STRUCTURAL PROPERTIES OF BRAID GRAPHS IN SIMPLY-LACED TRIANGLE-FREE COXETER SYSTEMS

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ABSTRACT

STRUCTURAL PROPERTIES OF BRAID GRAPHS IN SIMPLY-LACED TRIANGLE-FREE COXETER SYSTEMS

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Any two reduced expressions for the same Coxeter group element are related by a sequence of commutation and braid moves. We say that two reduced expressions are braid equivalent if they are related via a sequence of braid moves, and the corresponding equivalence classes are called braid classes. Each braid class can be encoded in terms of a braid graph in a natural way. In a recent paper, Awik et al. proved that every reduced expression in a simply-laced Coxeter group has a unique factorization as a product of so-called links, which in turn induces a decomposition of the braid graph into a box product of the braid graphs for each link factor. Moreover, the authors proved that when the Coxeter system is triangle free (i.e., the corresponding Coxeter graph has no three-cycles), the braid graph for a reduced expression is a partial cube (i.e., isometric to a subgraph of a hypercube). In this thesis, we study the structural properties of braid classes in simply-laced triangle-free Coxeter systems. In particular, we provide precise information about the local structure of reduced expressions in the braid class for a link and produce an alternate proof of the fact that every braid graph in simply-laced triangle-free Coxeter systems is a partial cube. Moreover, we outline the obstructions to proving the conjectures that every braid graph in a simply-laced triangle-free Coxeter system is median and corresponds to the Hasse diagram for a distributive lattice.

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Chapter 1

Graphs and partially ordered sets

This chapter introduces the necessary terminology and results regarding graphs and partially ordered sets. In this chapter, we will assume all graphs are connected and simple.

If G is a graph, let $V(G)$ denote the vertex set of G and let $E(G)$ denote the edge set of G. If $S \subseteq V(G)$ is any subset of vertices of G, then we define the *induced subgraph* $\langle S \rangle$ to be the graph whose vertex set is S and whose edge set consists of all of the edges in $E(G)$ that have endpoints in S.

Example 1.1. Consider the graphs depicted in Figures 1.1(a) and 1.1(b). The subgraph highlighted in Figure 1.1(a) is the induced subgraph generated by $S = \{a, b, c, d, e\}$. However, the subgraph highlighted in Figure 1.1(b) is not an induced subgraph since the edge joining a and f is not in the subgraph.

(a) (b)

Figure 1.1: An example and non-example of an induced subgraph.

An *embedding* of a graph G into a graph H is an injection $f : V(G) \to V(H)$ with the property that if u and v are adjacent vertices in G, then $f(u)$ and $f(v)$ are adjacent in H. That is, an embedding is an injective graph homomorphism. If in addition, $f(u)$ and $f(v)$ adjacent in H implies u and v adjacent in G, then we say that f is an *induced embedding*.

If f is an induced embedding, then G is isomorphic to the subgraph of H induced by the image of f.

Example 1.2. Figure 1.2(a) provides an example of an induced embedding while the embedding shown in Figure 1.2(b) is not an induced embedding due to the fact that $\{g(a), g(d)\}$ is an edge in the graph H_2 but $\{a, d\}$ is not an edge in the graph G_2.

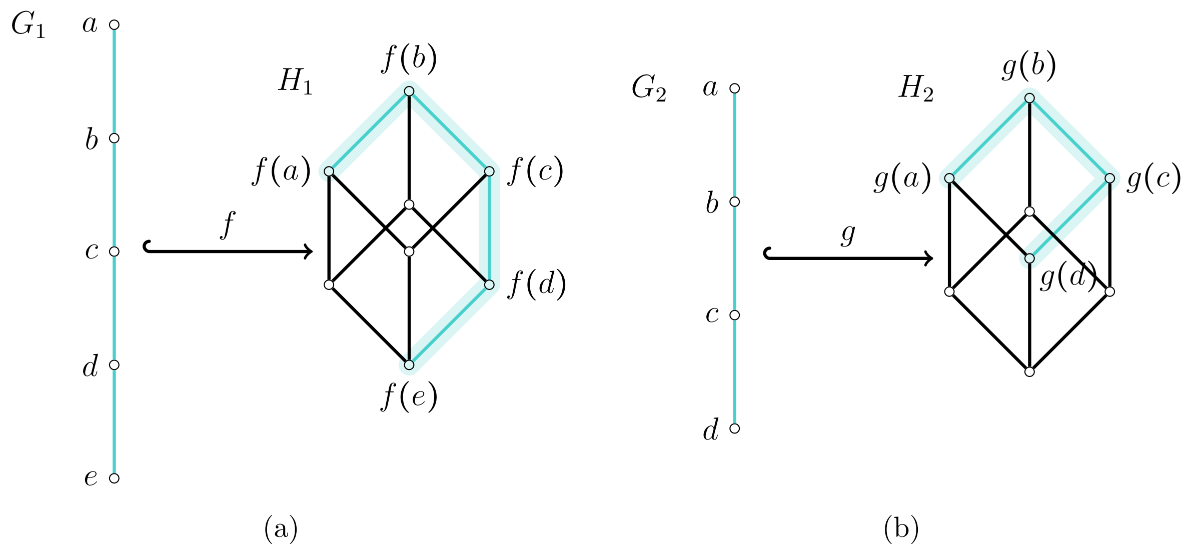


Figure 1.2: An induced embedding and an embedding that is not induced.

We can view any connected graph G as a metric space by taking the standard geodesic metric. That is, the distance between $u, v \in V(G)$ is defined via

$$d_G(u, v) \coloneqq \text{length of any minimal path between } u \text{ and } v.$$

Given the metric above, we define the *diameter* of G to be

$$\operatorname{diam}(G) \coloneqq \max\{d_G(u, v) \mid u, v \in V(G)\}.$$

That is, $\operatorname{diam}(G)$ is the longest of all shortest paths between any two vertices u and v. Two vertices u and v in $E(G)$ are said to be *diametrical* if $d(u, v) = \operatorname{diam}(G)$.

Example 1.3. Let G be the graph in Figure 1.3. We see that $\operatorname{diam}(G) = 4$. One path that yields the diameter has been highlighted in magenta. It follows that u and v are diametrical.

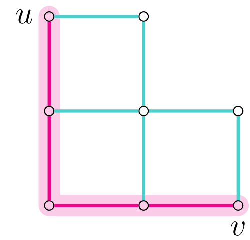


Figure 1.3: Example of a graph with diameter 4.

An *isometric embedding* of G into H is a function $f : V(G) \to V(H)$ with the property that $d_G(u,v) = d_H(f(u), f(v))$ for all $u, v \in V(G)$. Note that every isometric embedding is indeed injective. In this case, G is isometric to an induced subgraph of H. In other words, there exists a distance-preserving bijection between G and the subgraph of H induced by the image of f. Since an isometry is injective and two vertices are adjacent if and only if the distance between them is one, every isometric embedding is also an induced embedding. However, an induced embedding is not necessarily an isometric embedding.

Example 1.4. The induced embedding f seen in Figure 1.2(a) is not an isometric embedding since $d_G(a,e) = 4$ while $d_H(f(a), f(e)) = 2$.

Given two graphs G_1 and G_2, we define the *box product* (also referred to as the *Cartesian product*) of the two graphs, denoted $G_1 \square G_2$, to be the graph with vertex set $V(G_1) \times V(G_2)$ and there is an edge from (x_1, y_1) to (x_2, y_2) if and only if either

(a) $x_1 = x_2$ and there is an edge from y_1 to y_2 in G_2, or

(b) $y_1 = y_2$ and there is an edge from x_1 to x_2 in G_1.

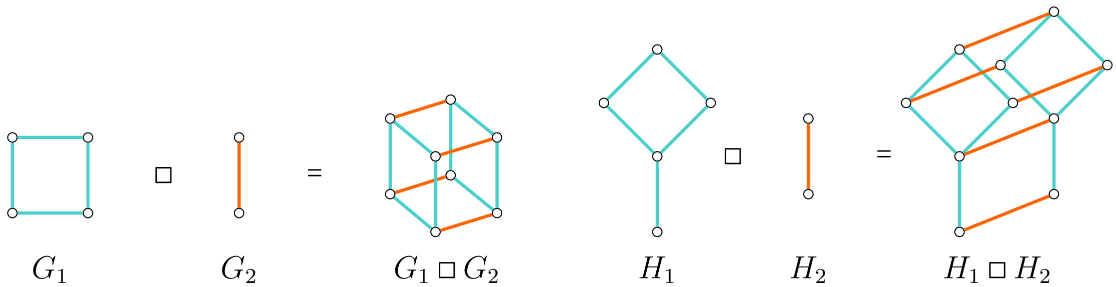
Note that the operation $\square$ is associative.

Example 1.5. Two examples of box products are depicted in Figure 1.4. We have used colors to aid the reader.

For a positive integer n we will denote the set of binary strings of length n by $\{0,1\}^n$. That is,

$$\{0,1\}^n := \{a_1 a_2 \cdots a_n \mid a_k \in \{0,1\}\}.$$

The *hypercube graph* of dimension $n \geq 0$, denoted by Q_n, is defined to be the graph whose vertices are elements of $\{0,1\}^n$ with two binary strings connected by an edge exactly when they differ by a single digit (i.e., the Hamming distance between the two vertices is equal to one). Note that Q_0 consists of a single vertex labeled by the empty string.



(a) (b)

Figure 1.4: Examples of the box product of graphs.

A *partial cube* is a graph that is isometric to a subgraph of a hypercube. The *isometric dimension* of a partial cube is the minimum dimension of a hypercube into which it may be isometrically embedded. That is, the isometric dimension of a partial cube G is the nonnegative integer

$$\dim_I(G) := \min\{m \in \mathbb{N} \cup \{0\} \mid \text{there exists an isometric embedding of } G \text{ into } Q_m\}.$$

Example 1.6. The graphs G_1 and G_2 given in Figures 1.5(a) and 1.5(b), respectively, are examples of partial cubes. In each case, we have provided possible isometric embeddings into the cube. It is easily seen that the isometric dimension is 3 for both graphs.

The following proposition is a result from [13].

Proposition 1.7. If G_1 and G_2 are partial cubes, then $G_1 \square G_2$ is a partial cube. Moreover, $\dim_I(G_1 \square G_2) = \dim_I(G_1) + \dim_I(G_2)$.

Example 1.8. Since each of the factors in both subfigures of Figure 1.4 are partial cubes, the resulting box products are partial cubes. Moreover, we see that

$$\dim_I(G_1 \square G_2) = \dim_I(G_1) + \dim_I(G_2) = 2 + 1 = 3$$

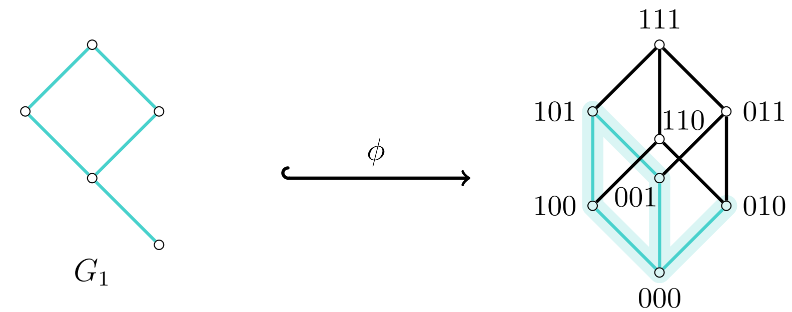
and

$$\dim_I(H_1 \square H_2) = \dim_I(H_1) + \dim_I(H_2) = 3 + 1 = 4,$$

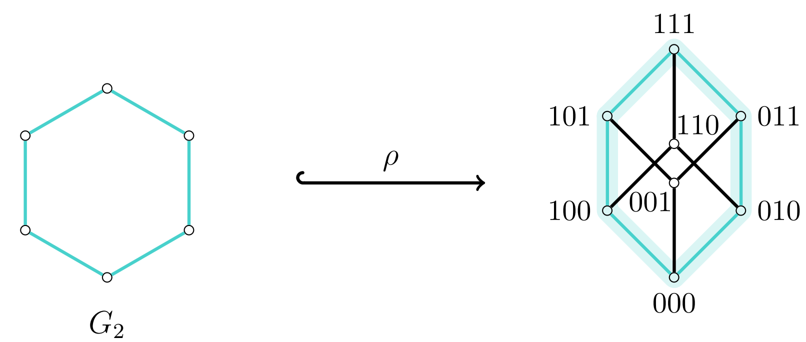
as expected.

We now mimic the development in Section 3 of [13]. Let G be a graph. For any two vertices $u, v \in V(G)$, we define W_{uv} to be the set of vertices that are closer to u than to v. That is,

$$W_{uv} := \{w \in V(G) \mid d(w,u) < d(w,v)\}.$$



(a)



(b)

Figure 1.5: Examples of partial cubes.

We refer to the set W_{uv} and the corresponding induced subgraph $\langle W_{uv} \rangle$ as a *semicube* of G. The semicubes W_{uv} and W_{vu} are called *opposite semicubes*. Notice that W_{uv} is defined even if $\{u, v\}$ is not an edge in G. The next two results appear in [13].

Proposition 1.9. A graph G is bipartite if and only if W_{uv} and W_{vu} form a partition of $V(G)$ for any edge $\{u, v\} \in E(G)$.

Proposition 1.10. If $w \in W_{uv}$ for some edge $\{u, v\} \in E(G)$, then $d(w, v) = d(w, u) + 1$. Moreover, $W_{uv} = \{w \in V(G) \mid d(w, v) = d(w, u) + 1\}$.

The previous proposition implies that all of the vertices that are in W_{uv} are exactly one step further from v than u in G.

We define the *Djoković–Winkler relation* $\boldsymbol{\theta}$ on $E(G)$ via $\{x, y\} \,\boldsymbol{\theta}\, \{u, v\}$ if and only if $\{u, v\}$ joins a vertex in W_{xy} with a vertex in W_{yx}. It is important to note that the relation $\boldsymbol{\theta}$ is always reflexive and symmetric, but not always transitive [13].

The following proposition from [13] will play an important role in a later chapter of this thesis.

Proposition 1.11. Let G be a connected graph. The following statements are equivalent:

(i) G is a partial cube.

(ii) G is bipartite and $\boldsymbol{\theta}$ is an equivalence relation.

(iii) G is bipartite and for all $\{x,y\},\{u,v\} \in E(G)$, if $\{x,y\}\,\boldsymbol{\theta}\,\{u,v\}$, then $\{W_{xy}, W_{yx}\} = \{W_{uv}, W_{vu}\}$.

(iv) G is bipartite and for any pair of adjacent vertices of G, there is a unique pair of opposite semicubes separating these two vertices.

Note that if G is a partial cube, there may be more than one pair of adjacent vertices that generate the unique semicubes mentioned in part (iv) of the previous proposition (see Example 1.12). In addition, if G is a partial cube, then by the previous result, G is bipartite and $\boldsymbol{\theta}$ is an equivalence relation on $E(G)$. In this case, if $\{u,v\} \in E(G)$, let F_{uv} denote the equivalence class of $\{u,v\}$ under $\boldsymbol{\theta}$. Then

$$F_{uv} = \{\{a,b\} \in E(G) \mid \{u,v\}\,\boldsymbol{\theta}\,\{a,b\}\} = \{\{a,b\} \in E(G) \mid a \in W_{uv}, b \in W_{vu}\},$$

where the second equality follows from Proposition 1.10. That is, F_{uv} is the set of edges joining the semicubes W_{uv} and W_{vu}.

Example 1.12. Consider the partial cube G in Figure 1.6. For the edge $\{u,v\}$, the semicubes W_{uv} and W_{vu} have been highlighted in blue while the equivalence class F_{uv} consists of the three magenta edges.

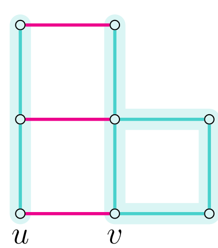


Figure 1.6: Example of semicubes and the corresponding equivalence class of edges.

The following result appears in [13].

Proposition 1.13. If G is a partial cube, then $\dim_I(G)$ is equal to the number of equivalence classes induced by the Djoković-Winkler relation $\boldsymbol{\theta}$.

Example 1.14. Given the same partial cube G from Example 1.12, it turns out that there are four equivalence classes of edges induced by the Djoković–Winkler relation $\boldsymbol{\theta}$ on G. The four equivalence classes are indicated by color in Figure 1.7. Therefore, $\dim_I(G) = 4$.

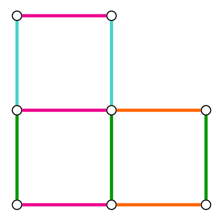


Figure 1.7: Equivalence classes of edges induced by the Djoković–Winkler relation.

Given two vertices u and v in a graph G, we define the *interval* between u and v, denoted $I(u,v)$, to be the collection of vertices on any shortest path between u and v. A connected graph is defined to be *median* if for any three vertices, u, v, and w,

$$|I(u,v) \cap I(u,w) \cap I(v,w)| = 1.$$

In other words, there is a unique vertex x that simultaneously lies on a shortest path between u and v, a shortest path between u and w, and a shortest path between v and w.

Example 1.15. Given the labeling in Figure 1.8(a), $I(u,v) \cap I(u,w) \cap I(v,w) = \{x\}$. Similar calculations for any three vertices of G_1 verify that G_1 is median. On the other hand, in Figure 1.8(b), we see that $I(u,v) \cap I(u,w) \cap I(v,w) = \varnothing$, so G_2 is not median. In both figures, the interval $I(u,v)$ has been highlighted in red, the interval $I(u,w)$ has been highlighted in blue, and the interval $I(v,w)$ has been highlighted in green.

The next proposition can be found in [14].

Proposition 1.16. If a graph G is median, then G is a partial cube.

Example 1.17. Since the graph G_2 given in Figure 1.8(b) is a partial cube (see Example 1.6) but is not median (see Example 1.15), the converse of Proposition 1.16 does not hold.

The following proposition is well known.

Proposition 1.18. If graphs G_1 and G_2 are median, then $G_1 \square G_2$ is median.

Figure 1.8: Examples of a median and non-median graph.

We now turn our attention to partially ordered sets. A *partially ordered set* (or poset for short) is a pair $(P, \leq)$ consisting of a set P together with a relation $\leq$ that is reflexive, antisymmetric, and transitive. For $x, y \in P$, we say x *covers* y, denoted $x \lessdot y$, if $x < y$ and there is no element $z \in P$ such that $x < z < y$. A Hasse diagram is a graphical representation of a poset $(P, \leq)$, where vertices are elements of P, x and y are connected by an edge if $x \lessdot y$, and there is an implied upward orientation (i.e., smaller elements are lower in the Hasse diagram). Examples of various Hasse diagrams for posets can be found in Figure 1.9. We will often abuse terminology and identify a poset with its Hasse diagram.

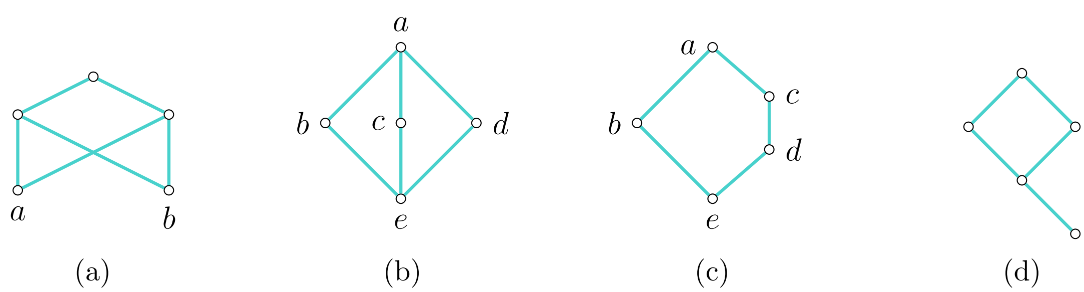


Figure 1.9: Examples of Hasse diagrams for posets.

