graph from a simply-laced Coxeter system, we observe the presence of two kinds of isometric cycles that span adjacent commutation classes. In particular, these isometric cycles appear as one of the subgraphs depicted in Figure 4.3. Note that the vertex labels exist "up to symmetry."

Since Matsumoto graphs are bipartite, the length of any isometric cycle must be even. We conjecture that in Matsumoto graphs for simply-laced Coxeter systems, all isometric cycles containing any combination of edges representing braid moves or commutation moves only have lengths four, six, or eight, as the next example demonstrates. Our claim is not true for more general Coxeter systems, however: in type B_4, the Matsumoto graph for a group element w with reduced expression $\alpha = 3412123$ contains a length-ten isometric cycle.

Example 4.7. Consider the Matsumoto graph depicted in Figure 4.4 for a group element w with reduced expression $\alpha = 212325$ in type A_4. Edges representing the braid moves

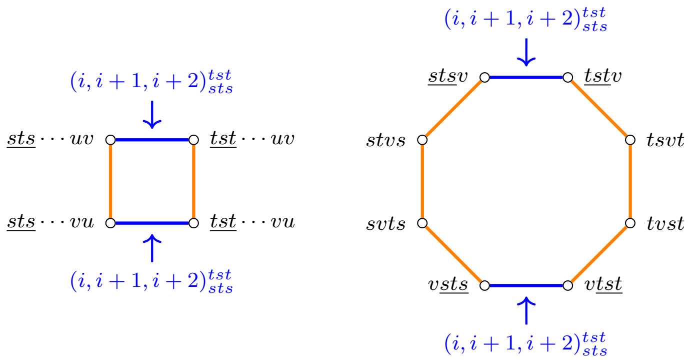


Figure 4.3: The two possible isometric cycles in a Matsumoto graph for a simply-laced Coxeter system involving both braid and commutation moves, where two adjacent commutation classes are connected by edges labeled by the same twin pair.

correspond to one of two twin pairs of braid admissible triples: $(1,2,3)^{212}_{121}$ or $(2,3,4)^{323}_{232}$. These edges are colored blue or violet, respectively. Note that all isometric cycles have length four, six, or eight, and opposite braid edges in an isometric cycle have the same twin pair label.

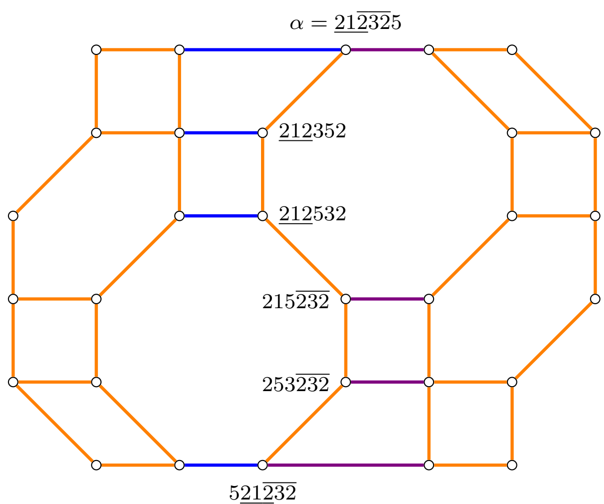


Figure 4.4: The Matsumoto graph in type A_5 from Example 4.7, where the edges representing braid moves are colored blue or violet according to their corresponding twin pair of braid admissible triples.

The following proposition has the same underlying assumptions as Proposition 4.6, but we will now only suppose that the distinct reduced expressions α_1 and α_2 belong to the same braid class, rather than differ by only a single braid move. Our approach for the proof involves tracking a "marked" generator that will appear in the center of a braid shadow for one or more equivalent reduced expressions. To illustrate how this mark is tracked through braid or commutation moves, consider the Matsumoto graph for a group element in type A_4 with reduced expression $\alpha_1 = 1\boxed{2}14$ (depicted in Figure 4.5), where the box around the generator 2 serves as our mark. By applying the braid move $121 \xmapsto{(1,2,3)} 212$, we will transfer the mark to 1, as in $\alpha_2 = 2\boxed{1}24$. In this manner, when a mark appears in the center of a braid shadow and we perform the corresponding braid move, we will transfer the mark to the generator in the center of the corresponding braid shadow of the resulting reduced expression. Since commutations only change the relative order of generators, we will maintain the mark on the corresponding generator over commutation moves. For example, as shown in Figure 4.5, we can obtain $\alpha_3 = 42\boxed{1}2$ from α_2 via a sequence of commutation moves, where the mark on 1 is maintained throughout the sequence.

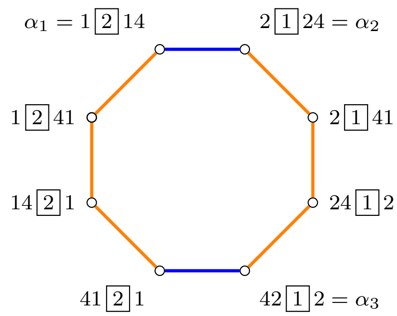


Figure 4.5: The Matsumoto graph for a group element in type A_4 with a marked generator.

Note that we include the type Λ hypothesis in the next result due to our proof's reliance on Proposition 3.7. We conjecture that this condition is not required (see Examples 5.2 and 5.3).

Proposition 4.8. Suppose (W, S) is of type Λ and let $w \in W$. For adjacent commutation classes $C_1, C_2 \in \text{Comm}(w)$, assume $\beta_1 \in C_1$ and $\beta_2 \in C_2$ are braid equivalent by applying the braid move $sts \xmapsto{(i,i+1,i+2)} tst$ to β_1. If $\alpha_1 \in C_1$, $\alpha_2 \in C_2$, and $\alpha_1 \approx_b \alpha_2$, then $\alpha_1 \sim_b \alpha_2$, whereby α_2 is obtained by applying the braid move $sts \xmapsto{(i,i+1,i+2)} tst$ to α_1.

Proof. Consider a geodesic in $\mathcal{B}(\alpha_1)$ from α_1 to α_2, which consists only of edges that represent braid moves. Let $\gamma \in [\alpha_1]_b$ be the first reduced expression (distinct from α_1) along this geodesic, and let $aba \xmapsto{(j,j+1,j+2)} bab$ be the braid move we apply on α_1 to obtain γ. We proceed with a case analysis based on the location in α_1 of the generators corresponding to $(i, i+1, i+2)_{sts}$ (which is present in α) and $(j, j+1, j+2)_{aba}$ (which is active in α).

Case 1. Suppose the positions in α corresponding to $(i, i+1, i+2)_{sts}$ are distinct from the positions corresponding to $(j, j+1, j+2)_{aba}$. Mark the generator in the center of the braid shadow corresponding to the active aba-braid admissible triple $(j, j+1, j+2)_{aba}$, as follows (where the instance of the subexpression sts could be anywhere).

$$\alpha_1 = \cdots \underline{a\boxed{b}a} \cdots s \cdots t \cdots s \cdots$$

Now, let's track the mark as we apply the sequence of commutation and braid moves from α_1 to α_2 via the braid move connecting β_1 and β_2. As we apply commutations from α_1 to β_1, the mark is maintained on b. It is still maintained after applying the braid move $sts \xmapsto{(i,i+1,i+2)} tst$ on β_1. And finally, the mark is again maintained as we apply commutations from β_1 to α_1. Thus, the mark on b that we identified in α_1 is preserved in α_2, as well. More generally, the mark on b is preserved for all $\alpha \in C_1 \cup C_2$.