A poset is *ranked* (also called *graded*) if there is a function $\rho : P \to \mathbb{N} \cup \{0\}$ such that $\rho(x) = 0$ if x is a minimal element, and $\rho(y) = \rho(x) + 1$ if $x \lessdot y$. The poset in Figure 1.9(c) is not a ranked poset while the other three posets in Figure 1.9 are ranked. A *lattice* is a special kind of poset where every pair of elements has a unique greatest lower bound (meet) and a unique least upper bound (join) in the poset. If P is a lattice, we denote the meet of x and y by $x \wedge y$ and the join of x and y by $x \vee y$. The posets in Figures 1.9(b), 1.9(c), and 1.9(d), are examples of lattices, while the poset in Figure 1.9(a) is not a lattice since there is not a unique least upper bound of a and b. A *distributive lattice* is a lattice in which the following identities hold for all $x, y, z \in P$:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \text{ and } x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

It turns out that it is enough to verify just one of the identities. Certainly, the poset in Figure 1.9(a) is not a distributive lattice since it is not a lattice. In Figure 1.9(b), we see that

$$b \wedge (c \vee d) = b \wedge a = b \neq e = e \vee e = (b \wedge c) \vee (b \wedge d),$$

which shows that the poset in Figure 1.9(b) is not a distributive lattice. The lattice in Figure 1.9(c) is also not distributive since

$$b \vee (c \wedge d) = b \vee d = e \neq a = a \wedge a = (b \vee c) \wedge (b \vee d).$$

However, one can show that the poset in Figure 1.9(d) is a distributive lattice.

The next proposition can be found in [15].

Proposition 1.19. A lattice is a distributive lattice if and only if it does not have either of the lattices from Figures 1.9(b) or 1.9(c) as a sublattice.

From [6] we have the following result.

Proposition 1.20. The underlying graph of the Hasse diagram for a finite distributive lattice is median.

Example 1.21. Since the poset given in Figure 1.9(d) is a distributive lattice, the underlying graph of the Hasse diagram is median according to the previous result. On the other hand, the underlying graph for the poset in Figure 1.9(b) is median, but the poset is not a distributive lattice, which shows the converse to Proposition 1.20 is false.

A similar result from [1][1] states the following.

Proposition 1.22. A graph G is the underlying graph of the Hasse diagram of a distributive lattice if and only if G is median and there exist two vertices u and v such that every vertex in G lies on a shortest path joining u and v.

Following [5], we define a *ring of sets* to be a family of sets that is closed under the operations of set union and set intersection. It is clear that every ring of sets is a ranked poset under inclusion. Every ring of sets ordered by inclusion is also a distributive lattice. The following proposition from [5], sometimes called Birkhoff's Representation Theorem or the Fundamental Theorem for Finite Distributive Lattices (although it is not usually stated in this form), states that every finite distributive lattice corresponds to a ring of sets.

[1]D: I think this should be *A Ternary Operation...*[6] by Birkhoff and Kiss.

Proposition 1.23. A finite poset is a distributive lattice if and only if it is isomorphic to a ring of sets ordered by inclusion.

Since every ring of sets is ranked, we immediately obtain the following corollary.

Corollary 1.24. Every finite distributive lattice is ranked.

Chapter 2

Coxeter systems and braid graphs

This chapter will discuss the necessary information regarding Coxeter systems and their overall structure. We will also introduce braid classes and their graphical representation into braid graphs, as well as related concepts of braid shadows and links.

A *Coxeter matrix* is an $n \times n$ symmetric matrix $M = (m_{ij})$ with entries $m_{ij} \in \{1, 2, 3, \ldots, \infty\}$ such that $m_{ii} = 1$ for all $1 \leq i \leq n$ and $m_{ij} \geq 2$ for $i \neq j$. A *Coxeter system* is a pair (W, S) consisting of a finite set $S = \{s_1, s_2, \ldots, s_n\}$ and a group W, called a *Coxeter group*, with presentation

$$W = \langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m(s_i, s_j)} = e \rangle,$$

where $m(s_i, s_j) := m_{ij}$ for some $n \times n$ Coxeter matrix $M = (m_{ij})$. For $s, t \in S$, the condition $m(s, t) = \infty$ means that there is no relation imposed between s and t. It turns out that the elements of S are distinct as group elements and $m(s, t)$ is the order of st [11]. Since elements of S have order two, the relation $(st)^{m(s,t)} = e$ can be written as

$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)}$$

with $m(s, t) \geq 2$ factors. When $m(s, t) = 2$, $st = ts$ is called a *commutation relation* and when $m(s, t) \geq 3$, the corresponding relation is called a *braid relation.* The replacement

$$\underbrace{sts\cdots}_{m(s,t)} \longmapsto \underbrace{tst\cdots}_{m(s,t)}$$

is called a *commutation move* if $m(s, t) = 2$ and a *braid move* if $m(s, t) \geq 3$.

A Coxeter system (W, S) can be encoded by a unique *Coxeter graph* Γ having vertex set S and edges $\{s, t\}$ for each $m(s, t) \geq 3$. Each edge is labeled with the corresponding $m(s, t)$. Typically, labels of 3 are omitted since they are the most common. In this case, we say that (W, S), or just W, is of type Γ, and we may denote the Coxeter group as $W(\Gamma)$ and the generating set as $S(\Gamma)$ for emphasis. We say that a Coxeter system is *simply laced* provided $m(s,t) \leq 3$ for all $s,t \in S$. In the case that the Coxeter system has no three-cycles, (W,S) is called *triangle free*. We say that a Coxeter system (W,S) (or the group W) is of type Λ if (W,S) is both simply laced and triangle free. This thesis will focus primarily on Coxeter systems of type Λ.

Example 2.1. The Coxeter graphs given in Figure 2.1 correspond to four common simply-laced Coxeter systems. The defining relations for the Coxeter systems are determined by the corresponding graphs. The Coxeter system of type A_n is given by the Coxeter graph in Figure 2.1(a). The Coxeter group $W(A_n)$ has generating set $S(A_n) = \{s_1, s_2, \ldots, s_n\}$ and has defining relations

- $s_i^2 = e$ for all i;
- $s_i s_j = s_j s_i$ when $|i-j| > 1$;
- $s_i s_j s_i = s_j s_i s_j$ when $|i-j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group S_{n+1} under the mapping that sends s_i to the adjacent transposition $(i, i+1)$.

The Coxeter system of type D_n is given by the Coxeter graph in Figure 2.1(b). The Coxeter group $W(D_n)$ has generating set $S(D_n) = \{s_1, s_2, \ldots, s_n\}$ and has defining relations

- $s_i^2 = e$ for all i;
- $s_i s_j = s_j s_i$ if $|i-j| > 1$ and $i, j \neq 1$;
- $s_i s_j = s_j s_i$ if $i = 1$ and $j \neq 3$;
- $s_1 s_3 s_1 = s_3 s_1 s_3$ and $s_i s_j s_i = s_j s_i s_j$ if $|i-j| = 1$.

The Coxeter group $W(D_n)$ is isomorphic to the index two subgroup of the group of signed permutations on n letters having an even number of sign changes.

It turns out that the Coxeter groups of types $\widetilde{A}_n$ and $\widetilde{D}_n$ are infinite. All of these Coxeter systems are of type Λ except type $\widetilde{A}_2$ due to the Coxeter graph being a three-cycle.

Given a Coxeter system (W,S), let S^* denote the free monoid on the alphabet S. An element $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is called a *word*. A *subword* of $\boldsymbol{\alpha}$ is a word of the form $s_{x_i} s_{x_{i+1}} \cdots s_{x_{j-1}} s_{x_j}$ for $1 \leq i \leq j \leq m$. The word $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is an *expression* for w if $\boldsymbol{\alpha}$ is equal to w when considered as an element of the group W. If m is minimal among all possible expressions for w, we say that $\boldsymbol{\alpha}$ is a *reduced expression* for w, and we call $\ell(w) := m$,

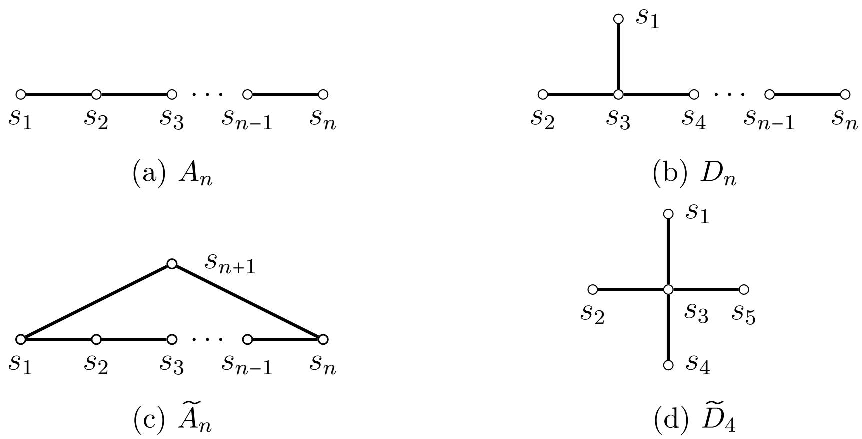


Figure 2.1: Examples of common simply-laced Coxeter graphs.

the *length* of w. Note that any subword of a reduced expression is also reduced. We will denote the set of all reduced expressions for $w \in W$ by $\mathcal{R}(w)$.

For the remainder of this thesis, if we are considering a particular labeling of a Coxeter graph, we will often replace s_i with i for brevity.

The relationship between reduced expressions for a given group element is characterized by the following theorem, called Matsumoto's Theorem [9].

Proposition 2.2 (Matsumoto's Theorem)**.** In a Coxeter system (W, S), any two reduced expressions for the same group element differ by a sequence of commutation and braid moves.

Following Matsumoto's Theorem, we can visually represent the relationships among reduced expressions for a given element in a Coxeter group. For $w \in W$, define the *Matsumoto graph* $\mathcal{G}(w)$ to be the graph having vertex set equal to $\mathcal{R}(w)$, where two reduced expressions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are connected by an edge if and only if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related via a single commutation or braid move. We will distinguish between commutation and braid moves by connecting $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with an orange edge if there is a commutation move between them and a blue edge if there is a braid move between them. Matsumoto's Theorem implies that $\mathcal{G}(w)$ is connected. Bergeron, Ceballos, and Labbé [4] proved that for finite Coxeter groups, every cycle in a Matsumoto graph has even length. Grinberg and Postnikov [10], extended this result to arbitrary Coxeter systems. The following proposition is an immediate consequence of these two facts.

Proposition 2.3. If (W, S) is a Coxeter system and $w \in W$, then $\mathcal{G}(w)$ is bipartite.

Matsumoto's Theorem allows us to define two different equivalence relations on the set of reduced expressions for a given element of a Coxeter group. Let (W, S) be a Coxeter system and let $w \in W$. For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{R}(w)$, we define $\boldsymbol{\alpha} \sim_c \boldsymbol{\beta}$ if we can obtain $\boldsymbol{\alpha}$ from $\boldsymbol{\beta}$ by applying a single commutation move of the form $st \mapsto ts$, where $m(s,t) = 2$. We define the equivalence relation $\approx_c$ by taking the reflexive and transitive closure of $\sim_c$. Each equivalence class under $\approx_c$ is called a *commutation class*, denoted $[\boldsymbol{\alpha}]_c$ for $\boldsymbol{\alpha} \in \mathcal{R}(w)$. Two reduced expressions are said to be *commutation equivalent* if they are in the same commutation class. Commutation classes have been studied extensively in the literature. In the Coxeter system of type A_n, Elnitsky [7] showed that the set of commutation classes for a given w is in one-to-one correspondence with the set of rhombic tilings of a certain polygon determined by w. Bedard [3] developed recursive formulas to find the number of reduced expressions in each commutation class while Meng [12] studied the number of commutation classes and their relationship via braid moves.

Similarly, we define $\boldsymbol{\alpha} \sim_b \boldsymbol{\beta}$ if we can obtain $\boldsymbol{\alpha}$ from $\boldsymbol{\beta}$ by applying a single braid move. The equivalence relation $\approx_b$ is defined by taking the reflexive and transitive closure of $\sim_b$. Each equivalence class under $\approx_b$ is called a *braid class*, denoted $[\boldsymbol{\alpha}]_b$ for $\boldsymbol{\alpha} \in \mathcal{R}(w)$. Two reduced expressions are said to be *braid equivalent* if they are in the same braid class. Although braid classes have not been studied as extensively in the literature as commutation classes, braid classes have appeared in the work of Bergeron, et al. [4], among others. In the Coxeter system of type A_n, Fishel et al. [8] provided upper and lower bounds on the number of reduced expressions for a fixed permutation by studying the commutation classes and braid classes in tandem. Also, in Coxeter systems of type A_n, Zollinger [16] provided formulas for the cardinality of braid classes.

Example 2.4. Consider the expression $\boldsymbol{\alpha} = 1321434$ for some w in the Coxeter system of type D_4. It turns out that $\boldsymbol{\alpha}$ is reduced so that $\ell(w) = 7$. There are 15 reduced expressions in $\mathcal{R}(w)$ and the corresponding Matsumoto graph is given in Figure 2.2. The edges of $\mathcal{G}(w)$ show how pairs of reduced expressions are related via commutation or braid moves. The set of 15 reduced expressions is partitioned into five commutation classes:

$$
\begin{aligned}
[1321434]_c &= \{1321434, 1324134, 1342134, 1341234, 1314234, 1312434\}\\
[3123243]_c &= \{3123243, 3213243, 3213423, 3123423\}\\
[3134234]_c &= \{3134234, 3132434\}\\
[1321343]_c &= \{1321343, 1312343\}\\
[3132343]_c &= \{3132343\}
\end{aligned}
$$

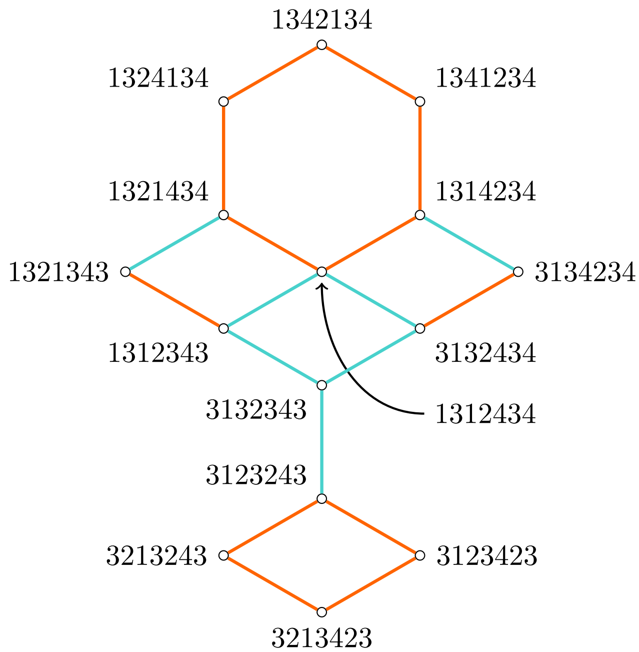


Figure 2.2: Example of a Matsumoto graph in the Coxeter system of type D_4.

and nine braid classes:

$$
\begin{aligned}
[1312343]_b &= \{1312343, 1312434, 3132434, 3132343, 3123243\}\\
[1321343]_b &= \{1321343, 1321434\}\\
[1314234]_b &= \{1314234, 3134234\}\\
[1324134]_b &= \{1324134\}\\
[1342134]_b &= \{1342134\}\\
[1341234]_b &= \{1341234\}\\
[3213243]_b &= \{3213243\}\\
[3213423]_b &= \{3213423\}\\
[3123423]_b &= \{3123423\}
\end{aligned}
$$

Notice that the braid classes of sizes 2 and 5 correspond to the vertices in the blue connected components of the Matsumoto graph given in Figure 2.2 while the singleton braid classes correspond to the six vertices that are not incident to any blue edges. A similar structure holds for the commutation classes.

Since we will focus solely on braid classes for the remainder of this thesis, we will write

$[\boldsymbol{\alpha}]$ in place of $[\boldsymbol{\alpha}]_b$. We can see the relationship among reduced expressions in a fixed braid class by looking at the corresponding maximal blue connected components of the Matsumoto graph. This leads to the following definition. Let $\boldsymbol{\alpha}$ be a reduced expression for $w \in W$, we define the *braid graph* of $\boldsymbol{\alpha}$, denoted $\mathcal{B}(\boldsymbol{\alpha})$, to be the graph with vertex set $[\boldsymbol{\alpha}]$, where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in [\boldsymbol{\alpha}]$ are connected by an edge if and only if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related via a single braid move. Note that braid graphs are defined with respect to a fixed reduced expression (or equivalence class) as opposed to the corresponding group element. Moreover, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are braid related, then $\mathcal{B}(\boldsymbol{\alpha}) = \mathcal{B}(\boldsymbol{\beta})$.

Example 2.5. Below we describe three different braid classes and illustrate their corresponding braid graphs, where we have used underlines and overlines to indicate where braid moves may occur.

(a) In the Coxeter system of type A_6, the expression 1213243565 is reduced. Its braid class consists of the following reduced expressions:

$$\boldsymbol{\alpha}_1 = \underline{121}3243\underline{565},\ \boldsymbol{\alpha}_2 = \underline{21\overline{2}}\overline{32}43\underline{565},\ \boldsymbol{\alpha}_3 = 21\underline{32\overline{3}}\overline{43}\underline{565},\ \boldsymbol{\alpha}_4 = 2132\overline{434}\underline{565},$$

$$\boldsymbol{\alpha}_5 = \underline{121}3243\underline{656},\ \boldsymbol{\alpha}_6 = \underline{21\overline{2}}\overline{32}43\underline{656},\ \boldsymbol{\alpha}_7 = 21\underline{32\overline{3}}\overline{43}\underline{656},\ \boldsymbol{\alpha}_8 = 2132\overline{434}\underline{656}.$$

(b) In the Coxeter system of type D_4, the expression 4341232 is reduced and its braid class consists of the following reduced expressions:

$$\boldsymbol{\beta}_1 = \underline{434}1\underline{232},\ \boldsymbol{\beta}_2 = \underline{343}1\underline{232},\ \boldsymbol{\beta}_3 = \underline{434}1\underline{323},\ \boldsymbol{\beta}_4 = \underline{34\overline{3}}\overline{1}\underline{\overline{3}23},\ \boldsymbol{\beta}_5 = 34\underline{131}23.$$

(c) In the Coxeter system of type D_4, the expression 343132343 is reduced and its braid class consists of the following reduced expressions:

$$\boldsymbol{\gamma}_1 = \underline{34\overline{3}}\overline{1}\underline{\overline{3}2\overline{3}}\overline{43},\ \boldsymbol{\gamma}_2 = 34\underline{131}2\underline{343},\ \boldsymbol{\gamma}_3 = \underline{434}1\underline{32\overline{3}}\overline{43},\ \boldsymbol{\gamma}_4 = \underline{343}1\underline{232}43,$$

$$\boldsymbol{\gamma}_5 = \underline{434}1\underline{232}43,\ \boldsymbol{\gamma}_6 = \underline{343}132\underline{434},\ \boldsymbol{\gamma}_7 = 34\underline{131}2\underline{434},\ \boldsymbol{\gamma}_8 = \underline{434}132\underline{434}.$$

The braid graphs $\mathcal{B}(\boldsymbol{\alpha}_1), \mathcal{B}(\boldsymbol{\beta}_1)$, and $\mathcal{B}(\boldsymbol{\gamma}_1)$ are depicted in Figures 2.3(a), 2.3(b), and 2.3(c), respectively.

The next result follows immediately from Proposition 2.3.

Proposition 2.6. If (W, S) is a Coxeter system and $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$, then $\mathcal{B}(\boldsymbol{\alpha})$ is a bipartite.