Similarly, track the mark as we apply the sequence of braid moves from α_1 to α_2 along the geodesic in $\mathcal{B}(\alpha_1)$ between them. After applying the braid move $aba \xmapsto{(j,j+1,j+2)} bab$ on α_1, we transfer the mark from b, the center of a braid shadow in $\mathcal{S}(\alpha_1)$, to a, the "new" center of the braid shadow in $\mathcal{S}(\gamma)$, as shown below:

$$\alpha_1 = \cdots \underline{a\boxed{b}a} \cdots \quad \xmapsto{(j,j+1,j+2)_{aba}} \quad \gamma = \cdots \underline{b\boxed{a}b} \cdots .$$

By Proposition 3.7, the mark on a in γ is maintained as we follow the geodesic to α_2. But instead of b, the marked generator in α_2 is a, a contradiction since $a \neq b$.

Case 2. Suppose there is one position in α_1 corresponding to an end of $(i, i+1, i+2)_{sts}$ in common with the positions in α_1 corresponding to an end of $(j, j+1, j+2)_{aba}$. Here, it must be the case that $a = s$. Without loss of generality, suppose the common position comes from $i+2$ and j. Note that we can substitute the subexpression aba with sbs in the appropriate positions in α_1. Mark the central generator corresponding to the active sbs-braid admissible triple $(j, j+1, j+2)_{sbs}$ in α_1, as shown below:

$$\alpha_1 = \cdots \underline{s\boxed{b}s} \cdots \quad \xmapsto{(j,j+1,j+2)_{sbs}} \quad \gamma = \cdots \underline{b\boxed{s}b} \cdots .$$

As in Case 1, the mark is maintained on b across $C_1 \cup C_2$, even after applying the braid move $sts \xmapsto{(i,i+1,i+2)} tst$ from β_1 to β_2. But after applying the braid move $sbs \xmapsto{(j,j+1,j+2)} bsb$ from α_1 to γ, we transfer the mark on b to s, as shown above. By Proposition 3.7, the mark on s in γ is maintained as we follow the geodesic to α_2. But instead of b, the marked generator in α_2 is s, a contradiction since $s \neq b$.

Case 3. Suppose the one position in α_1 corresponding to the middle of $(i, i+1, i+2)_{sts}$ is in common with the position in α_1 corresponding to the middle of $(j, j+1, j+2)_{aba}$. Here, it must be the case that $b = t$. Mark the central generator corresponding to the active ata-braid admissible triple $(j, j+1, j+2)_{ata}$, as shown below:

$$\alpha_1 = \cdots s \cdots \underline{a\boxed{t}a} \cdots s \cdots \quad \xmapsto{(j,j+1,j+2)_{ata}} \quad \gamma = \cdots s \cdots \underline{t\boxed{a}t} \cdots s \cdots .$$

As in Case 1, the mark is maintained on t across $C_1 \cup C_2$, even after applying the braid move $sts \xmapsto{(i,i+1,i+2)} tst$ from β_1 to β_2. But after applying the braid move $ata \xmapsto{(j,j+1,j+2)} tat$ from α_1 to γ, we transfer the mark on t to a, as shown above. By Proposition 3.7, the mark on a in γ is maintained as we follow the geodesic in $\mathcal{B}(\alpha_1$ to α_2. But instead of a, the marked generator in α_2 is t, a contradiction since $t \neq a$.

Case 4. Lastly, suppose there are two positions in α_1 corresponding to $(i, i+1, i+2)_{sts}$ in common with the positions in α_1 corresponding to $(j, j+1, j+2)_{aba}$, so that $a = t$ and $b = s$. Without loss of generality, suppose $(j, j+1, j+2)_{tst} = (i+1, i+2, i+3)_{tst}$. Since $(i, i+1, i+2)_{sts}$ is active in β_1 and $(i+1, i+2, i+3)_{tst}$ is present in β_1, we must have

$$\beta_1 = \cdots sts \cdots t \cdots .$$

However, since $(i+1, i+2, i+3)_{sts}$ is a braid admissible triple, the generators between the second and third appearance of t in β_1 must commute with t. So there exists $\gamma = \cdots stst \cdots \in C_1$, which is not reduced, a contradiction.

Therefore, with all cases exhausted, the braid move from α_1 to γ_1 must be $sts \xmapsto{(i,i+1,i+2)} tst$, and the result follows by Proposition 4.6. □

In summary, we have added st-subexpressions, sts-braid admissible triples, and twin pairs of braid admissible triples as tools for understanding local relationships between commutation classes as they appear in Matsumoto graphs. Using these notions, the above results show that the labels on braid edges between adjacent commutation classes are consistent in Coxeter systems of type Λ. We conjecture that the definitions and results in this chapter generalize to arbitrary Coxeter systems.

Chapter 5

On the relationship between quotient commutation graphs and braid graphs

Following [10], we define the *quotient commutation graph* $\overline{\mathcal{G}(w)}$ to be the graph with vertex set equal to the collection of commutation classes for $w \in W$, and two commutation classes $[\alpha]_c$ and $[\beta]_c$ are connected by an edge if there exists $\alpha_0 \in [\alpha]_c$ and $\beta_0 \in [\beta]_c$ such that $\alpha_0 \sim_b \beta_0$. That is, $\overline{\mathcal{G}(w)}$ is obtained by "contracting" the edges in $\mathcal{G}(w)$ that correspond to commutation moves. The resulting graph has edges that represent braid moves between a pair of representatives in the adjacent commutation classes.

Quotient commutation graphs have been studied by several authors, such as [13], though many open problems about them still remain. For example, excluding a few trivial cases, we do not know how many vertices the quotient commutation graph has. As discussed in [6], the problem of enumerating the commutation classes for the so-called longest element in type A_n has been posed in a variety of contexts [15, 16, 21].

Recall that for a pair of adjacent commutation classes, all the edges joining representatives share the same label, namely the corresponding twin pair of braid admissible triples that describe the relation between a pair of representatives. As a consequence of this, we can label edges connecting vertices in the quotient commutation graph with these same labels.

A *transversal* (also called a *cross section*) of a collection of sets $\mathcal{A} = \{A_i\}_{i \in I}$ is a set T that contains exactly one element from each set in $\mathcal{A}$, i.e., $|T \cap A_i| = 1$ for all i. We say that $w \in W$ has the *transversal property* with $\alpha \in \mathcal{R}(w)$ provided that $[\alpha]_b$ is a transversal for the collection of commutation classes of w. By Proposition 4.2, this implies $|[\alpha]_b \cap [\beta]_c| = 1$ for all $\beta \in \mathcal{R}(w)$, and since the collection of commutation classes is a partition of $\mathcal{R}(w)$, there exists a bijection between $\mathrm{Comm}(w)$ and $[\alpha]_b$. Note that in some contexts, a transversal is defined by the condition $T \cap A_i \neq \emptyset$ for all i, though for our purposes, either approach is sufficient and yields the same outcome.

Often, the quotient commutation graph of a Coxeter group element is isomorphic to the braid graph of one of its reduced expressions. In Coxeter systems of type Λ, this happens exactly when w has the transversal property (see Theorem 5.8).

Example 5.1. Consider the Matsumoto graph depicted in Figure 5.1 for a group element w in type D_4 with reduced expression $\alpha = 3132343$. Observe that the braid class $[\alpha]_b$ is a transversal (highlighted in blue) for the five commutation classes of w, so w has the transversal property with α. Moreover, we see that $\mathcal{B}(\alpha) \cong \overline{\mathcal{G}(w)}$.

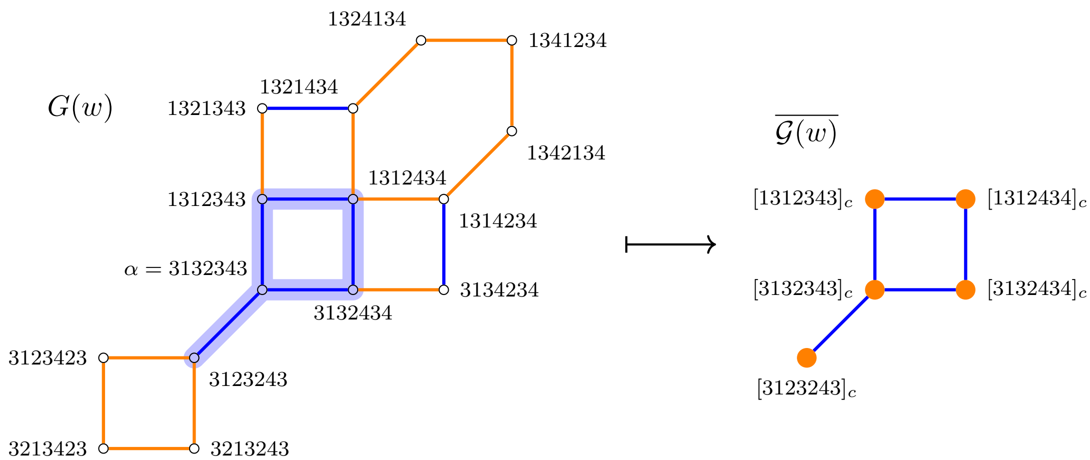


Figure 5.1: The Matsumoto graph (left) in type D_4 from Example 5.1 and its quotient commutation graph (right) obtained by contracting the orange edges.

We conjecture that a group element with the transversal property has this isomorphism for more general Coxeter systems. The next example exhibits this for a group element in the Coxeter system of type B_4, which is triangle-free but not simply laced.

Example 5.2. Consider the Matsumoto graph depicted in Figure 5.2 for a group element w in type B_4 with reduced expression $\alpha = 1212324$. Observe that the braid class $[\alpha]_b$ is a transversal (highlighted in blue) for the three commutation classes of w, so w has the transversal property with α. Moreover, we see that $\mathcal{B}(\alpha) \cong \overline{\mathcal{G}(w)}$.

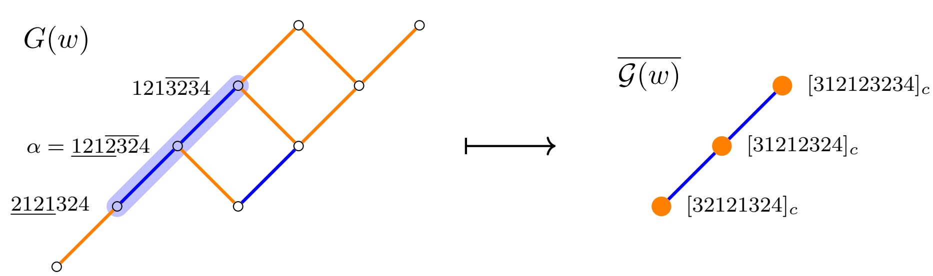


Figure 5.2: The Matsumoto graph (left) in type B_4 from Example 5.2 and its quotient commutation graph (right) obtained by contracting the orange edges.

Likewise, we believe the isomorphism holds for Coxeter systems that are simply laced, but not triangle-free, as the next example demonstrates.

Example 5.3. Let (W, S) be the simply-laced Coxeter system with Coxeter graph Γ_1 shown in Figure 5.3. Consider the Matsumoto graph depicted in Figure 5.4 for a group element $w \in W$ with reduced expression $\alpha = 12131214$. Observe that the braid class $[\alpha]_b$ is a transversal (highlighted in blue) for the six commutation classes of w, so w has the transversal property with α. Moreover, we see that $\mathcal{B}(\alpha) \cong \overline{\mathcal{G}(w)}$.

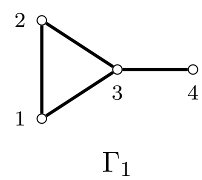


Figure 5.3: The Coxeter graph for Example 5.3.

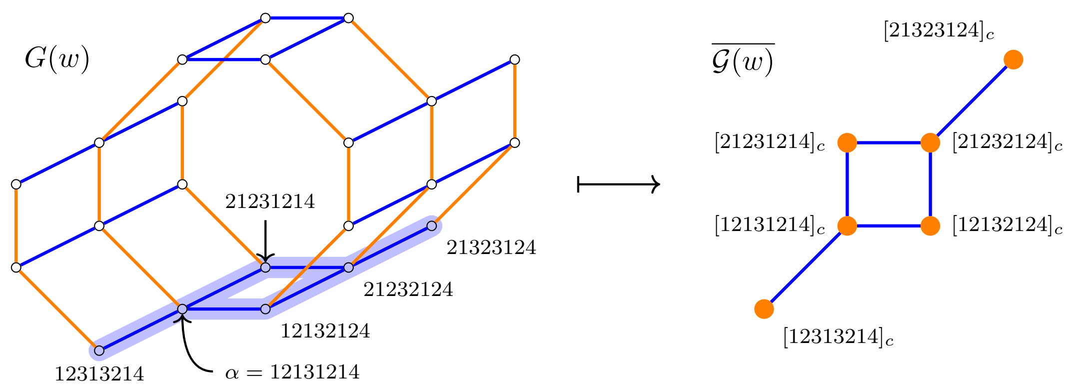


Figure 5.4: The Matsumoto graph (left) in type Γ_1 from Example 5.3 and its quotient commutation graph (right) obtained by contracting the orange edges.

The following example illustrates that the quotient commutation graph of a group element is not always isomorphic to the braid graph of any of its reduced expressions. However, the braid graph of any reduced expression is always realized in the corresponding quotient commutation graph. In this sense, even if a group element w does not have the transversal property, we may describe every braid class of a reduced expression for w as a "local" transversal of the commutation classes with which it has a nonempty intersection. This isomorphism can be expressed in terms of induced subgraphs, as we will soon discuss.

Example 5.4. Consider the Matsumoto graph depicted in Figure 5.5 for a group element w in type A_3 with reduced expression $\alpha = 121321$. Observe that no braid class is a transversal for the four commutation classes of w. Furthermore, by contracting the orange edges, we obtain a cycle with eight vertices, which is not isomorphic to any braid graph of a reduced expression for the group element. However, the braid graph of α appears as a subgraph in the quotient commutation graph, where each reduced expression represented by a vertex in $\mathcal{B}(\alpha)$ corresponds to its commutation class.