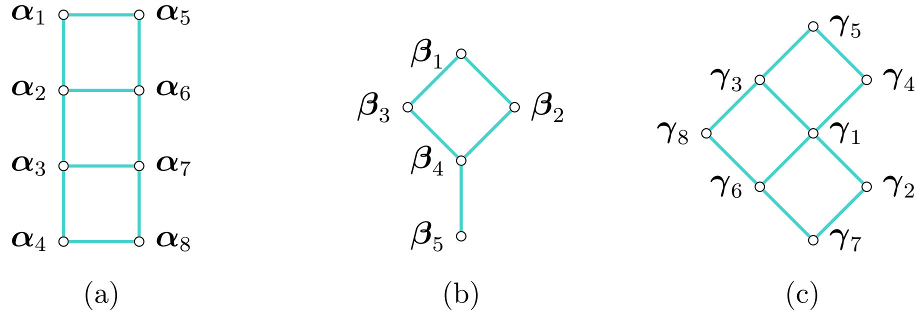


Figure 2.3: Braid graphs generated by various reduced expressions.

We will now discuss more terminology that will allow us to introduce the notions of braid shadow and link. Throughout the remainder of this chapter, we assume that (W, S) is a simply-laced Coxeter system. This condition is necessary for a majority of the following results, although the results likely generalize with the appropriate modifications.

If $i, j \in \mathbb{N}$ with $i < j$, then we define the interval $[\![i, j]\!] := \{i, i+1, \ldots, j-1, j\}$. We define the degenerate interval $[\![i, i]\!]$ to be the singleton set $\{i\}$. We will use the intervals $[\![i, j]\!]$ to represent positions in a reduced expression. If $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression for $w \in W$, we define the *local support* of $\boldsymbol{\alpha}$ over $[\![i, j]\!]$ via

$$\mathrm{supp}_{[\![i,j]\!]}(\boldsymbol{\alpha}) := \{s_{x_k} \mid k \in [\![i, j]\!]\}.$$

The *local support of the braid class* $[\boldsymbol{\alpha}]$ over $[\![i, j]\!]$ is defined by

$$\mathrm{supp}_{[\![i,j]\!]}([\boldsymbol{\alpha}]) := \bigcup_{\boldsymbol{\beta} \in [\boldsymbol{\alpha}]} \mathrm{supp}_{[\![i,j]\!]}(\boldsymbol{\beta}).$$

In other words, $\mathrm{supp}_{[\![i,j]\!]}(\boldsymbol{\alpha})$ is the set consisting of the generators that appear in positions $i, i+1, \ldots, j$ of $\boldsymbol{\alpha}$ while $\mathrm{supp}_{[\![i,j]\!]}([\boldsymbol{\alpha}])$ is the set of generators that appear in positions $i, i+1, \ldots, j$ of any reduced expression in $[\boldsymbol{\alpha}]$. In the case of the degenerate interval $[\![i, i]\!]$, we will use the notation $\mathrm{supp}_{[\![i]\!]}(\boldsymbol{\alpha})$ and $\mathrm{supp}_{[\![i]\!]}([\boldsymbol{\alpha}])$, and we will simply write $\mathrm{supp}(\boldsymbol{\alpha})$ for the set of generators that appear in $\boldsymbol{\alpha}$. We will also let $\boldsymbol{\alpha}_{[\![i,j]\!]}$ denote the subword $s_{x_i} s_{x_{i+1}} \cdots s_{x_{j-1}} s_{x_j}$ of $\boldsymbol{\alpha}$.

Following [2], if $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression for $w \in W$, then the interval $[\![i, i+2]\!]$ is a *braid shadow* for $\boldsymbol{\alpha}$ if $s_{x_i} = s_{x_{i+2}}$ and $m(s_{x_i}, s_{x_{i+1}}) = 3$. The collection of braid shadows for $\boldsymbol{\alpha}$ is denoted by $\mathcal{S}(\boldsymbol{\alpha})$ and the set of braid shadows for the braid class $[\boldsymbol{\alpha}]$ is given by

$$\mathcal{S}([\boldsymbol{\alpha}]) := \bigcup_{\boldsymbol{\beta} \in [\boldsymbol{\alpha}]} \mathcal{S}(\boldsymbol{\beta}).$$

The cardinality of $\mathcal{S}([\boldsymbol{\alpha}])$ is called the *rank* of $\boldsymbol{\alpha}$, which we denote by $\mathrm{rank}(\boldsymbol{\alpha})$.

In summary, a braid shadow for a reduced expression $\boldsymbol{\alpha}$ refers to a location in $\boldsymbol{\alpha}$ where we can apply a braid move. A reduced expression may have many braid shadows, or none at all. The set $\mathcal{S}(\boldsymbol{\alpha})$ is the collection of the braid shadows for a specific $\boldsymbol{\alpha}$, while $\mathcal{S}([\boldsymbol{\alpha}])$ captures the braid shadows for all reduced expressions that are braid equivalent to $\boldsymbol{\alpha}$. If $[\![i, i+2]\!]$ is a braid shadow for $[\boldsymbol{\alpha}]$, then we will refer to position $i+1$ in any reduced expression in $[\boldsymbol{\alpha}]$ as the *center* of the braid shadow.

Example 2.7. Consider the reduced expressions given in Example 2.5. We see that:

(a) $\mathcal{S}(\boldsymbol{\alpha}_1) = \{[\![1,3]\!], [\![8,10]\!]\}$ and $\mathcal{S}([\boldsymbol{\alpha}_1]) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!], [\![8,10]\!]\}$,

(b) $\mathcal{S}(\boldsymbol{\beta}_1) = \{[\![1,3]\!], [\![5,7]\!]\}$ and $\mathcal{S}([\boldsymbol{\beta}_1]) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!]\}$,

(c) $\mathcal{S}(\boldsymbol{\gamma}_1) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!], [\![7,9]\!]\}$ and $\mathcal{S}([\boldsymbol{\gamma}_1]) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!], [\![7,9]\!]\}$.

If $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$, then a pair of braid shadows for $\boldsymbol{\alpha}$ are either disjoint or overlap by a single position. Section 2.1 of [8] states this explicitly for Coxeter systems of type A_n. The following proposition from [2] implies the result for all simply-laced Coxeter systems.

Proposition 2.8. Suppose (W, S) is a simply-laced Coxeter system. If $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$ with $[\![i, i+2]\!] \in \mathcal{S}([\boldsymbol{\alpha}])$, then $[\![i+1, i+3]\!] \notin \mathcal{S}([\boldsymbol{\alpha}])$.

Reduced expressions with the property that there are no gaps between consecutive braid shadows play a pivotal role. The previous result motivates the next definition. Let $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m}$ be a reduced expression for some w in a simply-laced Coxeter system with $m \geq 1$. We define $\boldsymbol{\alpha}$ to be a *link* if either $m = 1$ or m is odd and

$$\mathcal{S}([\boldsymbol{\alpha}]) = \{[\![1,3]\!], [\![3,5]\!], \dots, [\![m-4, m-2]\!], [\![m-2, m]\!]\}.$$

If $\boldsymbol{\alpha}$ is a link, then the corresponding braid class $[\boldsymbol{\alpha}]$ is called a *braid chain*. Note that every reduced expression in a braid chain is a link.

Example 2.9. Consider the reduced expressions given in Example 2.5. Since $\mathcal{S}([\boldsymbol{\alpha}_1]) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!], [\![8,10]\!]\}$, $\boldsymbol{\alpha}_1$ is a not a link, and hence $[\boldsymbol{\alpha}_1]$ is not a braid chain. However, it turns out that the subwords 1213243 and 565 of $\boldsymbol{\alpha}_1$ are links in their own right. On the other hand, since $\mathcal{S}([\boldsymbol{\beta}_1]) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!]\}$, it follows that $\boldsymbol{\beta}_1$ is a link and $[\boldsymbol{\beta}_1]$ is a braid chain. Lastly, since $\mathcal{S}([\boldsymbol{\gamma}_1]) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!], [\![7,9]\!]\}$, $\boldsymbol{\gamma}_1$ is a link and $[\boldsymbol{\gamma}_1]$ is a braid chain.

If $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$ with $\ell(w) \geq 1$, then we say that $\boldsymbol{\beta}$ is a *link factor* of $\boldsymbol{\alpha}$ provided that

(a) $\boldsymbol{\beta}$ is a subword of $\boldsymbol{\alpha}$,

(b) $\boldsymbol{\beta}$ is a link, and

(c) for every subword $\boldsymbol{\gamma}$ of $\boldsymbol{\alpha}$, if $\boldsymbol{\beta}$ is a subword of $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}$ is a link, then $\boldsymbol{\beta} = \boldsymbol{\gamma}$.

It follows from this definition that every reduced expression $\boldsymbol{\alpha}$ for a nonidentity group element can be written as a unique product of link factors, say $\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\cdots\boldsymbol{\alpha}_k$, where each $\boldsymbol{\alpha}_i$ is a link factor of $\boldsymbol{\alpha}$. This product will be referred to as the *link factorization* of $\boldsymbol{\alpha}$. For emphasis, we may denote the link factorization as $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 \mid \boldsymbol{\alpha}_2 \mid \cdots \mid \boldsymbol{\alpha}_k$.

Using the link factorization of a reduced expression, every braid graph for a reduced expression can be written as a box product of the braid graphs of the corresponding link factors in the link factorization. The decomposition is unique if the ordering of the link factors in respected. The following proposition is an immediate consequence of the previous definitions and appeared in [2].

Proposition 2.10. Suppose (W, S) is a simply-laced Coxeter system. If $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$ with link factorization $\boldsymbol{\alpha}_1 \mid \boldsymbol{\alpha}_2 \mid \cdots \mid \boldsymbol{\alpha}_k$, then

(a) $[\boldsymbol{\alpha}] = \left\{\boldsymbol{\beta}_1 \mid \boldsymbol{\beta}_2 \mid \cdots \mid \boldsymbol{\beta}_k : \boldsymbol{\beta}_i \in [\boldsymbol{\alpha}_i] \text{ for } 1 \le i \le k\right\}$,

(b) The cardinality of the braid class for $\boldsymbol{\alpha}$ is given by $\operatorname{card}([\boldsymbol{\alpha}]) = \displaystyle\prod_{i=1}^{k} \operatorname{card}([\boldsymbol{\alpha}_i])$,

(c) The rank of $\boldsymbol{\alpha}$ is given by $\operatorname{rank}(\boldsymbol{\alpha}) = \displaystyle\sum_{i=1}^{k} \operatorname{rank}(\boldsymbol{\alpha}_i)$,

(d) $\mathcal{B}(\boldsymbol{\alpha}) \cong \mathcal{B}(\boldsymbol{\alpha}_1) \,\square\, \mathcal{B}(\boldsymbol{\alpha}_2) \,\square \cdots \square\, \mathcal{B}(\boldsymbol{\alpha}_k)$.

Example 2.11. Consider the reduced expression $\boldsymbol{\alpha}_1 = 1213243565$ given in Example 2.5. The link factorization for $\boldsymbol{\alpha}_1$ is $1213243 \mid 565$. The decomposition $\mathcal{B}(\boldsymbol{\alpha}_1) \cong \mathcal{B}(1213243) \,\square\, \mathcal{B}(565)$ is illustrated in Figure 2.4. We have utilized colors to help distinguish the link factors.

Example 2.12. Consider the reduced expression $\boldsymbol{\alpha} = 21234356576$ for a group element in a Coxeter system of type A_7. The link factorization for $\boldsymbol{\alpha}$ is $212 \mid 343 \mid 56576$. The decomposition $\mathcal{B}(\boldsymbol{\alpha}) \cong \mathcal{B}(212) \,\square\, \mathcal{B}(343) \,\square\, \mathcal{B}(56576)$ is illustrated in Figure 2.5. Again, we have utilized colors to help distinguish the link factors.

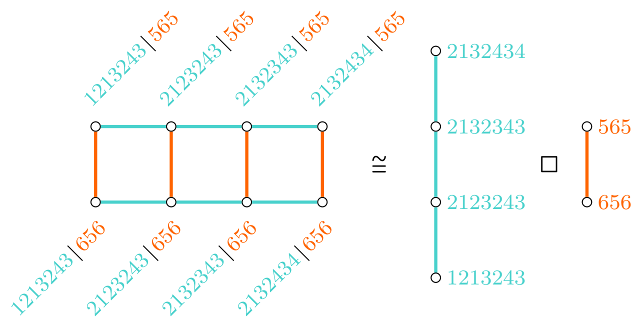


Figure 2.4: Decomposition of the braid graph for the reduced expression in Example 2.11.

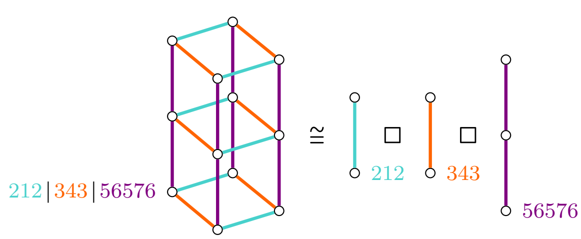


Figure 2.5: Decomposition of the braid graph for the reduced expression in Example 2.12.

Chapter 3

Local structure of links and braid chains

The goal of this chapter is to provide descriptions of the local structure of links in Coxeter systems of type Λ. We will begin by recalling several results from [2]. We will then introduce new results regarding the local structure of links.

The first proposition, found in [2], tells us that in a Coxeter system of type Λ the support of a braid shadow for a reduced expression is constant across the entire braid class.

Proposition 3.1. Suppose (W, S) is type Λ. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two braid equivalent links of rank at least one, then for all $[\![2i-1, 2i+1]\!] \in \mathcal{S}(\boldsymbol{\alpha}) \cap \mathcal{S}(\boldsymbol{\beta})$, $\operatorname{supp}_{[\![2i-1,2i+1]\!]}(\boldsymbol{\alpha}) = \operatorname{supp}_{[\![2i-1,2i+1]\!]}(\boldsymbol{\beta})$.

The previous result requires the assumption that the Coxeter system is triangle free. The following example illustrates that without this assumption the result does not hold.

Example 3.2. Consider the Coxeter system of type $\widetilde{A}_2$, which is determined by the Coxeter graph in Figure 2.1(c). Given the reduced expression $\boldsymbol{\alpha} = 1213121$, it is clear that $\boldsymbol{\beta} = 2123212 \in [\boldsymbol{\alpha}]$. However, $\operatorname{supp}_{[\![3,5]\!]}(\boldsymbol{\alpha}) = \{1, 3\}$ while $\operatorname{supp}_{[\![3,5]\!]}(\boldsymbol{\beta}) = \{2, 3\}$. Hence Proposition 3.1 does not necessarily hold if the Coxeter system is not triangle free.

The following proposition from [2] tells us that when a reduced expression has a braid shadow, the support of that braid shadow completely determines the collection of generators that can appear at the center of that braid shadow across the entire braid chain. Moreover, if there are overlapping braid shadows in a reduced expression, the supports of the braid shadows intersect at a single generator.

Proposition 3.3. Suppose (W, S) is type Λ. If $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$, then $[\![2i-1, 2i+1]\!] \in \mathcal{S}(\boldsymbol{\alpha})$ if and only if $[\![2i-1, 2i+1]\!] \in \mathcal{S}([\boldsymbol{\alpha}])$ and $\operatorname{supp}_{[\![2i-1,2i+1]\!]}(\boldsymbol{\alpha}) = \operatorname{supp}_{[\![2i]\!]}([\boldsymbol{\alpha}])$. If $[\![2i-1, 2i+1]\!], [\![2i+1, 2i+3]\!] \in \mathcal{S}([\boldsymbol{\alpha}])$, then $\operatorname{card}(\operatorname{supp}_{[\![2i]\!]}([\boldsymbol{\alpha}]) \cap \operatorname{supp}_{[\![2i+2]\!]}([\boldsymbol{\alpha}])) = 1$.

The previous proposition allows us to assume $\operatorname{supp}_{[\![2i]\!]}([\boldsymbol{\alpha}]) = \{s,t\}$ whenever we have $[\![2i-1, 2i+1]\!] \in \mathcal{S}(\boldsymbol{\alpha})$ with $\operatorname{supp}_{[\![2i-1,2i+1]\!]}(\boldsymbol{\alpha}) = \{s,t\}$. Moreover, if we also have $[\![2i+1, 2i+3]\!] \in \mathcal{S}(\boldsymbol{\alpha})$, then we can conclude that $\operatorname{supp}_{[\![2i+2]\!]}([\boldsymbol{\alpha}]) = \{t,u\}$. Note that in this situation $m(s,t) = 3$, $m(t,u) = 3$, and $m(s,u) = 2$. These facts will be used frequently for the remainder of this thesis and we may do so without explicitly mentioning the proposition.

The next four propositions from [2] provide explicit structure for braid shadows in a link specifically. The first result tells us that for any two overlapping braid shadows, the support of the common position has cardinality three.

Proposition 3.4. Suppose (W,S) is type Λ. If $\boldsymbol{\alpha}$ is a link of rank $r \geq 2$ such that $\operatorname{supp}_{[\![2i]\!]}([\boldsymbol{\alpha}]) = \{s,t\}$ and $\operatorname{supp}_{[\![2i+2]\!]}([\boldsymbol{\alpha}]) = \{t,u\}$ for $1 \leq i \leq r-1$, then $\operatorname{supp}_{[\![2i+1]\!]}([\boldsymbol{\alpha}]) = \{s,t,u\}$.

The next proposition tells us that the left and right ends of a link have a fairly rigid structure.

Proposition 3.5. Suppose (W,S) is type Λ and let $\boldsymbol{\alpha}$ be a link of rank at least two.

(a) If $\operatorname{supp}_{[\![2]\!]}([\boldsymbol{\alpha}]) = \{s,t\}$, then $\boldsymbol{\alpha}_{[\![1,2]\!]} = st$ or $\boldsymbol{\alpha}_{[\![1,2]\!]} = ts$.

(b) If $\operatorname{supp}_{[\![2r]\!]}([\boldsymbol{\alpha}]) = \{s,t\}$, then $\boldsymbol{\alpha}_{[\![2r,2r+1]\!]} = st$ or $\boldsymbol{\alpha}_{[\![2r,2r+1]\!]} = ts$.

Proposition 3.6. Suppose (W,S) is type Λ and let $\boldsymbol{\alpha} = s_{x_1}s_{x_2}\cdots s_{x_m}$ be a link of rank at least 2 such that $\operatorname{supp}_{[\![2i]\!]}([\boldsymbol{\alpha}]) = \{s,t\}$ and $\operatorname{supp}_{[\![2i+2]\!]}([\boldsymbol{\alpha}]) = \{t,u\}$. If $s_{x_{2i}} \in \{s,t\}$ and $s_{x_{2i+2}} \in \{t,u\} \setminus \{s_{x_{2i}}\}$, then $s_{x_{2i+1}} \in \{s,t,u\} \setminus \{s_{x_{2i}}, s_{x_{2i+2}}\}$.