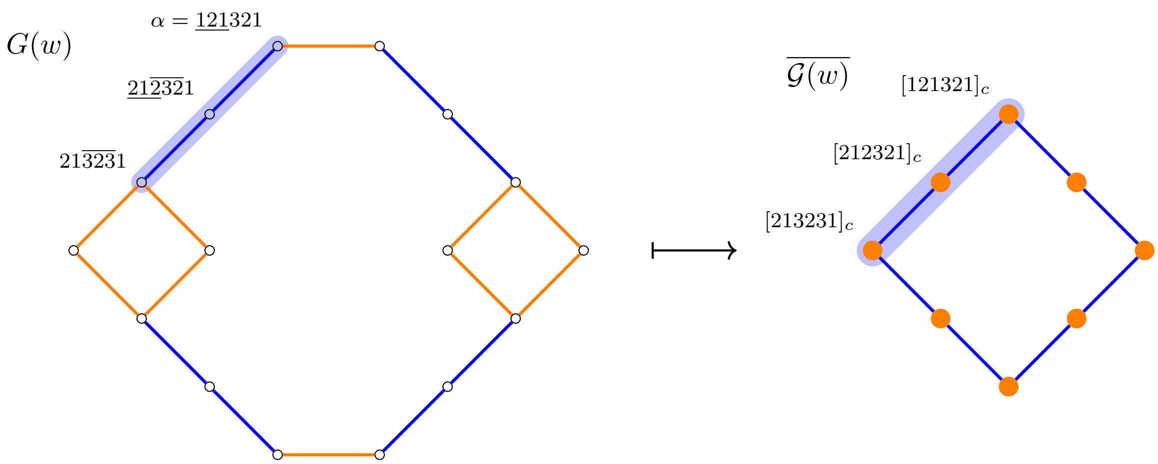


Figure 5.5: The Matsumoto graph (left) in type A_3 from Example 5.4 and its quotient commutation graph (right) obtained by contracting the orange edges.

Let $U \subseteq V(\mathcal{G}(w))$ for some group element w and consider the induced subgraph $\mathcal{G}(w)[U]$. We define $\overline{\mathcal{G}(w)[U]}$ to be the graph with vertex set $\{[v]_c \cap U \mid v \in U\}$, where two vertices are connected by an edge if there exists $\alpha_0 \in [\alpha]_c \cap U$ and $\beta_0 \in [\beta]_c \cap U$ such that $\alpha_0 \sim_b \beta_0$. Loosely speaking, one obtains $\overline{\mathcal{G}(w)[U]}$ by "contracting" the edges in $\mathcal{G}(w)[U]$ that correspond to commutation moves.

For a reduced expression $\alpha \in \mathcal{R}(w)$, define the *district* of α as $D(\alpha) := \bigcup_{\beta \in [\alpha]_b} [\beta]_c$. In other words, $D(\alpha)$ is the collection of vertices that are commutation equivalent to any $\beta \in [\alpha]_b$. Let $\overline{D(\alpha)} := \{[\beta]_c \mid \beta \in D(\alpha)\} \subseteq \mathrm{Comm}(w)$, and note that by an application of definitions and Proposition 4.2, $\overline{D(\alpha)} = \{[\beta]_c \mid \beta \in [\alpha]_b\}$.

We provide the following example to illustrate how districts relate to Matsumoto graphs. Despite the Coxeter system being of type B_4, which is not type Λ, the essential idea we seek to convey is that if $\alpha \in \mathcal{R}(w)$ for a group element w in any Coxeter system, then $[\alpha]_b$ serves as a "local transversal" for commutation classes that intersect $D(\alpha)$.

Example 5.5. Consider the Matsumoto graph depicted in Figure 5.6 for a group element w in type B_4 with reduced expression $\alpha = 31212324$. Observe that no braid class is a transversal for the four commutation classes of w. However, the union of the three commutation classes $[32121324]_c$, $[31212324]_c$, and $[31213234]_c$ is precisely $D(\alpha)$. Furthermore, if we view just the induced subgraph $G(w)[D(\alpha)]$ (highlighted in gray), we observe $[\alpha]_b$ is a transversal for the three commutation classes that are subsets of $D(\alpha)$. Then by contracting the orange edges in $\mathcal{G}(w)[D(\alpha)]$, we see that $\mathcal{B}(\alpha) \cong \overline{\mathcal{G}(w)}\left[\overline{D(\alpha)}\right] = \overline{G(w)\,[D(\alpha)]}$ (highlighted in blue in the quotient commutation graph on the right.

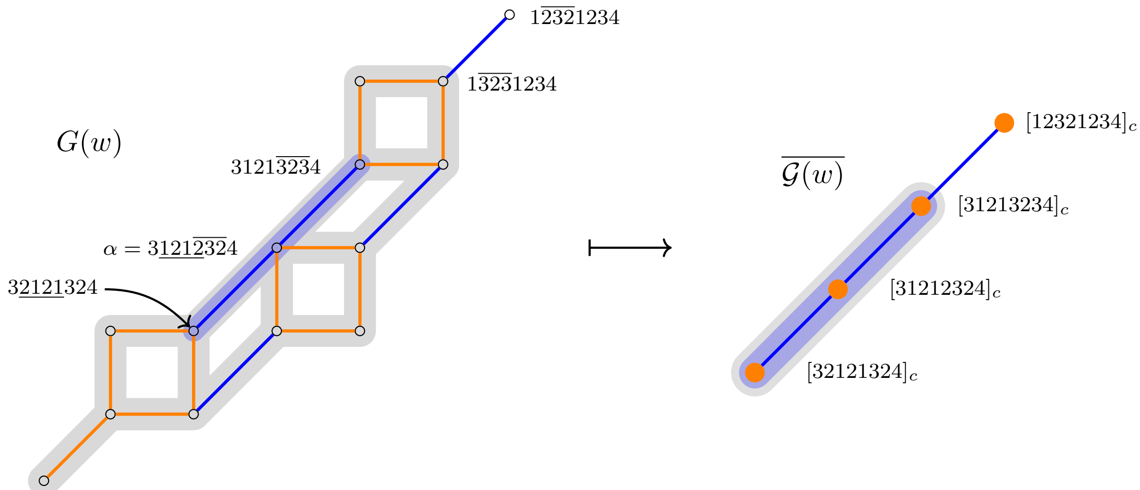


Figure 5.6: The Matsumoto graph (left) in type B_4 from Example 5.5 and its quotient commutation graph (right) obtained by contracting the orange edges.

The following theorem states that for any Coxeter system of type Λ, if we quotient the induced subgraph $\mathcal{G}(w)[D(\alpha)]$ by edges corresponding to commutation moves, the resulting graph is isomorphic to $\mathcal{B}(\alpha)$. Additionally, the subgraph of $\overline{\mathcal{G}(w)}$ induced by $\overline{D(\alpha)}$ is the same as what we obtain by quotienting the induced subgraph $\mathcal{G}(w)[D(\alpha)]$. That is, "quotient-then-induce" obtains the same as "induce-then-quotient." Recall that Proposition 4.8 requires the assumption that the Coxeter system is of type Λ. Since the proof of our theorem invokes this proposition, we include the type Λ assumption among our hypotheses.

Theorem 5.6. Suppose (W, S) is of type Λ and $w \in W$. For any $\alpha \in \mathcal{R}(w)$,

$$\mathcal{B}(\alpha) \cong \overline{\mathcal{G}(w)\,[D(\alpha)]} = \overline{\mathcal{G}(w)}\left[\overline{D(\alpha)}\right].$$

Proof. Let $\alpha \in \mathcal{R}(w)$. First, note that $|V(\mathcal{B}(\alpha))| = |\overline{\mathcal{G}(w)[D(\alpha)]}|$ since the number of elements in $[\alpha]_b$ is the same as the number of commutation classes whose representatives are elements in $[\alpha]_b$. We now seek to show that if an edge exists between two reduced expressions in $\mathcal{B}(\alpha)$, then there exists a corresponding edge in $\overline{\mathcal{G}(w)[D(\alpha)]}$, and vice versa.

Assume α_1 and α_2 are adjacent in $\mathcal{B}(\alpha)$. By the definition of quotient commutation graph, $[\alpha_1]_c$ and $[\alpha_2]_c$ are adjacent in $\overline{\mathcal{G}(w)[D(\alpha)]}$. Conversely, suppose $[\alpha_1]_c$ and $[\alpha_2]_c$ are adjacent in $\overline{\mathcal{G}(w)[D(\alpha)]}$ for some $\alpha_1, \alpha_2 \in [\alpha]_b$. Then there exist $\beta_1 \in [\alpha_1]_c$ and $\beta_2 \in [\alpha_2]_c$ such that β_1 and β_2 are related by a single braid move. By Proposition 4.8, every geodesic in $\mathcal{B}(\alpha)$ consisting of braid moves between α_1 and α_2 has length one, so α_1 and α_2 are adjacent in $\mathcal{B}(\alpha)$.

To show equality between $\overline{\mathcal{G}(w)\,[D(\alpha)]}$ and $\overline{\mathcal{G}(w)}\left[\overline{D(\alpha)}\right]$, first consider their vertex sets. Vertices in $\overline{\mathcal{G}(w)\,[D(\alpha)]}$ are the commutation classes of elements in $D(\alpha) \subseteq \mathcal{R}(w)$, which are represented by elements in $[\alpha]_b$. That is, the vertex set of $\overline{\mathcal{G}(w)\,[D(\alpha)]}$ is the collection of commutation classes $\overline{D(\alpha)}$, which by definition is exactly the vertex set of $\overline{\mathcal{G}(w)}\left[\overline{D(\alpha)}\right]$. One can easily see that the corresponding edge sets are also equal by applying the definition of quotient commutation graph. $\square$

Based on experimental data collected with Python code throughout the development of this thesis, we conjecture that the above theorem holds for arbitrary Coxeter systems. But to show this, one would need to generalize beyond simply-laced Coxeter systems and find a proof approach that does not require Proposition 3.7.

The next result follows from Theorem 5.6, together with Proposition 2.12.

Corollary 5.7. If (W, S) is of type Λ, $w \in W$, and $\alpha \in \mathcal{R}(w)$, then $\overline{\mathcal{G}(w)[D(\alpha)]}$ is a median graph and a partial cube with isometric dimension $\operatorname{rank}(\alpha)$.

The following result describes a special case of our previous theorem, providing a strong equivalence for quotient commutation graphs when the ambient group element has the transversal property. We consider this to be the main result of this thesis.

Theorem 5.8. Suppose (W, S) is of type Λ and $w \in W$. Then w has the transversal property with α if and only if $\mathcal{B}(\alpha) \cong \overline{\mathcal{G}(w)}$.

Proof. The forward implication is immediate by Theorem 5.6. For the reverse implication, we have that $[\alpha]_b$ is a transversal for Comm(w) by applying the definition of quotient commutation graph: since $\overline{\mathcal{G}(w)} \cong \mathcal{B}(\alpha)$, every vertex in $\mathcal{B}(\alpha)$ is a unique representative of the elements in $\operatorname{Comm}(w)$. So, w has the transversal property with α. $\square$

We immediately have the following corollary.

Corollary 5.9. If (W, S) is of type Λ and $w \in W$ has the transversal property with $\alpha \in \mathcal{R}(w)$, then $\overline{\mathcal{G}(w)}$ is a median graph and a partial cube with isometric dimension $\operatorname{rank}(\alpha)$.

Note that the converse of the previous result is not true. For instance, the quotient commutation graph from Example 5.5 is a median graph (and hence a partial cube) despite the corresponding group element not having the transversal property.

A method for computing the median of three braid-equivalent reduced expressions is detailed in [2]. So, if w has the transversal property with α, then we may compute the median of three commutation classes in $\overline{\mathcal{G}(w)}$ by computing the median of their corresponding representatives in $[\alpha]_b$.

The union of all elements in $\operatorname{Comm}(w)$, which are the vertices in $\overline{\mathcal{G}(w)}$, is $\mathcal{R}(w)$. So, if w has the transversal property with α, the above corollary implies that any reduced expression in $\mathcal{R}(w)$ is obtainable from some element of the transversal $[\alpha]_b$ only by commutations. That is, if one seeks a reduced expression in $\mathcal{R}(w)$ that is equivalent to α, one may perform *all* required braid moves on α followed by *all* required commutation moves to obtain it. We conjecture this idea could be helpful for finding $\mathcal{R}(w)$ computationally: with a transversal $[\alpha]_b$

in hand, since commutation moves seem less memory-intensive than braid moves, computing the remaining elements of $\mathcal{R}(w)$ could follow quickly by considering the commutation classes of each element in $[\alpha]_b$.

We know $\overline{\mathcal{G}(w)}\left[\overline{D(\alpha)}\right] \cong \mathcal{B}(\alpha)$ by Theorem 5.6. Recall from Chapter 1 that we conjectured $\mathcal{B}(\alpha)$ is an isometric subgraph of $\mathcal{G}(w)$. Although induced subgraphs are generally not isometric subgraphs, we conjecture the following.

Conjecture 5.10. If (W, S) is any Coxeter system and $w \in W$, then $\overline{\mathcal{G}(w)}\left[\overline{D(\alpha)}\right]$ is an isometric subgraph of $\overline{\mathcal{G}(w)}$ for any $\alpha \in \mathcal{R}(w)$.

Recall Example 5.1, which discussed a group element w with the transversal property in type D_4. The braid graph of the reduced expression is a transversal for $\mathrm{Comm}(w)$ and is isomorphic to the quotient commutation graph of the corresponding group element w. Below, we offer additional examples of group elements in simply-laced Coxeter systems that have the transversal property and hence satisfy the conditions of Theorem 5.6.

Example 5.11. Consider the Matsumoto graph depicted in Figure 5.7 for a group element w in type A_4 with reduced expression $\alpha = 1213243$. Here, $[1213243]_b$ is a transversal for the four commutation classes of w (highlighted in cyan), and so $\mathcal{B}(\alpha) \cong \overline{\mathcal{G}(w)}$. Two other braid classes containing more than one element exist, however: $[2312343]_b$ and $[1213423]_b$. Their braid graphs are isometrically embedded in $\overline{\mathcal{G}(w)}$, as highlighted in magenta and yellow, respectively.

Example 5.12. Consider the Matsumoto graph depicted in Figure 5.8 for a group element w in type A_5 with reduced expression $\alpha = 12135$. Here, there exist three transversals for the two commutation classes of the corresponding group element: $[12135]_b$, $[12153]_b$, and $[51213]_b$. At least one is sufficient to show that $\mathcal{B}(\alpha) \cong \overline{\mathcal{G}(w)}$, which means a group element does not need to have the transversal property with respect to a unique braid class for the isomorphism to hold.

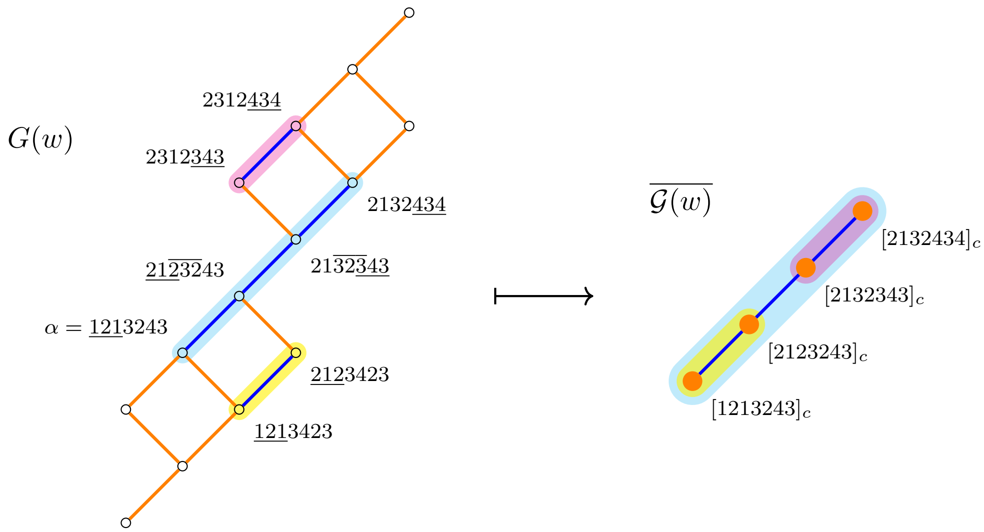


Figure 5.7: The Matsumoto graph (left) in type A_4 from Example 5.11 with three distinct braid classes highlighted in blue, magenta, and yellow, and its quotient commutation graph (right) obtained by contracting the orange edges.