One consequence of Proposition 3.6 is that for any two overlapping braid shadows in a braid chain $[\boldsymbol{\alpha}]$, there are three possible forms that $\boldsymbol{\alpha}_{[\![2i-1,2i+3]\!]}$ may take:

(a) $$\cdots\underbrace{\underset{2i-1}{?}\quad\underset{2i}{s}\quad\underset{2i+1}{u}\quad\underset{2i+2}{t}\quad\underset{2i+3}{?}}_{\boldsymbol{\alpha}}\cdots$$

(b) $$\cdots\underbrace{\underset{2i-1}{?}\quad\underset{2i}{s}\quad\underset{2i+1}{t}\quad\underset{2i+2}{u}\quad\underset{2i+3}{?}}_{\boldsymbol{\alpha}}\cdots$$

(c) $$\cdots\underbrace{\underset{2i-1}{?}\quad\underset{2i}{t}\quad\underset{2i+1}{s}\quad\underset{2i+2}{u}\quad\underset{2i+3}{?}}_{\boldsymbol{\alpha}}\cdots$$

where $m(s,t) = 3$, $m(t,u) = 3$, and $m(s,u) = 2$. We have highlighted in orange the generators in the center of each braid shadow in $[\boldsymbol{\alpha}]$ to aid the reader. We will continue to do this for the remainder of this chapter. Note that position $2i+1$ is the position where the two braid shadows in $[\boldsymbol{\alpha}]$ overlap. It is important to note that the form

$$\cdots\underbrace{\underset{2i-1}{\underline{\ ?\ }}\ \underset{2i}{\underline{\ t\ }}\ \underset{2i+1}{\underline{\ ?\ }}\ \underset{2i+2}{\underline{\ t\ }}\ \underset{2i+3}{\underline{\ ?\ }}}_{\boldsymbol{\alpha}}\cdots$$

is not possible since Proposition 3.6 tells us $\mathrm{supp}_{[\![2i+1]\!]} = \{s,t,u\} \smallsetminus \{t\} = \{s,u\}$, but either generator will cause a braid shadow in positions $[\![2i, 2i+2]\!]$ since $m(s,t) = 3$ and $m(t,u) = 3$, which is not possible by Proposition 2.8.

The following proposition concludes that every link is uniquely determined by the generators appearing in the even positions of the reduced expression.

Proposition 3.7. Suppose (W,S) is type Λ and let $\boldsymbol{\alpha} = s_{x_1}s_{x_2}\cdots s_{x_{2r+1}}$ and $\boldsymbol{\beta} = s_{y_1}s_{y_2}\cdots s_{y_{2r+1}}$ be two braid equivalent links of rank r. Then $\boldsymbol{\alpha} = \boldsymbol{\beta}$ if and only if $s_{x_{2j}} = s_{y_{2j}}$ for all $1 \le j \le r$.

Since the previous result is so significant, we developed the following definition. If (W,S) is type Λ and $\boldsymbol{\alpha} = s_{x_1}\cdots s_{x_{2r+1}}$ is a link of rank r, the *signature* of $\boldsymbol{\alpha}$, denoted $\mathrm{sig}(\boldsymbol{\alpha})$, is the ordered list of generators appearing in the even positions of $\boldsymbol{\alpha}$. That is, $\mathrm{sig}(\boldsymbol{\alpha}) = (s_{x_2}, s_{x_4}, \ldots, s_{x_{2r}})$. We will use $\mathrm{sig}_i(\boldsymbol{\alpha})$ to represent the ith position of $\mathrm{sig}(\boldsymbol{\alpha})$. Note that the ith position of the signature corresponds to the ith braid shadow in $[\boldsymbol{\alpha}]$. We define $\Delta(\mathrm{sig}(\boldsymbol{\alpha}), \mathrm{sig}(\boldsymbol{\beta}))$ to be the number of entries that differ between the signatures of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

This allows us to restate Proposition 3.7 as follows.

Proposition 3.8. Suppose (W,S) is type Λ and let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two braid equivalent links. Then $\boldsymbol{\alpha} = \boldsymbol{\beta}$ if and only if $\mathrm{sig}(\boldsymbol{\alpha}) = \mathrm{sig}(\boldsymbol{\beta})$.

The above results provide a lot of information regarding which generators can be in which positions across a braid chain. For the remainder of this chapter, we focus our attention on the local structure of links.

The following series of propositions can be viewed as a strengthening of Propositions 3.1, 3.3, 3.5, and 3.6, where we provide precise information about what generators appear in the odd positions of a link based on the relations of the corresponding generators and what generators appear in the even positions. The next result follows immediately from Propositions 3.5 and 3.6.

Proposition 3.9. Suppose (W,S) is type Λ and let $\boldsymbol{\alpha}$ be a link with $\mathrm{rank}(\boldsymbol{\alpha}) = 1$. If $\mathrm{supp}_{[\![2]\!]}([\boldsymbol{\alpha}]) = \{s,t\}$, then $\boldsymbol{\alpha} = sts$ or $\boldsymbol{\alpha} = tst$.

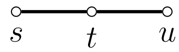


Figure 3.1: Induced subgraphs of the Coxeter graph for Propositions 3.10 and 3.11.

Proposition 3.10. Suppose (W, S) is type Λ and let $\boldsymbol{\alpha}$ be a link with $\operatorname{rank}(\boldsymbol{\alpha}) \geq 2$. If $\operatorname{supp}_{[\![2]\!]}([\boldsymbol{\alpha}]) = \{s, t\}$ and $\operatorname{supp}_{[\![4]\!]}([\boldsymbol{\alpha}]) = \{t, u\}$, where the relationships among s, t, and u are depicted in Figure 3.1, then $\boldsymbol{\alpha}_{[\![1,4]\!]}$ is equal to one of the following:

(a) $\underbrace{\underset{1}{t}\ \underset{2}{s}\ \underset{3}{u}\ \underset{4}{t}\cdots}_{\boldsymbol{\alpha}}$

(b) $\underbrace{\underset{1}{t}\ \underset{2}{s}\ \underset{3}{t}\ \underset{4}{u}\cdots}_{\boldsymbol{\alpha}}$

(c) $\underbrace{\underset{1}{s}\ \underset{2}{t}\ \underset{3}{s}\ \underset{4}{u}\cdots}_{\boldsymbol{\alpha}}$

Proof. By Proposition 3.6, there are three possibilities to consider for $\boldsymbol{\alpha}_{[\![2,4]\!]}$. In each case, there is a unique choice for which generator appears in the first position according to Propositions 3.5. Thus we obtain the three forms for $\boldsymbol{\alpha}_{[\![1,4]\!]}$. $\square$

The following result is the right-handed version of Proposition 3.10.

Proposition 3.11. Suppose (W, S) is type Λ and let $\boldsymbol{\alpha}$ be a link with $\operatorname{rank}(\boldsymbol{\alpha}) \geq 2$. If $\operatorname{supp}_{[\![2r-2]\!]}([\boldsymbol{\alpha}]) = \{t, u\}$ and $\operatorname{supp}_{[\![2r]\!]}([\boldsymbol{\alpha}]) = \{s, t\}$, where the relationships among s, t, and u are depicted in Figure 3.1, then $\boldsymbol{\alpha}_{[\![2r-2,2r+1]\!]}$ is equal to one of the following:

(a) $\underbrace{\cdots\underset{2r-2}{t}\ \underset{2r-1}{u}\ \underset{2r}{s}\ \underset{2r+1}{t}}_{\boldsymbol{\alpha}}$

(b) $\underbrace{\cdots\underset{2r-2}{u}\ \underset{2r-1}{t}\ \underset{2r}{s}\ \underset{2r+1}{t}}_{\boldsymbol{\alpha}}$

(c) $\underbrace{\cdots\underset{2r-2}{u}\ \underset{2r-1}{s}\ \underset{2r}{t}\ \underset{2r+1}{s}}_{\boldsymbol{\alpha}}$

Proof. This follows from a symmetric argument to the proof of Proposition 3.10. $\square$

The next result follows immediately from the previous results together with Proposition 3.5.

Corollary 3.12. If $\operatorname{rank}(\boldsymbol{\alpha}) = 2$, $\operatorname{supp}_{[\![2]\!]}([\boldsymbol{\alpha}]) = \{s,t\}$, and $\operatorname{supp}_{[\![4]\!]}([\boldsymbol{\alpha}]) = \{t,u\}$, where the relationships among s,t,u are depicted in Figure 3.1, then $\boldsymbol{\alpha}$ is equal to one of the following:

(a) $\underbrace{\underset{1}{t}\;\underset{2}{s}\;\underset{3}{u}\;\underset{4}{t}\;\underset{5}{u}}_{\boldsymbol{\alpha}}$

(b) $\underbrace{\underset{1}{t}\;\underset{2}{s}\;\underset{3}{t}\;\underset{4}{u}\;\underset{5}{t}}_{\boldsymbol{\alpha}}$

(c) $\underbrace{\underset{1}{s}\;\underset{2}{t}\;\underset{3}{s}\;\underset{4}{u}\;\underset{5}{t}}_{\boldsymbol{\alpha}}$

In what follows, we will use a dashed line in a Coxeter graph to denote an edge that may or may not be present.

(a) (b)

Figure 3.2: Induced subgraphs of the Coxeter graphs for Propositions 3.13 and 3.14.

Proposition 3.13. Suppose (W,S) is type Λ and let $\boldsymbol{\alpha}$ be a link with $\operatorname{rank}(\boldsymbol{\alpha}) \geq 5$. If $[\![2i-3, 2i-1]\!]$ is not the leftmost braid shadow and $[\![2i+1, 2i+3]\!]$ is not the rightmost braid shadow in $\mathcal{S}([\boldsymbol{\alpha}])$ such that $\operatorname{supp}_{[\![2i-2]\!]}([\boldsymbol{\alpha}]) = \{s,v\}$, $\operatorname{supp}_{[\![2i]\!]}([\boldsymbol{\alpha}]) = \{s,t\}$, and $\operatorname{supp}_{[\![2i+2]\!]}([\boldsymbol{\alpha}]) = \{t,u\}$, where the relationships among v,s,t,u are depicted in Figure 3.2(a), then $\boldsymbol{\alpha}_{[\![2i-2,2i+2]\!]}$ is equal to one of the following:

(a) $\underbrace{\cdots\underset{2i-2}{v}\;\underset{2i-1}{t}\;\underset{2i}{s}\;\underset{2i+1}{u}\;\underset{2i+2}{t}\cdots}_{\boldsymbol{\sigma}}$

(b) $\underbrace{\cdots\underset{2i-2}{v}\;\underset{2i-1}{t}\;\underset{2i}{s}\;\underset{2i+1}{t}\;\underset{2i+2}{u}\cdots}_{\boldsymbol{\sigma}}$

(c) $\cdots \underbrace{\underset{2i-2}{\underline{\;s\;}}\,\underset{2i-1}{\underline{\;v\;}}\,\underset{2i}{\underline{\;t\;}}\,\underset{2i+1}{\underline{\;s\;}}\,\underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\sigma}} \cdots$

(d) $\cdots \underbrace{\underset{2i-2}{\underline{\;v\;}}\,\underset{2i-1}{\underline{\;s\;}}\,\underset{2i}{\underline{\;t\;}}\,\underset{2i+1}{\underline{\;s\;}}\,\underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}} \cdots$

Proof. Based on the consequence of Proposition 3.6, we have three possibilities for $\boldsymbol{\alpha}_{[\![2i,2i+2]\!]}$. In each case, we consider all possibilities for $\boldsymbol{\alpha}_{[\![2i-2]\!]}$ and then apply Proposition 3.6 to obtain the generators in position $2i-1$. We see that

(a) $\cdots \underbrace{\underset{2i-2}{\underline{\;?\;}}\,\underset{2i-1}{\underline{\;?\;}}\,\underset{2i}{\underline{\;s\;}}\,\underset{2i+1}{\underline{\;u\;}}\,\underset{2i+2}{\underline{\;t\;}}}_{\boldsymbol{\alpha}} \cdots \overset{3.6}{\Longrightarrow} \cdots \underbrace{\underset{2i-2}{\underline{\;v\;}}\,\underset{2i-1}{\underline{\;t\;}}\,\underset{2i}{\underline{\;s\;}}\,\underset{2i+1}{\underline{\;u\;}}\,\underset{2i+2}{\underline{\;t\;}}}_{\boldsymbol{\alpha}} \cdots$

(b) $\cdots \underbrace{\underset{2i-2}{\underline{\;?\;}}\,\underset{2i-1}{\underline{\;?\;}}\,\underset{2i}{\underline{\;s\;}}\,\underset{2i+1}{\underline{\;t\;}}\,\underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}} \cdots \begin{cases} \boldsymbol{\alpha}_{[\![2i-2]\!]} = v \overset{3.6}{\Longrightarrow} \cdots \underbrace{\underset{2i-2}{\underline{\;v\;}}\,\underset{2i-1}{\underline{\;t\;}}\,\underset{2i}{\underline{\;s\;}}\,\underset{2i+1}{\underline{\;t\;}}\,\underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}} \cdots \\ \boldsymbol{\alpha}_{[\![2i-2]\!]} = s \overset{3.6}{\Longrightarrow} \cdots \underbrace{\underset{2i-2}{\underline{\;s\;}}\,\underset{2i-1}{\underline{\;v\;}}\,\underset{2i}{\underline{\;t\;}}\,\underset{2i+1}{\underline{\;s\;}}\,\underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}} \cdots \end{cases}$

(c) $\cdots \underbrace{\underset{2i-2}{\underline{\;?\;}}\,\underset{2i-1}{\underline{\;?\;}}\,\underset{2i}{\underline{\;t\;}}\,\underset{2i+1}{\underline{\;s\;}}\,\underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}} \cdots \begin{cases} \boldsymbol{\alpha}_{[\![2i-2]\!]} = v \overset{3.6}{\Longrightarrow} \cdots \underbrace{\underset{2i-2}{\underline{\;v\;}}\,\underset{2i-1}{\underline{\;s\;}}\,\underset{2i}{\underline{\;t\;}}\,\underset{2i+1}{\underline{\;s\;}}\,\underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}} \cdots \\ \boldsymbol{\alpha}_{[\![2i-2]\!]} = s \overset{3.6}{\Longrightarrow} \cdots \underbrace{\underset{2i-2}{\underline{\;s\;}}\,\underset{2i-1}{\underline{\;v\;}}\,\underset{2i}{\underline{\;t\;}}\,\underset{2i+1}{\underline{\;s\;}}\,\underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}} \cdots \end{cases}$

□

The proof for following proposition will use a similar approach to the proof of Proposition 3.13.

Proposition 3.14. Suppose (W,S) is type Λ and let $\boldsymbol{\alpha}$ be a link with $\mathrm{rank}(\boldsymbol{\alpha}) \geq 5$. If $[\![2i-3,2i-1]\!]$ is not the leftmost braid shadow and $[\![2i+1,2i+3]\!]$ is not the rightmost braid shadow in $\mathcal{S}([\boldsymbol{\alpha}])$ such that $\mathrm{supp}_{[\![2i-2]\!]}([\boldsymbol{\alpha}]) = \{v,t\}$, $\mathrm{supp}_{[\![2i]\!]}([\boldsymbol{\alpha}]) = \{s,t\}$, and $\mathrm{supp}_{[\![2i+2]\!]}([\boldsymbol{\alpha}]) = \{t,u\}$, where the relationships among v,s,t,u are depicted in Figure 3.2(b), then $\boldsymbol{\alpha}_{[\![2i-2,2i+2]\!]}$ is equal to one of the following:

(a) $\cdots \underbrace{\underset{2i-2}{\underline{\;v\;}}\,\underset{2i-1}{\underline{\;t\;}}\,\underset{2i}{\underline{\;s\;}}\,\underset{2i+1}{\underline{\;u\;}}\,\underset{2i+2}{\underline{\;t\;}}}_{\boldsymbol{\alpha}} \cdots$

$$\text{(b)}\quad \cdots\underbrace{\underset{2i-2}{\underline{\;v\;}}\ \underset{2i-1}{\underline{\;t\;}}\ \underset{2i}{\underline{\;s\;}}\ \underset{2i+1}{\underline{\;t\;}}\ \underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}}\cdots$$

$$\text{(c)}\quad \cdots\underbrace{\underset{2i-2}{\underline{\;v\;}}\ \underset{2i-1}{\underline{\;s\;}}\ \underset{2i}{\underline{\;t\;}}\ \underset{2i+1}{\underline{\;s\;}}\ \underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}}\cdots$$

$$\text{(d)}\quad \cdots\underbrace{\underset{2i-2}{\underline{\;t\;}}\ \underset{2i-1}{\underline{\;v\;}}\ \underset{2i}{\underline{\;s\;}}\ \underset{2i+1}{\underline{\;u\;}}\ \underset{2i+2}{\underline{\;t\;}}}_{\boldsymbol{\alpha}}\cdots$$

$$\text{(e)}\quad \cdots\underbrace{\underset{2i-2}{\underline{\;t\;}}\ \underset{2i-1}{\underline{\;v\;}}\ \underset{2i}{\underline{\;s\;}}\ \underset{2i+1}{\underline{\;t\;}}\ \underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}}\cdots$$

Proof. As in the previous proof, based on the consequence of Proposition 3.6, we have three possibilities for $\boldsymbol{\alpha}_{[\![2i,2i+2]\!]}$. In each case, we consider all available possibilities for $\boldsymbol{\alpha}_{[\![2i-2]\!]}$ and then apply Proposition 3.6 to obtain the generators in position $2i-1$. We see that

$$\text{(a)}\quad \cdots\underbrace{\underset{2i-2}{\underline{\;?\;}}\ \underset{2i-1}{\underline{\;?\;}}\ \underset{2i}{\underline{\;s\;}}\ \underset{2i+1}{\underline{\;u\;}}\ \underset{2i+2}{\underline{\;t\;}}}_{\boldsymbol{\alpha}}\cdots \begin{cases} \boldsymbol{\alpha}_{[\![2i-2]\!]} = v \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{\underline{\;v\;}}\ \underset{2i-1}{\underline{\;t\;}}\ \underset{2i}{\underline{\;s\;}}\ \underset{2i+1}{\underline{\;u\;}}\ \underset{2i+2}{\underline{\;t\;}}}_{\boldsymbol{\alpha}}\cdots \\[2ex] \boldsymbol{\alpha}_{[\![2i-2]\!]} = t \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{\underline{\;t\;}}\ \underset{2i-1}{\underline{\;v\;}}\ \underset{2i}{\underline{\;s\;}}\ \underset{2i+1}{\underline{\;u\;}}\ \underset{2i+2}{\underline{\;t\;}}}_{\boldsymbol{\alpha}}\cdots \end{cases}$$

$$\text{(b)}\quad \cdots\underbrace{\underset{2i-2}{\underline{\;?\;}}\ \underset{2i-1}{\underline{\;?\;}}\ \underset{2i}{\underline{\;s\;}}\ \underset{2i+1}{\underline{\;t\;}}\ \underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}}\cdots \begin{cases} \boldsymbol{\alpha}_{[\![2i-2]\!]} = v \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{\underline{\;v\;}}\ \underset{2i-1}{\underline{\;t\;}}\ \underset{2i}{\underline{\;s\;}}\ \underset{2i+1}{\underline{\;t\;}}\ \underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}}\cdots \\[2ex] \boldsymbol{\alpha}_{[\![2i-2]\!]} = t \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{\underline{\;t\;}}\ \underset{2i-1}{\underline{\;v\;}}\ \underset{2i}{\underline{\;s\;}}\ \underset{2i+1}{\underline{\;t\;}}\ \underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}}\cdots \end{cases}$$

$$\text{(c)}\quad \cdots\underbrace{\underset{2i-2}{\underline{\;?\;}}\ \underset{2i-1}{\underline{\;?\;}}\ \underset{2i}{\underline{\;t\;}}\ \underset{2i+1}{\underline{\;s\;}}\ \underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}}\cdots \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{\underline{\;v\;}}\ \underset{2i-1}{\underline{\;s\;}}\ \underset{2i}{\underline{\;t\;}}\ \underset{2i+1}{\underline{\;s\;}}\ \underset{2i+2}{\underline{\;u\;}}}_{\boldsymbol{\alpha}}\cdots$$

$\square$

In summary, Proposition 3.9 and Corollary 3.12 tell us precisely what links of rank one and two look like. Also, Propositions 3.10 and 3.11 indicate the structure of the four leftmost and rightmost positions of a link, respectively. Propositions 3.13 and 3.14 indicate the structure of the middle five positions in a link in the location of three overlapping braid shadows for a braid chain. An immediate consequence of Propositions 3.9–3.14 is that if $\boldsymbol{\alpha}$ is a link, then there are four possibilities for the subwords appearing in a braid shadow for $[\boldsymbol{\alpha}]$. We state this precisely in the following corollary.

Corollary 3.15. Suppose (W, S) is type Λ. If $\boldsymbol{\alpha}$ is a link of rank at least 1, then $\boldsymbol{\alpha}_{[\![2i-1,2i+1]\!]}$ is equal to one of the following:

(a) sts, where $m(s,t) = 3$;

(b) sut, where $m(s,u) = 2$, $m(u,t) = 3$, and $m(s,t) = 3$;

(c) tsu, where $m(s,t) = 3$, $m(s,u) = 2$, and $m(t,u) = 3$;

(d) vsu, where $m(v,s) = 2$, $m(s,u) = 2$, and $m(v,u) = 2$.

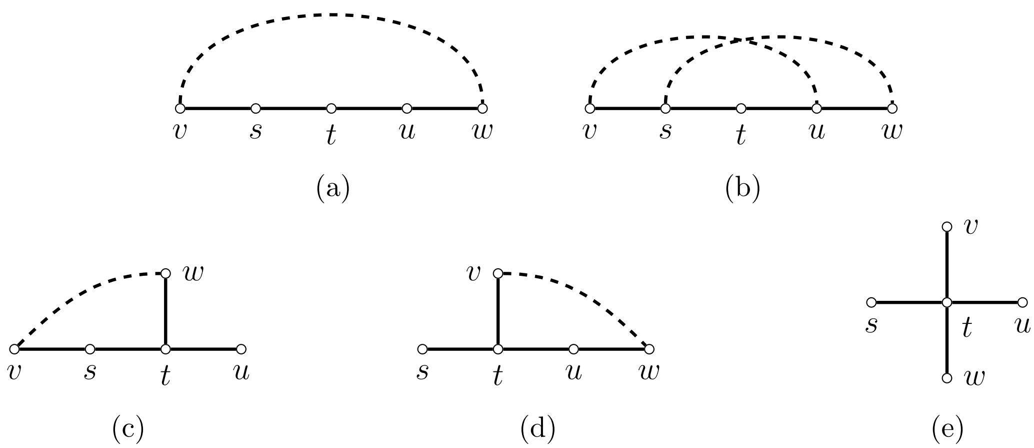


Figure 3.3: Induced subgraphs of the Coxeter graphs for Propositions 3.16–3.19.

In light of Proposition 3.14, the fourth type in Corollary 3.15 can only occur if the corresponding generators come from a Coxeter system whose Coxeter graph contains the Coxeter graph of type D_4 as a subgraph.

We can expand on Propositions 3.13 and 3.14 to find the possible configurations of two overlapping braid shadows in a link. We will need five generators in order to extend the previous lemmas, so we will reference Figure 3.3, which depicts the possible induced subgraphs of Coxeter graphs that describe the relationships among the relevant generators.

Proposition 3.16. Suppose (W,S) is type Λ and let $\boldsymbol{\alpha}$ be a link with $\operatorname{rank}(\boldsymbol{\alpha}) \geq 4$. If $[\![2i-3,2i-1]\!]$ is not the leftmost braid shadow and $[\![2i+1,2i+3]\!]$ is not the rightmost braid shadow in $\mathcal{S}([\boldsymbol{\alpha}])$ such that $\operatorname{supp}_{[\![2i-2]\!]}([\boldsymbol{\alpha}]) = \{v,t\}$, $\operatorname{supp}_{[\![2i]\!]}([\boldsymbol{\alpha}]) = \{s,t\}$, $\operatorname{supp}_{[\![2i+2]\!]}([\boldsymbol{\alpha}]) = \{t,u\}$, and $\operatorname{supp}_{[\![2i+4]\!]}([\boldsymbol{\alpha}]) = \{u,w\}$, where the relationships among v,s,t,u,w are depicted in Figures 3.3(a) and 3.3(b), then the overlapping braid shadows in positions $[\![2i-1,2i+1]\!]$ and $[\![2i+1,2i+3]\!]$ are equal to one of the following:

(a) $$\cdots\underbrace{\underset{2i-1}{\underline{t}}\ \underset{2i}{\underline{s}}\ \underset{2i+1}{\underline{u}}\ \underset{2i+2}{\underline{t}}\ \underset{2i+3}{\underline{w}}}_{\boldsymbol{\alpha}}\cdots$$

(b) $$\cdots\underbrace{\underset{2i-1}{\underline{t}}\ \underset{2i}{\underline{s}}\ \underset{2i+1}{\underline{u}}\ \underset{2i+2}{\underline{t}}\ \underset{2i+3}{\underline{u}}}_{\boldsymbol{\alpha}}\cdots$$

(c) $$\cdots\underbrace{\underset{2i-1}{\underline{t}}\ \underset{2i}{\underline{s}}\ \underset{2i+1}{\underline{t}}\ \underset{2i+2}{\underline{u}}\ \underset{2i+3}{\underline{t}}}_{\boldsymbol{\alpha}}\cdots$$

(d) $$\cdots\underbrace{\underset{2i-1}{\underline{v}}\ \underset{2i}{\underline{t}}\ \underset{2i+1}{\underline{s}}\ \underset{2i+2}{\underline{u}}\ \underset{2i+3}{\underline{t}}}_{\boldsymbol{\alpha}}\cdots$$

(e) $$\cdots\underbrace{\underset{2i-1}{\underline{s}}\ \underset{2i}{\underline{t}}\ \underset{2i+1}{\underline{s}}\ \underset{2i+2}{\underline{u}}\ \underset{2i+3}{\underline{t}}}_{\boldsymbol{\alpha}}\cdots$$

Proof. We use the results from Proposition 3.13 for our case analysis.

(a) $$\cdots\underbrace{\underset{2i-2}{\underline{v}}\ \underset{2i-1}{\underline{t}}\ \underset{2i}{\underline{s}}\ \underset{2i+1}{\underline{u}}\ \underset{2i+2}{\underline{t}}\ \underset{2i+3}{\underline{?}}\ \underset{2i+4}{\underline{?}}}_{\boldsymbol{\alpha}}\cdots \begin{cases} \boldsymbol{\alpha}_{[\![2i+4]\!]} = u \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{\underline{v}}\ \underset{2i-1}{\underline{t}}\ \underset{2i}{\underline{s}}\ \underset{2i+1}{\underline{u}}\ \underset{2i+2}{\underline{t}}\ \underset{2i+3}{\underline{w}}\ \underset{2i+4}{\underline{u}}}_{\boldsymbol{\alpha}}\cdots \\ \boldsymbol{\alpha}_{[\![2i+4]\!]} = w \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{\underline{v}}\ \underset{2i-1}{\underline{t}}\ \underset{2i}{\underline{s}}\ \underset{2i+1}{\underline{u}}\ \underset{2i+2}{\underline{t}}\ \underset{2i+3}{\underline{u}}\ \underset{2i+4}{\underline{w}}}_{\boldsymbol{\alpha}}\cdots \end{cases}$$

(b) $$\cdots\underbrace{\underset{2i-2}{\underline{v}}\ \underset{2i-1}{\underline{t}}\ \underset{2i}{\underline{s}}\ \underset{2i+1}{\underline{t}}\ \underset{2i+2}{\underline{u}}\ \underset{2i+3}{\underline{?}}\ \underset{2i+4}{\underline{w}}}_{\boldsymbol{\alpha}}\cdots \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{\underline{v}}\ \underset{2i-1}{\underline{t}}\ \underset{2i}{\underline{s}}\ \underset{2i+1}{\underline{t}}\ \underset{2i+2}{\underline{u}}\ \underset{2i+3}{\underline{t}}\ \underset{2i+4}{\underline{w}}}_{\boldsymbol{\alpha}}\cdots$$

(c) $$\cdots\underbrace{\underset{2i-2}{\underline{s}}\ \underset{2i-1}{\underline{v}}\ \underset{2i}{\underline{t}}\ \underset{2i+1}{\underline{s}}\ \underset{2i+2}{\underline{u}}\ \underset{2i+3}{\underline{?}}\ \underset{2i+4}{\underline{w}}}_{\boldsymbol{\alpha}}\cdots \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{\underline{s}}\ \underset{2i-1}{\underline{v}}\ \underset{2i}{\underline{t}}\ \underset{2i+1}{\underline{s}}\ \underset{2i+2}{\underline{u}}\ \underset{2i+3}{\underline{t}}\ \underset{2i+4}{\underline{w}}}_{\boldsymbol{\alpha}}\cdots$$

(d) $\cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{s}\ \underset{2i}{t}\ \underset{2i+1}{s}\ \underset{2i+2}{u}\ \underset{2i+3}{?}\ \underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{s}\ \underset{2i}{t}\ \underset{2i+1}{s}\ \underset{2i+2}{u}\ \underset{2i+3}{t}\ \underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots$

□

Proposition 3.17. Suppose (W, S) is type Λ and let $\boldsymbol{\alpha}$ be a link with $\text{rank}(\boldsymbol{\alpha}) \geq 4$. If $[\![2i-3, 2i-1]\!]$ is not the leftmost braid shadow and $[\![2i+1, 2i+3]\!]$ is not the rightmost braid shadow in $\mathcal{S}([\boldsymbol{\alpha}])$ such that $\text{supp}_{[\![2i-2]\!]}([\boldsymbol{\alpha}]) = \{v, t\}$, $\text{supp}_{[\![2i]\!]}([\boldsymbol{\alpha}]) = \{s, t\}$, $\text{supp}_{[\![2i+2]\!]}([\boldsymbol{\alpha}]) = \{t, u\}$, and $\text{supp}_{[\![2i+4]\!]}([\boldsymbol{\alpha}]) = \{t, w\}$, where the relationships among v, s, t, u, w are depicted in Figure 3.3(c), then the overlapping braid shadows in positions $[\![2i-1, 2i+1]\!]$ and $[\![2i+1, 2i+3]\!]$ are equal to one of the following:

(a) $\cdots\underbrace{\underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{u}\ \underset{2i+2}{t}\ \underset{2i+3}{u}}_{\boldsymbol{\alpha}}\cdots$

(b) $\cdots\underbrace{\underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{t}\ \underset{2i+2}{u}\ \underset{2i+3}{w}}_{\boldsymbol{\alpha}}\cdots$

(c) $\cdots\underbrace{\underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{t}\ \underset{2i+2}{u}\ \underset{2i+3}{t}}_{\boldsymbol{\alpha}}\cdots$

(d) $\cdots\underbrace{\underset{2i-1}{v}\ \underset{2i}{s}\ \underset{2i+1}{t}\ \underset{2i+2}{u}\ \underset{2i+3}{w}}_{\boldsymbol{\alpha}}\cdots$

(e) $\cdots\underbrace{\underset{2i-1}{v}\ \underset{2i}{s}\ \underset{2i+1}{t}\ \underset{2i+2}{u}\ \underset{2i+3}{t}}_{\boldsymbol{\alpha}}\cdots$

(f) $\cdots\underbrace{\underset{2i-1}{s}\ \underset{2i}{t}\ \underset{2i+1}{s}\ \underset{2i+2}{u}\ \underset{2i+3}{w}}_{\boldsymbol{\alpha}}\cdots$

(g) $\cdots\underbrace{\underset{2i-1}{s}\ \underset{2i}{t}\ \underset{2i+1}{s}\ \underset{2i+2}{u}\ \underset{2i+3}{t}}_{\boldsymbol{\alpha}}\cdots$

Proof. We use the results from Proposition 3.13 for our case analysis, and once again utilize Proposition 3.6.

$$\text{(a)}\quad \cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{u}\ \underset{2i+2}{t}\ \underset{2i+3}{?}\ \underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{u}\ \underset{2i+2}{t}\ \underset{2i+3}{u}\ \underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots$$

$$\text{(b)}\quad \cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{t}\ \underset{2i+2}{u}\ \underset{2i+3}{?}\ \underset{2i+4}{?}}_{\boldsymbol{\alpha}}\cdots \begin{cases} \boldsymbol{\alpha}_{\llbracket 2i+4\rrbracket} = t \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{t}\ \underset{2i+2}{u}\ \underset{2i+3}{w}\ \underset{2i+4}{t}}_{\boldsymbol{\alpha}}\cdots \\ \boldsymbol{\alpha}_{\llbracket 2i+4\rrbracket} = w \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{t}\ \underset{2i+2}{u}\ \underset{2i+3}{t}\ \underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots \end{cases}$$

$$\text{(c)}\quad \cdots\underbrace{\underset{2i-2}{s}\ \underset{2i-1}{v}\ \underset{2i}{t}\ \underset{2i+1}{s}\ \underset{2i+2}{u}\ \underset{2i+3}{?}\ \underset{2i+4}{?}}_{\boldsymbol{\alpha}}\cdots \begin{cases} \boldsymbol{\alpha}_{\llbracket 2i+4\rrbracket} = t \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{s}\ \underset{2i-1}{v}\ \underset{2i}{t}\ \underset{2i+1}{s}\ \underset{2i+2}{u}\ \underset{2i+3}{w}\ \underset{2i+4}{t}}_{\boldsymbol{\alpha}}\cdots \\ \boldsymbol{\alpha}_{\llbracket 2i+4\rrbracket} = w \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{s}\ \underset{2i-1}{v}\ \underset{2i}{t}\ \underset{2i+1}{s}\ \underset{2i+2}{u}\ \underset{2i+3}{t}\ \underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots \end{cases}$$

$$\text{(d)}\quad \cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{s}\ \underset{2i}{t}\ \underset{2i+1}{s}\ \underset{2i+2}{u}\ \underset{2i+3}{?}\ \underset{2i+4}{?}}_{\boldsymbol{\alpha}}\cdots \begin{cases} \boldsymbol{\alpha}_{\llbracket 2i+4\rrbracket} = t \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{s}\ \underset{2i}{t}\ \underset{2i+1}{s}\ \underset{2i+2}{u}\ \underset{2i+3}{w}\ \underset{2i+4}{t}}_{\boldsymbol{\alpha}}\cdots \\ \boldsymbol{\alpha}_{\llbracket 2i+4\rrbracket} = w \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{s}\ \underset{2i}{t}\ \underset{2i+1}{s}\ \underset{2i+2}{u}\ \underset{2i+3}{t}\ \underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots \end{cases}$$

□

Proposition 3.18. Suppose (W, S) is type Λ and let $\boldsymbol{\alpha}$ be a link with $\operatorname{rank}(\boldsymbol{\alpha}) \geq 4$. If $\llbracket 2i-3, 2i-1\rrbracket$ is not the leftmost braid shadow and $\llbracket 2i+1, 2i+3\rrbracket$ is not the rightmost braid shadow in $\mathcal{S}([\boldsymbol{\alpha}])$ such that $\operatorname{supp}_{\llbracket 2i-2\rrbracket}([\boldsymbol{\alpha}]) = \{v, t\}$, $\operatorname{supp}_{\llbracket 2i\rrbracket}([\boldsymbol{\alpha}]) = \{s, t\}$, $\operatorname{supp}_{\llbracket 2i+2\rrbracket}([\boldsymbol{\alpha}]) = \{t, u\}$, and $\operatorname{supp}_{\llbracket 2i+4\rrbracket}([\boldsymbol{\alpha}]) = \{u, w\}$, where the relationships among v, s, t, u, w are depicted in Figure 3.3(d), then the overlapping braid shadows in positions $\llbracket 2i-1, 2i+1\rrbracket$ and $\llbracket 2i+1, 2i+3\rrbracket$ are equal to one of the following:

$$\text{(a)}\quad \cdots\underbrace{\underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{u}\ \underset{2i+2}{t}\ \underset{2i+3}{w}}_{\boldsymbol{\alpha}}\cdots$$

$$\text{(b)}\quad \cdots\underbrace{\underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{u}\ \underset{2i+2}{t}\ \underset{2i+3}{u}}_{\boldsymbol{\alpha}}\cdots$$

(c) $$\cdots\underbrace{\underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{t}\ \underset{2i+2}{u}\ \underset{2i+3}{t}}_{\boldsymbol{\alpha}}\cdots$$

(d) $$\cdots\underbrace{\underset{2i-1}{s}\ \underset{2i}{t}\ \underset{2i+1}{s}\ \underset{2i+2}{u}\ \underset{2i+3}{t}}_{\boldsymbol{\alpha}}\cdots$$

(e) $$\cdots\underbrace{\underset{2i-1}{v}\ \underset{2i}{s}\ \underset{2i+1}{u}\ \underset{2i+2}{t}\ \underset{2i+3}{w}}_{\boldsymbol{\alpha}}\cdots$$

(f) $$\cdots\underbrace{\underset{2i-1}{v}\ \underset{2i}{s}\ \underset{2i+1}{u}\ \underset{2i+2}{t}\ \underset{2i+3}{u}}_{\boldsymbol{\alpha}}\cdots$$

(g) $$\cdots\underbrace{\underset{2i-1}{v}\ \underset{2i}{s}\ \underset{2i+1}{t}\ \underset{2i+2}{u}\ \underset{2i+3}{t}}_{\boldsymbol{\alpha}}\cdots$$

Proof. We use the results from Proposition 3.14 for our case analysis, and then apply Proposition 3.6.