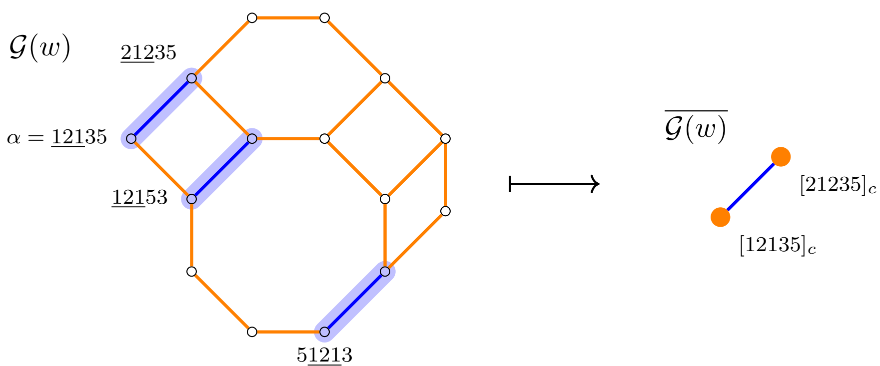


Figure 5.8: The Matsumoto graph (left) in type A_5 from Example 5.12 with the three transversal braid classes highlighted in blue, and its quotient commutation graph (right) obtained by contracting the orange edges.

Chapter 6

Classes of elements with the transversal property

We now proceed to discuss several classes of group elements that exhibit the transversal property and highlight their quotient commutation graphs. We begin with some definitions.

A group element $w \in W$ is *fully commutative* if it has only one commutation class. According to Stembridge [20], this happens exactly when no reduced expression for w has the opportunity to apply a braid move.

Example 6.1. Consider the following examples of fully commutative group elements.

(a) In type A_4, the group element w with reduced expression $\alpha = 1324$ is fully commutative since it has only one commutation class: $[1324]_c = \{1324, 3124, 1342, 3142, 3412\}$. Note that there is never an opportunity to apply a braid move among these reduced expressions.

(b) In type D_4, the group element w with reduced expression $\alpha = 31234$ is fully commutative since it has only one commutation class: $[31234]_c = \{31234, 32134\}$. Observe that even though $\alpha(1,3) = 313$, there is no 313-braid admissible triple for α. Indeed, there is no opportunity to apply a braid move in any reduced expression for w.

If w is fully commutative, both the braid graph of any of its reduced expressions and the quotient commutation graph of w are a single vertex. Any choice of $\alpha \in \mathcal{R}(w)$ provides a transversal $[\alpha]_b$ for the sole commutation class of w, and we thus have the following result.

Proposition 6.2. If (W, S) is a Coxeter system and $w \in W$ is a fully commutative group element, then w has the transversal property and $\mathcal{B}(\alpha) \cong \overline{\mathcal{G}(w)}$ is a single vertex for all $\alpha \in \mathcal{R}(w)$.

As a special case of fully commutative elements, we define a group element w to be *trivially commutative* if every reduced expression for w consists of pairwise commuting generators.

Example 6.3. In type A_5, the group element w with reduced expression $\alpha = 135$ is trivially commutative since it has only one commutation class, $[135]_c = \{135, 315, 351, 531, 513, 153\}$, and all generators appearing in α pairwise commute.

Observe that the structure of the Matsumoto graph for a trivially commutative element depends only on the length of the element, as the following example illustrates.

Example 6.4. Figure 6.1 shows the Matsumoto graphs of trivially commutative elements whose reduced expressions have length $n \in \{1, 2, 3, 4\}$. The quotient commutation graph for any of these group elements is just a single vertex, which is clearly isomorphic to the braid graph of any of their reduced expressions.

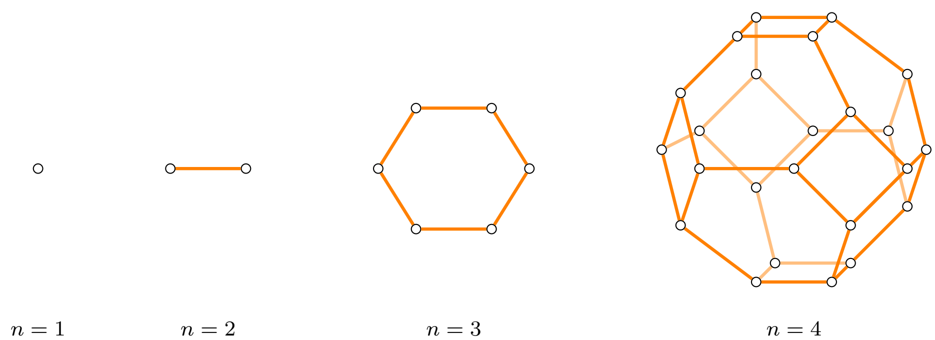


Figure 6.1: The Matsumoto graphs of trivially commutative group elements of length n.

There is a natural connection between the set of reduced expressions for a trivially commutative element and the symmetric group. A *permutohedron* of order n is the Cayley graph of the symmetric group S_n with respect to the generating set of adjacent transpositions $(i, i + 1)$. Its vertices are the permutations of $\{1, 2, \ldots, n\}$, and two permutations are connected by an edge when they differ by left-multiplication of a single adjacent transposition. Indeed, each of the Matsumoto graphs from the previous example is isomorphic to a permutohedron of order n, as we show with the following proposition.

Proposition 6.5. If (W, S) is a Coxeter system and $w \in W$ is a trivially commutative element of length n, then $\mathcal{G}(w)$ is isomorphic to a permutohedron of order n.

Proof. Fix a reduced expression $\alpha \in \mathcal{R}(w)$. Since w is trivially commutative, any other reduced expression $\beta \in \mathcal{R}(w)$ is obtainable from α by a sequence of commutation moves, i.e., by a sequence of moves that swap adjacent commuting generators. Note that in a reduced expression for a trivially commutative element, all pairs of adjacent generators commute.

Label the positions of the generators in α by the indices $1, 2, \ldots, n$, and assign the identity permutation $12 \cdots n$ to α. Any commutation move swaps two adjacent generators and hence corresponds to applying a transposition to the indices. In this way, any reduced expression

$\beta \in \mathcal{R}(w)$ is in correspondence with a permutation in S_n that records how its generators have been reordered relative to α.

This defines a map from the vertices of $\mathcal{G}(w)$ to the vertices of a permutohedron of order n. By construction, edges in $\mathcal{G}(w)$ representing commutative moves correspond exactly to edges in the permutohedron representing a transposition of indices. Thus, the map is an isomorphism. $\square$

Recall Proposition 2.12, which states that braid graphs of reduced expressions in Coxeter systems of type Λ are partial cubes, i.e., such braid graphs can be isometrically embedded into a hypercube with a sufficiently large dimension. In this vein, we conjecture that the Matsumoto graphs of any fully commutative element can be isometrically embedded into a permutohedron of sufficiently large order.

In contrast to Stembridge's characterization of fully commutative elements, we call a group element $w \in W$ *anti-commutative* when no reduced expression for w has the opportunity to apply a commutation move. We believe this happens exactly when w has only one braid class, which may serve as an equivalent definition that better aligns with Stembridge's definition for fully commutative elements. If this is indeed true for all anti-commutative elements, we suggest this class of elements instead be called *fully braided*.

Example 6.6. Consider the Matsumoto graph depicted in Figure 6.2 for a group element w in type $\tilde{A}_2$ with the reduced expression $\alpha = 12131$, which is anti-commutative. Here, $[12131]_b$ is a transversal for the three singleton commutation classes of the corresponding group element, and so $\mathcal{B}(\alpha) \cong \overline{\mathcal{G}(w)}$.

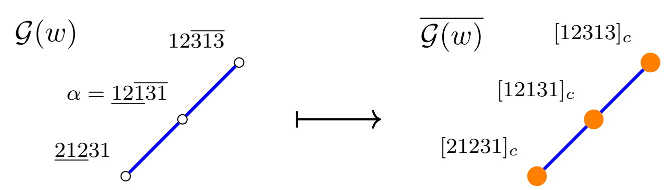


Figure 6.2: The Matsumoto graph (left) in type $\tilde{A}_2$ from Example 6.6 and its quotient commutation graph (right) obtained by contracting the orange edges.

Since no commutation moves are ever available in an anti-commutative element, each commutation class has size one and uniquely intersects a reduced expression in the sole braid class. In other words, the unique braid class is a transversal for the singleton commutation classes of w. We thus have the following result.

Proposition 6.7. If (W, S) is a Coxeter system and $w \in W$ is an anti-commutative group element, then w has the transversal property with any $\alpha \in \mathcal{R}(w)$ and $\mathcal{G}(w) = \mathcal{B}(\alpha) \cong \overline{\mathcal{G}(w)}$.

Finally, we consider freely braided elements, which were introduced and studied within the context of simply-laced Coxeter systems in [10, 11], then later generalized to arbitrary

Coxeter systems in [6]. We omit the original definition for freely braided because it is given in terms of root sequences, which is a concept beyond the scope of this thesis. However, inspired by Proposition 3.1.3 in [11], we offer the following definition for Property F, whose name was chosen because “F” is the first letter in “freely braided,” and we believe there is a deep connection between their definitions.

For a simply-laced Coxeter system (W, S), we say that $w \in W$ has *Property F* if there exists a reduced expression $\alpha \in \mathcal{R}(w)$ such that:

(a) $\mathcal{S}(\alpha) = \mathcal{S}([\alpha]_b)$,

(b) $I \cap J = \emptyset$ for all distinct $I, J \in \mathcal{S}([\alpha]_b)$, and

(c) $|\mathrm{Comm}(w)| = 2^{\mathrm{rank}(\alpha)}$.

We say that α is *central* if it satisfies the above conditions. Note that if a group element w has Property F and α is central, then β is central for all $\beta \in [\alpha]_b$.

Example 6.8. Consider the following examples.

(a) In type A_4, the group element w with reduced expression $\alpha = 31214$ has two commutation classes: $[31214]_c$ and $[32124]_c$. Hence, $|\mathrm{Comm}(w)| = 2$. Observe that $\mathcal{S}(\alpha) = \{[\![2, 4]\!]\} = \mathcal{S}([\alpha]_b)$, which implies that $\mathrm{rank}(\alpha) = 1$. Hence $2^{\mathrm{rank}(\alpha)} = 1 = \mathrm{Comm}(w)$, so w has Property F and α is a central reduced expression for w.

(b) In type A_4, the group element w with reduced expression $\alpha = 121343$ has four commutation classes: $[121343]_c, [212343]_c, [121434]_c$, and $[212434]_C$. Since $\mathcal{S}(\alpha) = \{[\![1, 3]\!], [\![4, 6]\!]\} = \mathcal{S}([\alpha]_b)$, no two braid shadows in $\mathcal{S}([\alpha]_b)$ overlap. So, $2^{\mathrm{rank}(\alpha)} = 4$, w has Property F, and α is a central reduced expression for w. The Matsumoto graph and quotient commutation graph for w are depicted in Figure 6.3.

(c) In type D_5, the group element w with reduced expression $\alpha = 313454323$ has nine commutation classes: $[313454323]_c$, $[131454323]_c$, $[313545323]_c$, $[313454232]_c$, $[131545323]_c$, $[131454232]_c$, $[313545232]_c$, $[131545232]_c$, and $[315343523]_c$. Hence $|\mathrm{Comm}(w)| = 9$. Although $\mathcal{S}(\alpha) = \mathcal{S}([\alpha]_b)$ and no two braid shadows in $\mathcal{S}([\alpha_b])$ overlap, $2^{\mathrm{rank}(\alpha)} = 8 \neq |\mathrm{Comm}(w)|$. So, w does not have Property F.

The first condition of Property F guarantees that all braid shadows for $[\alpha]_b$ are present in α, while the second condition indicates that all these braid shadows are pairwise disjoint. The third condition appears somewhat artificial, but our motivation for its inclusion is to ensure that α has the maximum possible number of braid shadows. We do not know for certain that the condition guarantees this property, but it is inspired by the following proposition.

Proposition 6.9. If (W, S) is Coxeter system of type Λ and $w \in W$ has the transversal property with $\alpha \in \mathcal{R}(w)$, then $|\mathrm{Comm}(w)| \leq 2^{\mathrm{rank}(\alpha)}$.

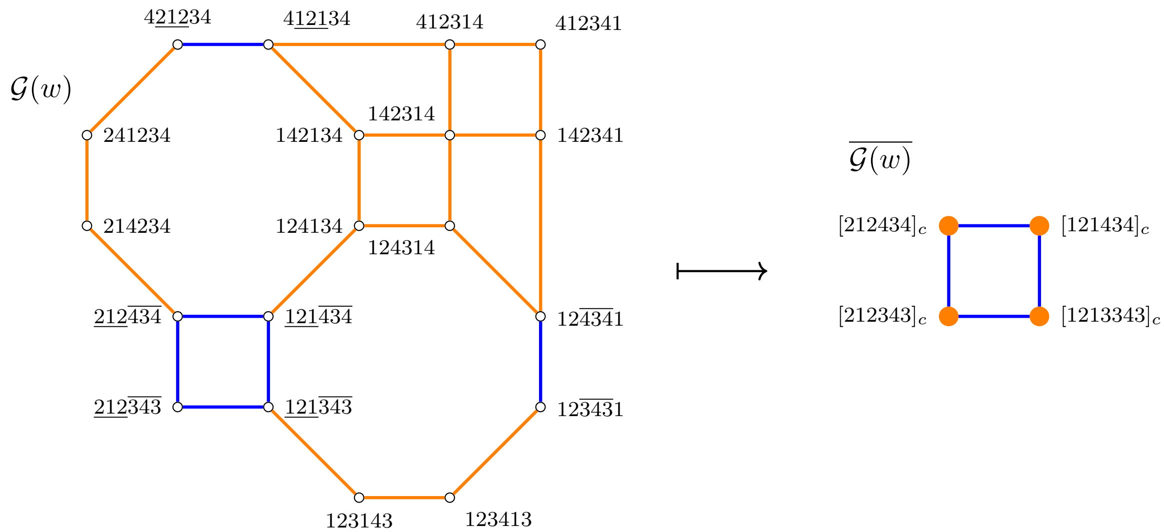


Figure 6.3: The Matsumoto graph (left) in type A_4 from Example 6.8 and its quotient commutation graph (right) obtained by contracting the orange edges.