(a) $$\cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{u}\ \underset{2i+2}{t}\ \underset{2i+3}{?}\ \underset{2i+4}{?}}_{\boldsymbol{\alpha}}\cdots \begin{cases} \boldsymbol{\alpha}_{[\![2i+4]\!]} = u \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{u}\ \underset{2i+2}{t}\ \underset{2i+3}{w}\ \underset{2i+4}{u}}_{\boldsymbol{\alpha}}\cdots \\ \boldsymbol{\alpha}_{[\![2i+4]\!]} = w \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{u}\ \underset{2i+2}{t}\ \underset{2i+3}{u}\ \underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots \end{cases}$$

(b) $$\cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{t}\ \underset{2i+2}{u}\ \underset{2i+3}{?}\ \underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{t}\ \underset{2i}{s}\ \underset{2i+1}{t}\ \underset{2i+2}{u}\ \underset{2i+3}{t}\ \underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots$$

(c) $$\cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{s}\ \underset{2i}{t}\ \underset{2i+1}{s}\ \underset{2i+2}{u}\ \underset{2i+3}{?}\ \underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{v}\ \underset{2i-1}{s}\ \underset{2i}{t}\ \underset{2i+1}{s}\ \underset{2i+2}{u}\ \underset{2i+3}{t}\ \underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots$$

(d) $$\cdots\underbrace{\underset{2i-2}{t}\ \underset{2i-1}{v}\ \underset{2i}{s}\ \underset{2i+1}{u}\ \underset{2i+2}{t}\ \underset{2i+3}{?}\ \underset{2i+4}{?}}_{\boldsymbol{\alpha}}\cdots \begin{cases} \boldsymbol{\alpha}_{[\![2i+4]\!]} = u \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{t}\ \underset{2i-1}{v}\ \underset{2i}{s}\ \underset{2i+1}{u}\ \underset{2i+2}{t}\ \underset{2i+3}{w}\ \underset{2i+4}{u}}_{\boldsymbol{\alpha}}\cdots \\ \boldsymbol{\alpha}_{[\![2i+4]\!]} = w \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{t}\ \underset{2i-1}{v}\ \underset{2i}{s}\ \underset{2i+1}{u}\ \underset{2i+2}{t}\ \underset{2i+3}{u}\ \underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots \end{cases}$$

(e) $$\cdots\underbrace{\underset{2i-2}{t}\;\underset{2i-1}{v}\;\underset{2i}{s}\;\underset{2i+1}{t}\;\underset{2i+2}{u}\;\underset{2i+3}{?}\;\underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots \overset{3.6}{\Longrightarrow} \cdots\underbrace{\underset{2i-2}{t}\;\underset{2i-1}{v}\;\underset{2i}{s}\;\underset{2i+1}{t}\;\underset{2i+2}{u}\;\underset{2i+3}{t}\;\underset{2i+4}{w}}_{\boldsymbol{\alpha}}\cdots$$

$\square$

Proposition 3.19. Suppose (W,S) is type Λ and let $\boldsymbol{\alpha}$ be a link with $\text{rank}(\boldsymbol{\alpha}) \geq 4$. If $[\![2i-3, 2i-1]\!]$ is not the leftmost braid shadow and $[\![2i+1, 2i+3]\!]$ is not the rightmost braid shadow in $\mathcal{S}([\boldsymbol{\alpha}])$ such that $\text{supp}_{[\![2i-2]\!]}([\boldsymbol{\alpha}]) = \{v,t\}$, $\text{supp}_{[\![2i]\!]}([\boldsymbol{\alpha}]) = \{s,t\}$, $\text{supp}_{[\![2i+2]\!]}([\boldsymbol{\alpha}]) = \{t,u\}$, and $\text{supp}_{[\![2i+4]\!]}([\boldsymbol{\alpha}]) = \{t,w\}$, where the relationships among v,s,t,u,w are depicted in Figure 3.3(e), then the overlapping braid shadows in positions $[\![2i-1,2i+1]\!]$ and $[\![2i+1,2i+3]\!]$ are equal to one of the following:

(a) $$\cdots\underbrace{\underset{2i-1}{t}\;\underset{2i}{s}\;\underset{2i+1}{u}\;\underset{2i+2}{t}\;\underset{2i+3}{u}}_{\boldsymbol{\alpha}}\cdots$$

(b) $$\cdots\underbrace{\underset{2i-1}{t}\;\underset{2i}{s}\;\underset{2i+1}{t}\;\underset{2i+2}{u}\;\underset{2i+3}{w}}_{\boldsymbol{\alpha}}\cdots$$

(c) $$\cdots\underbrace{\underset{2i-1}{t}\;\underset{2i}{s}\;\underset{2i+1}{t}\;\underset{2i+2}{u}\;\underset{2i+3}{t}}_{\boldsymbol{\alpha}}\cdots$$

(d) $$\cdots\underbrace{\underset{2i-1}{s}\;\underset{2i}{t}\;\underset{2i+1}{s}\;\underset{2i+2}{u}\;\underset{2i+3}{w}}_{\boldsymbol{\alpha}}\cdots$$

(e) $$\cdots\underbrace{\underset{2i-1}{s}\;\underset{2i}{t}\;\underset{2i+1}{s}\;\underset{2i+2}{u}\;\underset{2i+3}{t}}_{\boldsymbol{\alpha}}\cdots$$

(f) $$\cdots\underbrace{\underset{2i-1}{v}\;\underset{2i}{s}\;\underset{2i+1}{u}\;\underset{2i+2}{t}\;\underset{2i+3}{u}}_{\boldsymbol{\alpha}}\cdots$$

(g) $$\cdots\underbrace{\underset{2i-1}{v}\;\underset{2i}{s}\;\underset{2i+1}{u}\;\underset{2i+2}{t}\;\underset{2i+3}{w}}_{\boldsymbol{\alpha}}\cdots$$

(h) $$\cdots\underbrace{\underset{2i-1}{v}\;\underset{2i}{s}\;\underset{2i+1}{t}\;\underset{2i+2}{u}\;\underset{2i+3}{t}}_{\boldsymbol{\alpha}}\cdots$$

Proof. As in the previous proofs, we use the results from Proposition 3.14 for our case analysis, and again apply Proposition 3.6.

(a) $$\cdots\underbrace{\underset{2i-2}{\underline{v}}\;\underset{2i-1}{\underline{t}}\;\underset{2i}{\underline{s}}\;\underset{2i+1}{\underline{u}}\;\underset{2i+2}{\underline{t}}\;\underset{2i+3}{\underline{?}}\;\underset{2i+4}{\underline{w}}}_{\boldsymbol{\alpha}}\cdots \xRightarrow{3.6} \cdots\underbrace{\underset{2i-2}{\underline{v}}\;\underset{2i-1}{\underline{t}}\;\underset{2i}{\underline{s}}\;\underset{2i+1}{\underline{u}}\;\underset{2i+2}{\underline{t}}\;\underset{2i+3}{\underline{u}}\;\underset{2i+4}{\underline{w}}}_{\boldsymbol{\alpha}}\cdots$$

(b) $$\cdots\underbrace{\underset{2i-2}{\underline{v}}\;\underset{2i-1}{\underline{t}}\;\underset{2i}{\underline{s}}\;\underset{2i+1}{\underline{t}}\;\underset{2i+2}{\underline{u}}\;\underset{2i+3}{\underline{?}}\;\underset{2i+4}{\underline{?}}}_{\boldsymbol{\alpha}}\cdots \begin{cases} \boldsymbol{\alpha}_{[\![2i+4]\!]} = t \xRightarrow{3.6} \cdots\underbrace{\underset{2i-2}{\underline{v}}\;\underset{2i-1}{\underline{t}}\;\underset{2i}{\underline{s}}\;\underset{2i+1}{\underline{t}}\;\underset{2i+2}{\underline{u}}\;\underset{2i+3}{\underline{w}}\;\underset{2i+4}{\underline{t}}}_{\boldsymbol{\alpha}}\cdots \\ \boldsymbol{\alpha}_{[\![2i+4]\!]} = w \xRightarrow{3.6} \cdots\underbrace{\underset{2i-2}{\underline{v}}\;\underset{2i-1}{\underline{t}}\;\underset{2i}{\underline{s}}\;\underset{2i+1}{\underline{t}}\;\underset{2i+2}{\underline{u}}\;\underset{2i+3}{\underline{t}}\;\underset{2i+4}{\underline{w}}}_{\boldsymbol{\alpha}}\cdots \end{cases}$$

(c) $$\cdots\underbrace{\underset{2i-2}{\underline{v}}\;\underset{2i-1}{\underline{s}}\;\underset{2i}{\underline{t}}\;\underset{2i+1}{\underline{s}}\;\underset{2i+2}{\underline{u}}\;\underset{2i+3}{\underline{?}}\;\underset{2i+4}{\underline{?}}}_{\boldsymbol{\alpha}}\cdots \begin{cases} \boldsymbol{\alpha}_{[\![2i+4]\!]} = t \xRightarrow{3.6} \cdots\underbrace{\underset{2i-2}{\underline{v}}\;\underset{2i-1}{\underline{s}}\;\underset{2i}{\underline{t}}\;\underset{2i+1}{\underline{s}}\;\underset{2i+2}{\underline{u}}\;\underset{2i+3}{\underline{w}}\;\underset{2i+4}{\underline{t}}}_{\boldsymbol{\alpha}}\cdots \\ \boldsymbol{\alpha}_{[\![2i+4]\!]} = w \xRightarrow{3.6} \cdots\underbrace{\underset{2i-2}{\underline{v}}\;\underset{2i-1}{\underline{s}}\;\underset{2i}{\underline{t}}\;\underset{2i+1}{\underline{s}}\;\underset{2i+2}{\underline{u}}\;\underset{2i+3}{\underline{t}}\;\underset{2i+4}{\underline{w}}}_{\boldsymbol{\alpha}}\cdots \end{cases}$$

(d) $$\cdots\underbrace{\underset{2i-2}{\underline{t}}\;\underset{2i-1}{\underline{v}}\;\underset{2i}{\underline{s}}\;\underset{2i+1}{\underline{u}}\;\underset{2i+2}{\underline{t}}\;\underset{2i+3}{\underline{?}}\;\underset{2i+4}{\underline{w}}}_{\boldsymbol{\alpha}}\cdots \xRightarrow{3.6} \cdots\underbrace{\underset{2i-2}{\underline{t}}\;\underset{2i-1}{\underline{v}}\;\underset{2i}{\underline{s}}\;\underset{2i+1}{\underline{u}}\;\underset{2i+2}{\underline{t}}\;\underset{2i+3}{\underline{u}}\;\underset{2i+4}{\underline{w}}}_{\boldsymbol{\alpha}}\cdots$$

(e) $$\cdots\underbrace{\underset{2i-2}{\underline{t}}\;\underset{2i-1}{\underline{v}}\;\underset{2i}{\underline{s}}\;\underset{2i+1}{\underline{t}}\;\underset{2i+2}{\underline{u}}\;\underset{2i+3}{\underline{?}}\;\underset{2i+4}{\underline{?}}}_{\boldsymbol{\alpha}}\cdots \begin{cases} \boldsymbol{\alpha}_{[\![2i+4]\!]} = t \xRightarrow{3.6} \cdots\underbrace{\underset{2i-2}{\underline{t}}\;\underset{2i-1}{\underline{v}}\;\underset{2i}{\underline{s}}\;\underset{2i+1}{\underline{t}}\;\underset{2i+2}{\underline{u}}\;\underset{2i+3}{\underline{w}}\;\underset{2i+4}{\underline{t}}}_{\boldsymbol{\alpha}}\cdots \\ \boldsymbol{\alpha}_{[\![2i+4]\!]} = w \xRightarrow{3.6} \cdots\underbrace{\underset{2i-2}{\underline{t}}\;\underset{2i-1}{\underline{v}}\;\underset{2i}{\underline{s}}\;\underset{2i+1}{\underline{t}}\;\underset{2i+2}{\underline{u}}\;\underset{2i+3}{\underline{t}}\;\underset{2i+4}{\underline{w}}}_{\boldsymbol{\alpha}}\cdots \end{cases}$$

□

The following corollary is an immediate consequence of the previous propositions. Notice that this result is referring to braid shadows in $\boldsymbol{\alpha}$ as opposed to $[\boldsymbol{\alpha}]$.

Corollary 3.20. Suppose (W, S) is type Λ and let $\boldsymbol{\alpha}$ be a link of rank at least two. If $[\![2i-1, 2i+1]\!]$, $[\![2i+1, 2i+3]\!]$ are elements of $\mathcal{S}(\boldsymbol{\alpha})$, then $\boldsymbol{\alpha}_{[\![2i-1,2i+3]\!]} = tstut$, where the relationship among s, t, u is depicted in Figure 3.1.

To summarize, Corollary 3.15 tells us the possible forms a link can have in the location of a braid shadow in a braid chain. Propositions 3.16–3.19 indicate the structure of a link in the location of any two overlapping braid shadows that occur in the middle of a braid chain. Taken together, the results of this chapter completely characterize the local structure of a link.

Chapter 4

Structure of braid graphs

The goal of this chapter is to provide an alternative proof that braid graphs for links in Coxeter systems of type Λ are partial cubes (see Proposition 4.4), as well as to provide evidence that braid graphs are median and correspond to the underlying graph of the Hasse diagram for a distributive lattice.

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two braid equivalent reduced expressions in a Coxeter system of type Λ. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related via a single braid move, then this braid move occurred in a specific braid shadow. In this case, we can label the edge connecting $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in $\mathcal{B}(\boldsymbol{\alpha})$ with j, where $[\![2j-1, 2j+1]\!]$ is the corresponding braid shadow. We will write $b^j(\boldsymbol{\alpha}) = \boldsymbol{\beta}$ to represent that applying the braid move in the jth shadow of $\boldsymbol{\alpha}$ yields $\boldsymbol{\beta}$. We may denote a minimal sequence of braid moves from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ via $b_1^{j_1}, b_2^{j_2}, ..., b_k^{j_k}$, where $b_i^{j_i}$ is the ith braid move and this move occurs in the j_ith braid shadow, namely positions $[\![2j_i-1, 2j_i+1]\!]$. Note that a minimal braid sequence corresponds to the optimal path from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ in $\mathcal{B}(\boldsymbol{\alpha})$ along the edges labeled with $j_1, j_2, \ldots, j_k$.

Theorem 4.1. Suppose (W, S) is type Λ and let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two braid equivalent links of rank at least one. If $b_1^{j_1}, b_2^{j_2}, ..., b_k^{j_k}$ is a minimal braid sequence from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$, then each j_i appears exactly once.

Proof. For sake of a contradiction assume there exists $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{R}(w)$ such that there exists a minimal braid sequence $b_1^{j_1}, b_2^{j_2}, ..., b_k^{j_k}$ from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ where $j_i = j_{i^*}$ for some $i \neq i^*$. If such a pair exists, we may assume there exists $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ together with a minimal braid sequence $b_1^{j_1}, b_2^{j_2}, ..., b_k^{j_k}$ such that $j_1 = j_k$ and this is the only repeated braid shadow in the sequence. Choose $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that k is minimal among all such pairs.

There are two situations we must consider. First suppose j_1 and j_2 are disjoint braid shadows. Certainly, $b_1^{j_1}$ and $b_2^{j_2}$ can be applied in either order, so that both $b_1^{j_1}, b_2^{j_2}, \cdots, b_k^{j_1}$ and $b_2^{j_2}, b_1^{j_1}, \cdots, b_k^{j_1}$ yield $\boldsymbol{\beta}$. This contradicts the minimality of k.

Now suppose j_1 and j_2 are overlapping braid shadows. Note that j_1 corresponds to the braid shadow in positions $[\![2j_1-1, 2j_1+1]\!]$. Without loss of generality, let j_2 correspond to the braid shadow in positions $[\![2j_1+1, 2j_1+3]\!]$. By Corollary 3.20, it must be the case that $b_1^{j_1}(\boldsymbol{\alpha})_{[\![2j_1-1,2j_1+3]\!]} = tstut$, where the relationships among s, t, u are depicted in Figure 3.1. This implies that $\boldsymbol{\alpha}_{[\![2j_1-1,2j_1+3]\!]} = stsut$. Since j_1 and j_2 are braid shadows that are distinct from $j_3, \ldots, j_{k-1}$, as we continue to apply braid moves $b_3^{j_3}, \ldots, b_{k-1}^{j_{k-1}}$, we will not change positions $2j_1, 2j_1+1$, and $2j_1+2$. But by assumption, we should apply $b_k^{j_1}$ as our final braid move. However, a braid move will not be available to apply in braid shadow j_1 on the final step. This is a contradiction since the braid shadow j_1 was the only repeated braid shadow in the sequence. □

It follows from Theorem 4.1 that if $\boldsymbol{\alpha}$ is a link of rank r, then $\operatorname{diam}(\mathcal{B}(\boldsymbol{\alpha})) \leq r$. This also follows from the fact that $\mathcal{B}(\boldsymbol{\alpha})$ may be isometrically embedded in Q_r (see Proposition 4.4). The following theorem states that every minimal braid sequence from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ uses the same set of braid shadows.

Theorem 4.2. Suppose (W, S) is type Λ and let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two braid equivalent links. If $b_1^{j_1}, b_2^{j_2}, \ldots, b_k^{j_k}$ and $b_1^{l_1}, b_2^{l_2}, \ldots, b_k^{l_k}$ are minimal braid sequences from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$, then $\{j_1, \ldots, j_k\} = \{l_1, \ldots, l_k\}$.

Proof. Assume $b_1^{j_1}, b_2^{j_2}, \ldots, b_k^{j_k}$ and $b_1^{l_1}, b_2^{l_2}, \ldots, b_k^{l_k}$ are minimal braid sequences from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$. Let $j_i \in \{j_1, \ldots, j_k\}$. By Proposition 3.3, there exists $s, t \in S$ with $m(s,t) = 3$ such that $\operatorname{supp}_{[\![2j_i]\!]}([\boldsymbol{\alpha}]) = \{s, t\}$. Without loss of generality, assume $\boldsymbol{\alpha}_{[\![2j_i]\!]} = s$. By Theorem 4.1, we have $\operatorname{sig}_{2j_i}(\boldsymbol{\alpha}) \neq \operatorname{sig}_{2j_i}(\boldsymbol{\beta})$. Then it must be the case that there exists $1 \leq m \leq k$ such that $l_m = j_i$. We can conclude that $\{j_1, \ldots j_k\} = \{l_i, \ldots, l_k\}$. □

It follows from the previous theorems that every shortest path between a fixed pair of vertices in a braid graph utilizes edges with the same collection of labels, and moreover each label in this collection occurs exactly once. The next result follows from Theorems 4.1 and 4.2.

Corollary 4.3. If (W, S) is type Λ and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two braid equivalent links, then $d(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \Delta(\operatorname{sig}(\boldsymbol{\alpha}), \operatorname{sig}(\boldsymbol{\beta}))$.

In light of Proposition 3.7, we can define the following. Assume that (W, S) is a Coxeter system of type Λ and let $\boldsymbol{\alpha}$ be a link with rank $r \geq 1$. Define $\boldsymbol{\Phi}_{\boldsymbol{\alpha}} : [\boldsymbol{\alpha}] \to \{0,1\}^r$ via $\boldsymbol{\Phi}_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = a_1 a_2 \cdots a_r$, where

$$a_k = \begin{cases} 0, & \operatorname{sig}_k(\boldsymbol{\beta}) = \operatorname{sig}_k(\boldsymbol{\alpha}) \\ 1, & \text{otherwise.} \end{cases}$$

Note that the definition of the map $\mathbf{\Phi}_{\boldsymbol{\alpha}}$ above depends on the representative $\boldsymbol{\alpha}$. Choosing a different representative will necessarily result in a different mapping, however any two such embeddings differ only by an automorphism of the hypercube [2].

The following proposition is one of the main results from [2]. Later, we will provide an alternate proof of this fact.

Proposition 4.4. Suppose (W,S) is type Λ and let $\boldsymbol{\alpha}$ be a link of rank $r \geq 1$, then the map $\mathbf{\Phi}_{\boldsymbol{\alpha}}$ is an isometric embedding of $\mathcal{B}(\boldsymbol{\alpha})$ into Q_r. In particular, the braid graph for a link is a partial cube and $\dim_I(\mathcal{B}(\boldsymbol{\alpha})) \leq r$.

Example 4.5. Consider the reduced expressions $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_5$ given in Example 2.5. The braid graph $\mathcal{B}(\boldsymbol{\beta}_1)$ was shown in Figure 2.3(c). We know $\boldsymbol{\beta}_1$ is a link by Example 2.9 and $\operatorname{rank}(\boldsymbol{\beta}_1) = 3$. By Proposition 4.4, there are at least five distinct embeddings of $\mathcal{B}(\boldsymbol{\beta}_1)$ into Q_3, one for each representative of $[\boldsymbol{\beta}_1]$. One possible embedding of $\boldsymbol{\beta}_4$ is shown in Figure 4.1. It is easily seen that $\dim_I(\mathcal{B}(\boldsymbol{\beta}_1)) = 3 = \operatorname{rank}(\boldsymbol{\beta}_1)$.

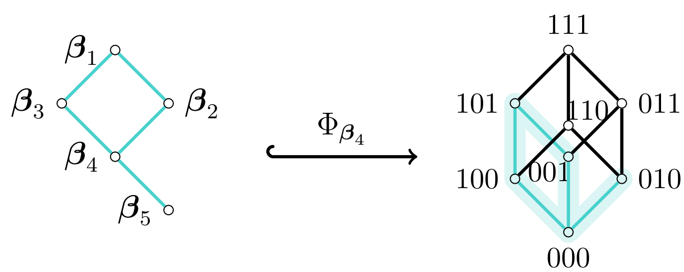


Figure 4.1: An induced embedding of $\mathcal{B}(\boldsymbol{\beta}_4)$ into Q_3 as in Example 4.5.

The next result follows from Proposition 4.4 together with Propositions 1.7 and 2.10 and also appeared in [2].

Proposition 4.6. If (W,S) is type Λ and $\boldsymbol{\alpha}$ is a reduced expression with link factorization $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 \mid \boldsymbol{\alpha}_2 \mid \cdots \mid \boldsymbol{\alpha}_k$, then $\mathcal{B}(\boldsymbol{\alpha})$ is a partial cube such that

$$\dim_I(\mathcal{B}(\boldsymbol{\alpha})) \leq \sum_{i=1}^{k} \operatorname{rank}(\boldsymbol{\alpha}_i).$$

In [2], the authors conjecture that if $\boldsymbol{\alpha}$ is a link in a Coxeter system of type Λ, then $\dim_I(\mathcal{B}(\boldsymbol{\alpha})) = \operatorname{rank}(\boldsymbol{\alpha})$. The following theorem leads to an alternate proof of Proposition 4.4. One consequence of our approach is that it verifies that the isometric dimension of $\mathcal{B}(\boldsymbol{\alpha})$ is equal to the rank of $\boldsymbol{\alpha}$.

First define $\overline{\mathrm{sig}}_i(\boldsymbol{\alpha}) := \{\boldsymbol{x} \mid \mathrm{sig}_i(\boldsymbol{x}) = \mathrm{sig}_i(\boldsymbol{\alpha})\}$. That is, $\overline{\mathrm{sig}}_i(\boldsymbol{\alpha})$ is the set of reduced expressions that are braid equivalent to $\boldsymbol{\alpha}$ and have signatures that agree with $\boldsymbol{\alpha}$ in the ith entry.

Theorem 4.7. Suppose (W,S) is type Λ and $\boldsymbol{\alpha}$ is a link of rank at least one. If $\{\boldsymbol{\alpha},\boldsymbol{\beta}\}$ is an edge in $\mathcal{B}(\boldsymbol{\alpha})$, then there exists a unique i such that $\mathrm{sig}_i(\boldsymbol{\alpha}) \neq \mathrm{sig}_i(\boldsymbol{\beta})$ and $W_{\boldsymbol{\alpha\beta}} = \overline{\mathrm{sig}}_i(\boldsymbol{\alpha})$.

Proof. Let $\boldsymbol{\alpha}$ be a link of rank $r \geq 1$ and let edge $\{\boldsymbol{\alpha},\boldsymbol{\beta}\}$ in $\mathcal{B}(\boldsymbol{\alpha})$. By Corollary 4.3, there exists a unique i such that $\mathrm{sig}_i(\boldsymbol{\alpha}) \neq \mathrm{sig}_i(\boldsymbol{\beta})$. Now, we will show that $W_{\boldsymbol{\alpha\beta}} = \overline{\mathrm{sig}}_i(\boldsymbol{\alpha})$.

For the forward containment, suppose $\boldsymbol{x} \in W_{\boldsymbol{\alpha\beta}}$. Then $d(\boldsymbol{x},\boldsymbol{\alpha}) < d(\boldsymbol{x},\boldsymbol{\beta})$. This implies that $\Delta(\mathrm{sig}(\boldsymbol{x}),\mathrm{sig}(\boldsymbol{\alpha})) < \Delta(\mathrm{sig}(\boldsymbol{x}),\mathrm{sig}(\boldsymbol{\beta}))$ by Corollary 4.3. For sake of a contradiction, suppose $\mathrm{sig}_i(\boldsymbol{x}) \neq \mathrm{sig}_i(\boldsymbol{\alpha})$. This implies that $\mathrm{sig}_i(\boldsymbol{x}) = \mathrm{sig}_i(\boldsymbol{\beta})$. Again by Corollary 4.3 $d(\boldsymbol{x},\boldsymbol{\beta}) = \Delta(\mathrm{sig}(\boldsymbol{x}),\mathrm{sig}(\boldsymbol{\beta}))$ while $d(\boldsymbol{x},\boldsymbol{\alpha}) = \Delta(\mathrm{sig}(\boldsymbol{x}),\mathrm{sig}(\boldsymbol{\alpha})) = \Delta(\mathrm{sig}(\boldsymbol{x}),\mathrm{sig}(\boldsymbol{\beta})) + 1 = d(\boldsymbol{x},\boldsymbol{\beta}) + 1$. Then $d(\boldsymbol{x},\boldsymbol{\beta}) < d(\boldsymbol{x},\boldsymbol{\alpha})$, which is a contradiction. Hence $\mathrm{sig}_i(\boldsymbol{x}) = \mathrm{sig}_i(\boldsymbol{\alpha})$ and $W_{\boldsymbol{\alpha\beta}} \subseteq \overline{\mathrm{sig}}_i(\boldsymbol{\alpha})$.

For the reverse containment, suppose $\boldsymbol{x} \in \overline{\mathrm{sig}}_i(\boldsymbol{\alpha})$. Then $d(\boldsymbol{x},\boldsymbol{\alpha}) = \Delta(\mathrm{sig}(\boldsymbol{x}),\mathrm{sig}(\boldsymbol{\alpha}))$ while $d(\boldsymbol{x},\boldsymbol{\beta}) = \Delta(\mathrm{sig}(\boldsymbol{x}),\mathrm{sig}(\boldsymbol{\beta})) = \Delta(\mathrm{sig}(\boldsymbol{x}),\mathrm{sig}(\boldsymbol{\alpha})) + 1 = d(\boldsymbol{x},\boldsymbol{\alpha}) + 1$. This implies that $d(\boldsymbol{x},\boldsymbol{\alpha}) < d(\boldsymbol{x},\boldsymbol{\beta})$ and hence $\boldsymbol{x} \in W_{\boldsymbol{\alpha\beta}}$. Thus $\overline{\mathrm{sig}}_i(\boldsymbol{\alpha}) \subseteq W_{\boldsymbol{\alpha\beta}}$.

Therefore $W_{\boldsymbol{\alpha\beta}} = \overline{\mathrm{sig}}_i(\boldsymbol{\alpha})$. □

If follows from Theorem 4.7 that if $\{\boldsymbol{\alpha},\boldsymbol{\beta}\}$ is an edge in $\mathcal{B}(\boldsymbol{\alpha})$ with $\mathrm{sig}_i(\boldsymbol{\alpha}) \neq \mathrm{sig}_i(\boldsymbol{\beta})$, then all of the edges that join the semicubes $W_{\boldsymbol{\alpha\beta}}$ and $W_{\boldsymbol{\beta\alpha}}$ correspond to the braid move involving the ith braid shadow. That is, $F_{\boldsymbol{\alpha\beta}}$ is the set of all edges labeled by i. So certainly $\boldsymbol{\theta}$ is transitive and thus an equivalence relation.

Example 4.8. Consider the link $\boldsymbol{\alpha} = 343123243$ in the Coxeter system of type D_4 and let $\boldsymbol{\beta} = 343132343$. Notice that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related by a single braid move in the third braid shadow in $[\boldsymbol{\alpha}]$. The semicubes $W_{\boldsymbol{\alpha\beta}}$ and $W_{\boldsymbol{\beta\alpha}}$ have been highlighted in orange and magenta, respectively, in Figure 4.2. We see that

$$\overline{\mathrm{sig}}_3(\boldsymbol{\alpha}) = \{343123243, 434123243\}$$

while

$$\overline{\mathrm{sig}}_3(\boldsymbol{\beta}) = \{343132343, 434132343, 434132434, 343132434, 341312434, 341312343\}.$$

As we expected from Theorem 4.7, $\overline{\mathrm{sig}}_3(\boldsymbol{\alpha}) = W_{\boldsymbol{\alpha\beta}}$ and $\overline{\mathrm{sig}}_3(\boldsymbol{\beta}) = W_{\boldsymbol{\beta\alpha}}$. We have highlighted the third signature position in each reduced expression in orange and magenta, respectively, to aid the reader.

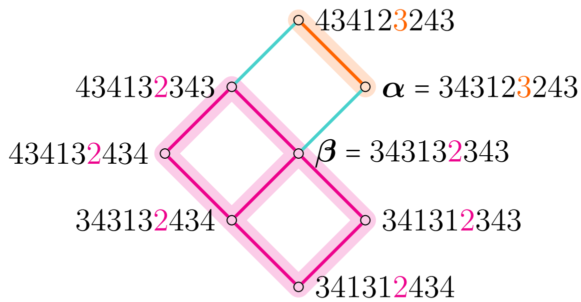


Figure 4.2: Example of Theorem 4.7 for a link in a Coxeter system of type D_4.

Recall from Proposition 2.6 that $\mathcal{B}(\boldsymbol{\alpha})$ is bipartite. Therefore by Proposition 1.11, $\mathcal{B}(\boldsymbol{\alpha})$ is a partial cube. This provides an alternate proof of Proposition 4.4. One consequence of our alternate proof is that the isometric dimension of $\mathcal{B}(\boldsymbol{\alpha})$ is equal to the number of equivalence classes of edges. This yields the following corollary, which settles the conjecture about the isometric dimension of $\mathcal{B}(\boldsymbol{\alpha})$ from [2].

Corollary 4.9. If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link, then $\mathcal{B}(\boldsymbol{\alpha})$ is a partial cube with $\dim_I(\mathcal{B}(\boldsymbol{\alpha})) = \operatorname{rank}(\boldsymbol{\alpha})$.

The next result follows from the previous corollary together with Propositions 1.7 and 2.10.

Corollary 4.10. If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a reduced expression with link factorization $\boldsymbol{\alpha}_1 \mid \boldsymbol{\alpha}_2 \mid \cdots \mid \boldsymbol{\alpha}_k$, then $\mathcal{B}(\boldsymbol{\alpha})$ is a partial cube with

$$\dim_I(\mathcal{B}(\boldsymbol{\alpha})) = \sum_{i=1}^{k} \operatorname{rank}(\boldsymbol{\alpha}_i).$$

We make the following conjecture regarding the diameter of a braid graph for a link.

Conjecture 4.11. If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link, then $\operatorname{diam}(\mathcal{B}(\boldsymbol{\alpha})) = \operatorname{rank}(\boldsymbol{\alpha})$.

If the conjecture is true, then we would have the following.

(1) If $\boldsymbol{\alpha}$ is a reduced expression with link factorization $\boldsymbol{\alpha}_1 \mid \boldsymbol{\alpha}_2 \mid \cdots \mid \boldsymbol{\alpha}_k$, then

$$\operatorname{diam}(\mathcal{B}(\boldsymbol{\alpha})) = \sum_{i=1}^{k} \operatorname{rank}(\boldsymbol{\alpha}_i).$$

(2) There exists reduced expressions $\boldsymbol{\mu}, \boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$ such that $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are diametrical.[1]

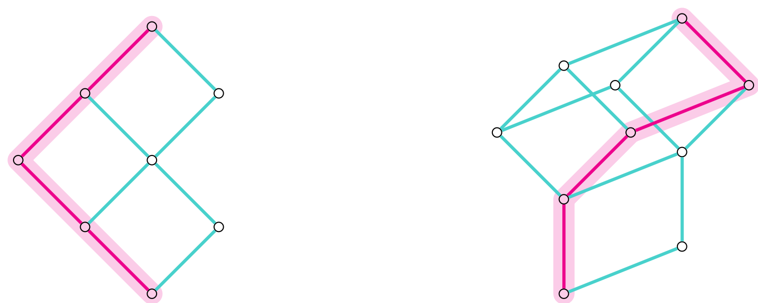
[1] Dana: Of course there is a pair that are diametrical. The uniqueness claim below is what matters.

In fact, we conjecture that the pair $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are unique.

Conjecture 4.12. If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link, then there exists a unique pair $\boldsymbol{\mu}, \boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$ such that $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are diametrical.

The previous conjecture is certainly not true if $\boldsymbol{\alpha}$ is not a link. For example, see Figure 4.3(a).

Example 4.13. The graph in Figure 4.3(a) is the braid graph for the link $\boldsymbol{\alpha} = 343132343$ in the Coxeter system of type D_4 while the graph in Figure 4.3(b) is the braid graph for the reduced expression $\boldsymbol{\beta} = 3132343676$ in the Coxeter system of type D_7. We see that $\text{rank}(\boldsymbol{\alpha}) = 4$. Notice that there is a unique pair that determines the diameter in $\mathcal{B}(\boldsymbol{\alpha})$. On the other hand, the link factorization of $\boldsymbol{\beta}$ is given by $3132343 \mid 676$, so $\text{rank}(\boldsymbol{\beta}) = 3 + 1 = 4$ by Proposition 2.10. In both graphs, the diameter agrees with the rank. We have highlighted paths that yield the diameter in magenta in Figure 4.3.



(a) (b)

Figure 4.3: Examples of braid graphs where diameter equals rank of the reduced expression.

Now we turn our attention towards intervals in braid graphs. The following definition is required for the next result. We define

$$\overline{\text{sig}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \coloneqq \{\boldsymbol{x} \in [\boldsymbol{\alpha}] \mid \text{sig}_i(\boldsymbol{x}) = \text{sig}_i(\boldsymbol{\alpha}) \text{ whenever } \text{sig}_i(\boldsymbol{\alpha}) = \text{sig}_i(\boldsymbol{\beta})\}.$$

That is, $\overline{\text{sig}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the set of reduced expressions whose signature equals the signature of $\boldsymbol{\alpha}$ in the ith entry when the signatures of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are equal in the ith entry.

Theorem 4.14. If (W, S) is type Λ and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are braid equivalent links, then $I(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \overline{\text{sig}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Proof. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be braid equivalent links. It is clear from Theorem 4.2 that $I(\boldsymbol{\alpha},\boldsymbol{\beta}) \subseteq \overline{\mathrm{sig}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ since every edge on a shortest path between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ corresponds to changing a single signature entry that differs between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ into a matching generator and every shortest path between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ must use the same set of braid shadows. For the reverse containment, suppose $\boldsymbol{x} \in \overline{\mathrm{sig}}(\boldsymbol{\alpha},\boldsymbol{\beta})$. Then $\mathrm{sig}_i(\boldsymbol{x}) = \mathrm{sig}_i(\boldsymbol{\alpha})$ whenever $\mathrm{sig}_i(\boldsymbol{\alpha}) = \mathrm{sig}_i(\boldsymbol{\beta})$. We know there exists an optimal path from $\boldsymbol{\alpha}$ to $\boldsymbol{x}$ consisting of $\Delta(\mathrm{sig}(\boldsymbol{\alpha}),\mathrm{sig}(\boldsymbol{x}))$ many edges and there exists an optimal path from $\boldsymbol{x}$ to $\boldsymbol{\beta}$ consisting of $\Delta(\mathrm{sig}(\boldsymbol{x}),\mathrm{sig}(\boldsymbol{\beta}))$ many edges by Corollary 4.3. So, there exists a path from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ passing through $\boldsymbol{x}$ consisting of $\Delta(\mathrm{sig}(\boldsymbol{\alpha}),\mathrm{sig}(\boldsymbol{x})) + \Delta(\mathrm{sig}(\boldsymbol{x}),\mathrm{sig}(\boldsymbol{\beta}))$ many edges. Certainly, $\Delta(\mathrm{sig}(\boldsymbol{\alpha}),\mathrm{sig}(\boldsymbol{x})) + \Delta(\mathrm{sig}(\boldsymbol{x}),\mathrm{sig}(\boldsymbol{\beta})) \geq \Delta(\mathrm{sig}(\boldsymbol{\alpha}),\mathrm{sig}(\boldsymbol{\beta}))$. We must show this path is optimal by showing $\Delta(\mathrm{sig}(\boldsymbol{\alpha}),\mathrm{sig}(\boldsymbol{x})) + \Delta(\mathrm{sig}(\boldsymbol{x}),\mathrm{sig}(\boldsymbol{\beta})) = \Delta(\mathrm{sig}(\boldsymbol{\alpha}),\mathrm{sig}(\boldsymbol{\beta}))$.

On each step from $\boldsymbol{\alpha}$ to $\boldsymbol{x}$, one position in the signature of $\boldsymbol{\alpha}$ that differs from the signature of $\boldsymbol{\beta}$ will change to match the signature of $\boldsymbol{\beta}$. There are only two options for the generators in each position of the signature from Proposition 3.3, so each edge from $\boldsymbol{\alpha}$ towards $\boldsymbol{x}$ will change a position in the signature of the current reduced expression to match the signature of $\boldsymbol{\beta}$. Since $\boldsymbol{x} \in \overline{\mathrm{sig}}(\boldsymbol{\alpha},\boldsymbol{\beta})$, the edges from $\boldsymbol{\alpha}$ to $\boldsymbol{x}$ will only change the positions in the signature where $\boldsymbol{\alpha}$ and $\boldsymbol{x}$ differ, so these edges will not affect the positions in the signature of $\boldsymbol{x}$ that differ from the signature of $\boldsymbol{\beta}$ while agreeing with the signature of $\boldsymbol{\alpha}$. Similarly, on each step from $\boldsymbol{x}$ to $\boldsymbol{\beta}$, one position in the signature of the current reduced expression will change to match a position in the signature of $\boldsymbol{\beta}$. Since $\mathrm{sig}_i(\boldsymbol{x}) = \mathrm{sig}_i(\boldsymbol{\alpha})$ whenever $\mathrm{sig}_i(\boldsymbol{\alpha}) = \mathrm{sig}_i(\boldsymbol{\beta})$, the optimal paths from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ do not share edges with the same label. So, $\Delta(\mathrm{sig}(\boldsymbol{\alpha}),\mathrm{sig}(\boldsymbol{x})) + \Delta(\mathrm{sig}(\boldsymbol{x}),\mathrm{sig}(\boldsymbol{\beta})) = \Delta(\mathrm{sig}(\boldsymbol{\alpha}),\mathrm{sig}(\boldsymbol{\beta}))$. Thus $\boldsymbol{x} \in I(\boldsymbol{\alpha},\boldsymbol{\beta})$. □

The previous theorem tells us that the reduced expressions on any shortest path between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ also share the same signature positions[2] that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ share. To expand on this result we introduce the following terminology.

Let $\boldsymbol{\alpha},\boldsymbol{\beta}$, and $\boldsymbol{\sigma}$ be braid equivalent links of rank $r \geq 1$. We define the ith *majority* of $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma}$ via

$$\mathrm{maj}_i(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma}) := \begin{cases} \mathrm{sig}_i(\boldsymbol{\alpha}), & \text{if } \mathrm{sig}_i(\boldsymbol{\alpha}) = \mathrm{sig}_i(\boldsymbol{\beta}) \text{ or } \mathrm{sig}_i(\boldsymbol{\alpha}) = \mathrm{sig}_i(\boldsymbol{\sigma}) \\ \mathrm{sig}_i(\boldsymbol{\beta}), & \text{otherwise.} \end{cases}$$

That is, the ith majority of a triple of braid equivalent links is the generator shared by at least two of the signatures in the ith position. Next we define the *majority* of $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma}$ via

$$\mathrm{maj}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma}) := (\mathrm{maj}_1(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma}),\ldots,\mathrm{maj}_r(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma})).$$

This results in an ordered list of generators.

[2] Dana: I don't think the wording of this is quite right.

Proposition 4.15. Suppose (W,S) is type Λ. If $\boldsymbol{\alpha},\boldsymbol{\beta}$, and $\boldsymbol{\sigma}$ are braid equivalent links, then $\text{med}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma}) = \{\boldsymbol{x}$[3] $\mid \text{sig}_i(\boldsymbol{x}) = \text{maj}_i(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma})\}$.

Proof. Let $\boldsymbol{\alpha},\boldsymbol{\beta}$, and $\boldsymbol{\sigma}$ be braid equivalent links. Recall that

$$\text{med}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma}) = I(\boldsymbol{\alpha},\boldsymbol{\beta}) \cap I(\boldsymbol{\beta},\boldsymbol{\sigma}) \cap I(\boldsymbol{\sigma},\boldsymbol{\alpha}).$$

By Theorem 4.14, we have

$$\text{med}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma}) = \overline{\text{sig}}(\boldsymbol{\alpha},\boldsymbol{\beta}) \cap \overline{\text{sig}}(\boldsymbol{\beta},\boldsymbol{\sigma}) \cap \overline{\text{sig}}(\boldsymbol{\sigma},\boldsymbol{\alpha}).$$

This intersection represents the set of reduced expressions $\boldsymbol{x}$ that satisfy $\text{sig}_i(\boldsymbol{x}) = \text{sig}_i(\boldsymbol{\alpha})$ whenever $\text{sig}_i(\boldsymbol{\alpha}) = \text{sig}_i(\boldsymbol{\beta})$, $\text{sig}_i(\boldsymbol{x}) = \text{sig}_i(\boldsymbol{\beta})$ whenever $\text{sig}_i(\boldsymbol{\beta}) = \text{sig}_i(\boldsymbol{\sigma})$, and $\text{sig}_i(\boldsymbol{x}) = \text{sig}_i(\boldsymbol{\sigma})$ whenever $\text{sig}_i(\boldsymbol{\sigma}) = \text{sig}_i(\boldsymbol{\alpha})$. It follows that

$$\text{med}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma}) = \{\boldsymbol{x} \mid \text{sig}_i(\boldsymbol{x}) = \text{maj}_i(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma})\},$$

as expected. □

We conjecture the following.