Proof. Since w has the transversal property with α, there is a bijection between $[\alpha]_b$ and $\mathrm{Comm}(w)$. Together with Proposition 2.13, we have $|\mathrm{Comm}(w)| = |[\alpha]_b| \leq 2^{\mathrm{rank}(\alpha)}$. ☐

Indeed, this conjecture holds for every example in this thesis, as well as all the ones we produced using Python. Moreover, this is consistent with one of the main results in [10], which states that $|\mathrm{Comm}(w)| \leq 2^{N(w)}$, where $N(w)$ is the number of so-called "contractible inversion triples" of w. In [11], the same authors prove that for all simply-laced Coxeter systems, if w is freely braided, then $|\mathrm{Comm}(w)| = 2^{N(w)}$. While $\mathrm{rank}(\alpha)$ is a statistic for a reduced expression that varies across $\mathcal{R}(w)$ and $N(w)$ is a statistic for w, we conjecture that these two concepts are inherently related. Moreover, we conjecture the following.

Conjecture 6.10. Let (W, S) be a simply-laced Coxeter system. A group element $w \in W$ has Property F if and only if it is freely braided.

Rather than needing to determine the number of commutation classes for a group element with Property F, we suspect a local combinatorial condition on reduced expressions exists to determine whether they are central. The condition is not apparent, however, and one may simply not exist.

If α is central, however, then all braid shadows in $\mathcal{S}([\alpha]_b)$ are represented and pairwise disjoint in α. Assuming it is a reduced expression for a nonidentity element, this implies that α is a product of length-3 and length-1 links. By Proposition 2.10, we immediately have the following.

Proposition 6.11. If (W, S) is a simply-laced Coxeter system and $w \in W$ has Property F with central reduced expression α, then $\mathcal{B}(\alpha) \cong Q_{\mathrm{rank}(\alpha)}$ and $|[\alpha]_b| = 2^{\mathrm{rank}(\alpha)}$.

The next proposition states that group elements with Property F also have the transversal property. This result is necessary for our approach to show $\mathcal{B}(\alpha) \cong \overline{\mathcal{G}(w)}$ for w with Property F and α a central reduced expression of w (see Proposition 6.13).

Proposition 6.12. If (W, S) is a simply-laced Coxeter system, $w \in W$ has Property F, and $\alpha \in \mathcal{R}(w)$ is central, then $[\alpha]_b$ is a transversal of $\text{Comm}(w)$.

Proof. Consider the district $D(\alpha) = \bigcup_{\beta \in [\alpha]_b} [\beta]_c$ for the central reduced expression α. By Proposition 4.2, $[\alpha]_b$ is a transversal of all commutation classes $[\beta]_c$ appearing in $D(\alpha)$, and by Proposition 6.11, the number of such commutation classes is $|[\alpha]_b| = 2^{\text{rank}(\alpha)}$. Since α is central, $|[\alpha]_b| = 2^{\text{rank}(\alpha)} = |\text{Comm}(w)|$, so $[\alpha]_b$ must be a transversal of all the commutation classes in $\text{Comm}(w)$. □

We thus have the following proposition. In contrast to the hypotheses of propositions so far in this chapter, we assume a Coxeter system of type Λ. This is due to our reliance on Theorem 5.8, whose proof requires the condition. However, we conjecture that the following result holds for more general Coxeter systems.

Proposition 6.13. If (W, S) is a Coxeter system of type Λ and $w \in W$ has Property F, then w has the transversal property with any central reduced expression $\alpha \in \mathcal{R}(w)$ and $\mathcal{B}(\alpha) \cong \overline{\mathcal{G}(w)}$.

Proof. The result follows by the definition of central, Proposition 6.12, and Corollary 5.8. □

If Conjecture 6.10 is true, then Proposition 6.13 implies that the quotient commutation graph of every freely braided element is a hypercube, which is consistent with the formula provided in [11] for the number of commutation classes of a freely braided element.

Bibliography

[1] F. Awik, J. Breland, Q. Cadman, and D.C. Ernst. Braid graphs in simply-laced triangle-free Coxeter systems are partial cubes. *European Journal of Combinatorics*, 118, 2024.

[2] J. Barnes, J.V. Breland, D.C. Ernst, and R. Perry. Braid graphs in simply-laced triangle-free Coxeter systems are median. `arXiv:2408.16839`, 2024.

[3] R. Bedard. On Commutation Classes of Reduced Words in Weyl Groups. *European Journal of Combinatorics*, 20, 1999.

[4] N. Bergeron, C. Ceballos, and J.P. Labbé. Fan realizations of subword complexes and multi-associahedra via Gale duality. *Discrete & Computational Geometry*, 54(1):195–231, 2015.

[5] B. Brešar, S. Klavžar, and R. Škrekovski. On cube-free median graphs. *Discrete Mathematics*, 307(3):345–351, 2007.

[6] H. Denoncourt, D.C. Ernst, and D. Story. On the number of commutation classes of the longest element in the symmetric group. *Open Problems in Mathematics*, 4, 2016.

[7] S. Elnitsky. Rhombic Tilings of Polygons and Classes of Reduced Words in Coxeter Groups. *The Journal of Combinatorial Theory, Series A*, 77(2), 1997.

[8] S. Fishel, E. Milićević, R. Patrias, and B.E. Tenner. Enumerations relating braid and commutation classes. *European Journal of Combinatorics*, 74, 2018.

[9] M. Geck and G. Pfeiffer. *Characters of finite Coxeter groups and Iwahori–Hecke algebras.* 2000.

[10] R.M. Green and J. Losonczy. Freely braided elements in Coxeter groups, I. *Annals of Combinatorics*, 6(3):337–348, 2002.

[11] R.M. Green and J. Losonczy. Freely braided elements in Coxeter groups, II. *Advances in Applied Mathematics*, 33(1):26–39, 2004.

[12] D. Grinberg and A. Postnikov. Proof of a conjecture of Bergeron, Ceballos and Labbé. *New York Journal of Mathematics*, 23:1581–1610, 2017.

[13] G. Gutierres, R. Mamede, and J.L. Santos. Commutation classes of the reduced words for the longest element of $\mathfrak{S}_n$. *The Electronic Journal of Combinatorics*, 27, 2020.

[14] J.E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, Cambridge, 1990.

[15] M.M. Kapranov and V.A. Voevodsky. Combinatorial-geometric aspects of polycategory theory: pasting schemes and higher Bruhat orders (list of results). *Cahiers de topologie et géométrie différentielle catégoriques*, 32(1), 1991.

[16] D. Knuth. *Axioms and Hulls*. Springer-Verlag, Berlin, 1992.

[17] D. Meng. Reduced decompositions and commutation classes. `arXiv:1009.0886`, 2010.

[18] S. Ovchinnikov. Partial cubes: Structures, characterizations, and constructions. *Discrete Mathematics*, 308(23):5597–5621, 2008.

[19] S. Ovchinnikov. *Graphs and Cubes*. Springer Science and Media, 2011.

[20] J.R. Stembridge. On the fully commutative elements of Coxeter groups. *J. Algebraic Combin.*, 5:353–385, 1996.

[21] B.E. Tenner. Reduced decompositions and permutation patterns. *Journal of Algebraic Combinatorics*, 24, 2006.

[22] D.M. Zollinger. *Equivalence classes of reduced words*. MS Thesis, University of Minnesota, 1994.