Conjecture 4.16. If (W,S) is type Λ, then for braid equivalent links $\boldsymbol{\alpha},\boldsymbol{\beta}$, and $\boldsymbol{\sigma}$, there exists $\boldsymbol{\mu} \in [\boldsymbol{\alpha}]$ such that $\text{sig}(\boldsymbol{\mu}) = \text{maj}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma})$.

If this conjecture is true, then for braid equivalent links $\boldsymbol{\alpha},\boldsymbol{\beta}$, and $\boldsymbol{\sigma}$, $\text{med}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma})$ is the link $\boldsymbol{\mu}$ mentioned in Conjecture 4.16. That is, if Conjecture 4.16 is true, it would imply that for a link $\boldsymbol{\alpha}$, the braid graph $\mathcal{B}(\boldsymbol{\alpha})$ is median.

Conjecture 4.17. If (W,S) is type Λ and $\boldsymbol{\alpha}$ is a link, then $\mathcal{B}(\boldsymbol{\alpha})$ is median.

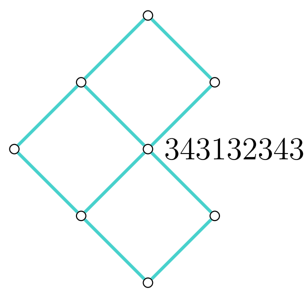
If $\mathcal{B}(\boldsymbol{\alpha})$ is median for a link $\boldsymbol{\alpha}$, it follows from Propositions 1.18 that the braid graph for any reduced expression is also median.

We outline a potential argument to prove Conjecture 4.16. Since $\text{maj}_i(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma})$ results in a single generator for each position i of the signature, by Proposition 4.15, $\text{med}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma})$ will either be the empty set or contain a single element of $[\boldsymbol{\alpha}]$ by Proposition 3.8. If $\text{med}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma}) \neq \varnothing$ (i.e., Conjecture 4.16), then $\mathcal{B}(\boldsymbol{\alpha})$ is median.

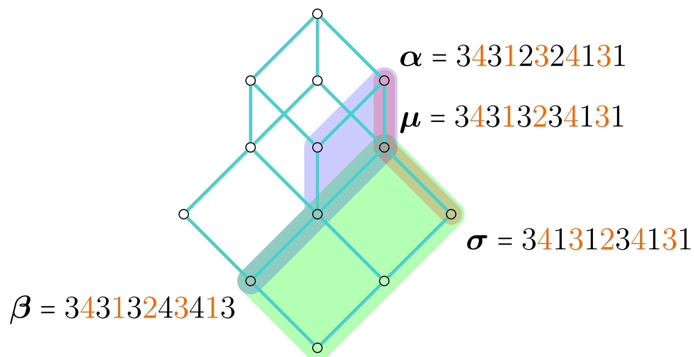
Example 4.18. Consider the link $\boldsymbol{\alpha} = 343132343$ in the Coxeter system of type D_4 whose braid graph is shown in Figure 4.4(a). We know by Example 1.15 that $\mathcal{B}(\boldsymbol{\alpha})$ is median.

[3] Dana: I think this should be $\boldsymbol{x} \in [\boldsymbol{\alpha}]$. Also, did we define med???

Example 4.19. In Figure 4.4(b), we see the braid graph for the link $\boldsymbol{\beta} = 34313243413$ in Coxeter system of type D_4. Given the reduced expressions $\boldsymbol{\alpha} = 34312324131$ and $\boldsymbol{\sigma} = 34131234131$ in $[\boldsymbol{\beta}]$, we can compute $\text{med}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})$ via Proposition 4.15. We see that $\text{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}) = (4, 1, 2, 4, 3)$, which corresponds to the signature of $\boldsymbol{\mu} = 34313234131$ in $[\boldsymbol{\beta}]$. This is in agreement with Conjecture 4.16. Moreover, $\mathcal{B}(\boldsymbol{\alpha})$ is median. We have colored the signatures of each reduced expression in the figure to aid the reader. We have also highlighted the intervals $I(\boldsymbol{\alpha}, \boldsymbol{\beta}), I(\boldsymbol{\beta}, \boldsymbol{\sigma})$, and $I(\boldsymbol{\sigma}, \boldsymbol{\alpha})$ in blue, green, and red, respectively, to depict the interval definition for $\text{med}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})$ and we see that $\text{med}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}) = \{\boldsymbol{\mu}\}$.



(a)



(b)

Figure 4.4: Examples of median braid graphs.

We now move on to constructing a poset whose Hasse diagram has $\mathcal{B}(\boldsymbol{\alpha})$ as its underlying graph. Let $\boldsymbol{\alpha}$ be a link of rank $r \geq 1$. By Theorem 4.1, $\text{diam}(\mathcal{B}(\boldsymbol{\alpha})) \leq r$. Identify a pair of vertices $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ of $\mathcal{B}(\boldsymbol{\alpha})$ such that $d(\boldsymbol{\mu}, \boldsymbol{\gamma}) = \text{diam}(\mathcal{B}(\boldsymbol{\alpha}))$. Note that by Corollary 4.3 $d(\boldsymbol{\mu}, \boldsymbol{\gamma}) = \Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\gamma}))$, so $\text{diam}(\mathcal{B}(\boldsymbol{\alpha})) = \Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\gamma}))$. Elect $\boldsymbol{\mu}$ to be the designated smallest vertex and define $\boldsymbol{\beta} \lessdot \boldsymbol{\sigma}$ if there exists a unique i such that $\text{sig}_i(\boldsymbol{\beta}) \neq \text{sig}_i(\boldsymbol{\sigma})$ and $\Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\beta})) + 1 = \Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\sigma}))$. Take $([\boldsymbol{\alpha}], \leq)$ to be the partial order induced by these covering relations. It is important to point out that the poset depends on the choice of $\boldsymbol{\mu}$. We will refer to both the poset and the Hasse diagram as $\mathcal{P}(\boldsymbol{\mu})$. It is easily seen that this relation is reflexive, antisymmetric, and transitive.

Certainly, if $\boldsymbol{\beta} \lessdot \boldsymbol{\sigma}$ in $\mathcal{P}(\boldsymbol{\mu})$, then $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ are adjacent in $\mathcal{B}(\boldsymbol{\alpha})$. Next we argue that if $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ are adjacent in $\mathcal{B}(\boldsymbol{\alpha})$, then either $\boldsymbol{\beta} \lessdot \boldsymbol{\sigma}$ or $\boldsymbol{\sigma} \lessdot \boldsymbol{\beta}$. Suppose there exists an edge $\boldsymbol{\beta}\boldsymbol{\sigma}$ labeled i in $\mathcal{B}(\boldsymbol{\alpha})$. This implies that $\text{sig}_i(\boldsymbol{\beta}) \neq \text{sig}_i(\boldsymbol{\sigma})$ and this is the only position of the signatures that differ. Then either $\text{sig}_i(\boldsymbol{\beta}) = \text{sig}_i(\boldsymbol{\mu})$ or $\text{sig}_i(\boldsymbol{\sigma}) = \text{sig}_i(\boldsymbol{\mu})$. Without loss of generality, assume $\text{sig}_i(\boldsymbol{\beta}) = \text{sig}_i(\boldsymbol{\mu})$. Then there exists an optimal path from $\boldsymbol{\mu}$ to $\boldsymbol{\sigma}$ that passes through $\boldsymbol{\beta}$ on the second to last step. So, we have $\Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\beta})) + 1 =$

$\Delta(\text{sig}(\boldsymbol{\mu}), (\boldsymbol{\sigma}))$ and hence $\boldsymbol{\beta} \lessdot \boldsymbol{\sigma}$. It follows that $\mathcal{B}(\boldsymbol{\alpha})$ is the underlying graph for the Hasse diagram of $\mathcal{P}(\boldsymbol{\mu})$. This yields the following result.

Theorem 4.20. Suppose (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link of rank r. Let $\boldsymbol{\mu}$ in $[\boldsymbol{\alpha}]$ be the designated smallest vertex. Then $\mathcal{P}(\boldsymbol{\mu})$ is ranked by $\Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\beta}))$ for $\boldsymbol{\beta}$ in $[\boldsymbol{\alpha}]$. Moreover, $\mathcal{B}(\boldsymbol{\alpha})$ is the underlying graph for the Hasse diagram of $\mathcal{P}(\boldsymbol{\mu})$.

Example 4.21. Consider the link $\boldsymbol{\alpha} = 3132343$ with $\text{rank}(\boldsymbol{\alpha}) = 3$ in the Coxeter system of type D_4. The braid graphs $\mathcal{B}(\boldsymbol{\alpha})$ is depicted in Figure 4.5. It is easily seen that $\text{diam}(\mathcal{B}(\boldsymbol{\alpha})) = 3$ and we can take $\boldsymbol{\mu} = 3123243$ to be the designated vertex. Then the graph given in Figure 4.5 is also the underlying graph of the Hasse diagram for $\mathcal{P}(\boldsymbol{\mu})$. Note that we could have also taken $\boldsymbol{\mu}$ to be 1312434, which would have resulted in an upside-down version of the graph in Figure 4.5 for the Hasse diagram of $\mathcal{P}(\boldsymbol{\mu})$.

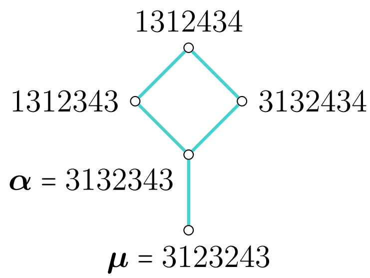


Figure 4.5: The Hasse diagram for $\mathcal{P}(\boldsymbol{\mu})$ in Example 4.21.

We conjecture the following.

Conjecture 4.22. If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link of rank r, then $\mathcal{B}(\boldsymbol{\alpha})$ is the underlying graph for the Hasse diagram of a distributive lattice.

We provide a few additional conjectures that may be useful in an attempt to prove Conjecture 4.22.

Conjecture 4.23. Suppose (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link of rank $r \geq 1$. Choose $\boldsymbol{\mu}$ to be the designated vertex of $\mathcal{P}(\boldsymbol{\mu})$. If $\text{sig}_i(\boldsymbol{\alpha}) = \text{sig}_i(\boldsymbol{\mu})$, then there exists $\boldsymbol{\beta} \in [\boldsymbol{\alpha}]$ such that $\boldsymbol{\alpha} \lessdot \boldsymbol{\beta}$.

Conjecture 4.23 states that we can always "go up" in the poset from an element that has at least one signature entry that agrees with the signature of $\boldsymbol{\mu}$.

If this conjecture is true, we would know the element $\boldsymbol{\gamma}$ that when paired with $\boldsymbol{\mu}$ yields the diameter of $\mathcal{B}(\boldsymbol{\alpha})$ would be the unique maximum element of $\mathcal{P}(\boldsymbol{\mu})$. In particular we would have, $\mathrm{diam}(\mathcal{B}(\boldsymbol{\alpha})) = \Delta(\mathrm{sig}(\boldsymbol{\mu}), \mathrm{sig}(\boldsymbol{\gamma})) = \mathrm{rank}(\boldsymbol{\alpha})$, which means that $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are diametrical. This result would settle Conjecture 4.11. On the other hand, if both Conjectures 4.17 and 4.23 are true, then Proposition 1.22 would imply Conjecture 4.22.

If there is a unique maximum element $\boldsymbol{\gamma}$ and unique minimum element $\boldsymbol{\mu}$, then another approach is to show $\mathcal{P}(\boldsymbol{\mu})$ is closed under meet and join and is isomorphic to a ring of sets. First we will describe an algorithm to find the meet of two braid equivalent reduced expressions in the poset. Let (W,S) be a Coxeter system of type Λ and $\boldsymbol{\alpha}$ be a link of rank $r \geq 1$. Let $\boldsymbol{\beta}, \boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$. To find $\boldsymbol{\beta} \wedge \boldsymbol{\sigma}$, first identify all positions in the signature such that $\mathrm{sig}_i(\boldsymbol{\beta}) = \mathrm{sig}_i(\boldsymbol{\sigma})$. We will not change any of these positions. Next identify all positions in the signature such that $\mathrm{sig}_i(\boldsymbol{\beta}) \neq \mathrm{sig}_i(\boldsymbol{\sigma})$. Then either $\mathrm{sig}_i(\boldsymbol{\beta}) = \mathrm{sig}_i(\boldsymbol{\mu})$ or $\mathrm{sig}_i(\boldsymbol{\sigma}) = \mathrm{sig}_i(\boldsymbol{\mu})$, so choose the generator in the differing position to match $\mathrm{sig}_i(\boldsymbol{\mu})$. Combining the positions in the signature that were common between $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ with the chosen generators matching $\mathrm{sig}(\boldsymbol{\mu})$, the result will be $\boldsymbol{\beta} \wedge \boldsymbol{\sigma}$. The issue we have is whether or not $\boldsymbol{\beta} \wedge \boldsymbol{\sigma}$ is in $[\boldsymbol{\alpha}]$.

We will follow a similar algorithm to find $\boldsymbol{\beta} \vee \boldsymbol{\sigma}$. First identify all positions in the signature such that $\mathrm{sig}_i(\boldsymbol{\beta}) = \mathrm{sig}_i(\boldsymbol{\sigma})$. We will not change any of these positions. Next identify all positions in the signature such that $\mathrm{sig}_i(\boldsymbol{\beta}) \neq \mathrm{sig}_i(\boldsymbol{\sigma})$. Then either $\mathrm{sig}_i(\boldsymbol{\beta}) = \mathrm{sig}_i(\boldsymbol{\gamma})$ in that position or $\mathrm{sig}_i(\boldsymbol{\sigma}) = \mathrm{sig}_i(\boldsymbol{\gamma})$, so choose the generator in the differing position to match $\mathrm{sig}_i(\boldsymbol{\gamma})$. Combining the positions in the signature that were common between $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ with the chosen generators matching $\mathrm{sig}(\boldsymbol{\gamma})$, the result will be $\boldsymbol{\beta} \vee \boldsymbol{\sigma}$. Again, the issue we have is whether or not $\boldsymbol{\beta} \vee \boldsymbol{\sigma}$ is in $[\boldsymbol{\alpha}]$.

If both $\boldsymbol{\beta} \wedge \boldsymbol{\sigma}$ and $\boldsymbol{\beta} \vee \boldsymbol{\sigma}$ are in $[\boldsymbol{\alpha}]$ and Conjecture 4.23 is true, then we are done. It is likely that the techniques used to prove Conjecture 4.23 would be helpful in resolving whether $\mathcal{P}(\boldsymbol{\mu})$ is closed under meet and join.

Example 4.24. Consider the braid equivalent links $\boldsymbol{\alpha} = 43412324313$ and $\boldsymbol{\beta} = 34131234131$ in a Coxeter system of type D_4. We will follow the algorithms described above to find $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \vee \boldsymbol{\beta}$. To find $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$, we see that $\mathrm{sig}_4(\boldsymbol{\alpha}) = \mathrm{sig}_4(\boldsymbol{\beta}) = 4$, so we will keep this signature position the same. Since $\mathrm{sig}_i(\boldsymbol{\alpha}) \neq \mathrm{sig}_i(\boldsymbol{\beta})$ for $i \in \{1,2,3,5\}$, we must choose the generators in these positions so that they match $\mathrm{sig}_i(\boldsymbol{\mu})$. Then $\mathrm{sig}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = (4,3,2,4,1)$, so we have $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = 34131234313$ which does occur in $[\boldsymbol{\alpha}]$.

Similarly, we will compute $\boldsymbol{\alpha} \vee \boldsymbol{\beta}$. We know that $\mathrm{sig}_4(\boldsymbol{\alpha}) = \mathrm{sig}_4(\boldsymbol{\beta}) = 4$ so we will keep this signature position the same. Since $\mathrm{sig}_i(\boldsymbol{\alpha}) \neq \mathrm{sig}_i(\boldsymbol{\beta})$ for $i \in \{1,2,3,5\}$, we must choose the the generators in these positions so that they match $\mathrm{sig}_i(\boldsymbol{\gamma})$. Then $\mathrm{sig}(\boldsymbol{\alpha} \vee \boldsymbol{\beta}) = (3,1,3,4,3)$, so we have $\boldsymbol{\alpha} \vee \boldsymbol{\beta} = 43412324131 = \boldsymbol{\gamma}$. We have highlighted the signatures of each reduced expression in orange in Figure 4.6 to aid the reader.

If our poset is closed under unique meet and join, then $\mathcal{P}(\boldsymbol{\mu})$ is a lattice. Lastly, we will define a potential bijection between $\mathcal{P}(\boldsymbol{\mu})$ and a certain ring of sets.

Assume $\boldsymbol{\alpha}$ is a link of rank $r \geq 1$ and $\boldsymbol{\mu}$ is the designated smallest vertex. Let $[r] =$

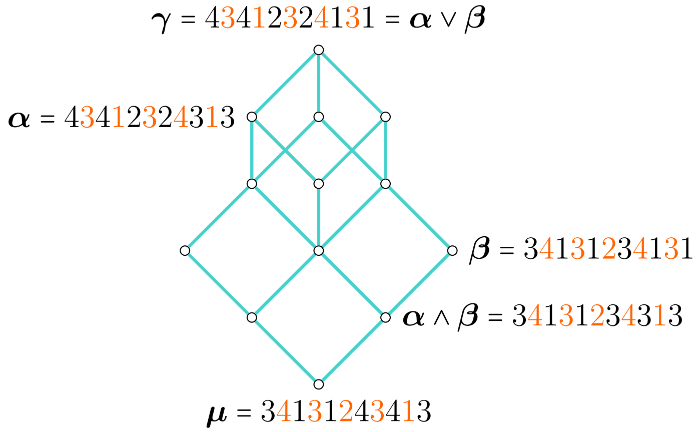


Figure 4.6: Example of meet and join for the two reduced expressions in Example 4.24.

$\{1, 2, \ldots, r\}$. Define $f : [\boldsymbol{\alpha}] \to [r]$ via $f(\boldsymbol{\beta}) = \{i \mid \text{sig}_i(\boldsymbol{\beta}) \neq \text{sig}_i(\boldsymbol{\mu})\}$. It is clear that $\boldsymbol{\beta} \lessdot \boldsymbol{\sigma}$ if and only if $f(\boldsymbol{\beta}) \subseteq f(\boldsymbol{\sigma})$. Therefore $\mathcal{P}(\boldsymbol{\mu})$ is isomorphic to the image of f, which is ordered by inclusion. If $\mathcal{P}(\boldsymbol{\mu})$ is closed under meet and join, the image of f is a ring of sets, and hence $\mathcal{P}(\boldsymbol{\mu})$ would be isomorphic to a ring of sets. Proposition 1.23 (Birkhoff's Representation Theorem) would imply that $\mathcal{P}(\boldsymbol{\mu})$ is a distributive lattice, which would then imply Conjecture 4.22.

Example 4.25. Consider the link $\boldsymbol{\alpha}$ = 343132343 in the Coxeter system of type D_4. The braid graph for $\boldsymbol{\alpha}$ is given on the left in Figure 4.7. The image of the function f defined above is given on the right.

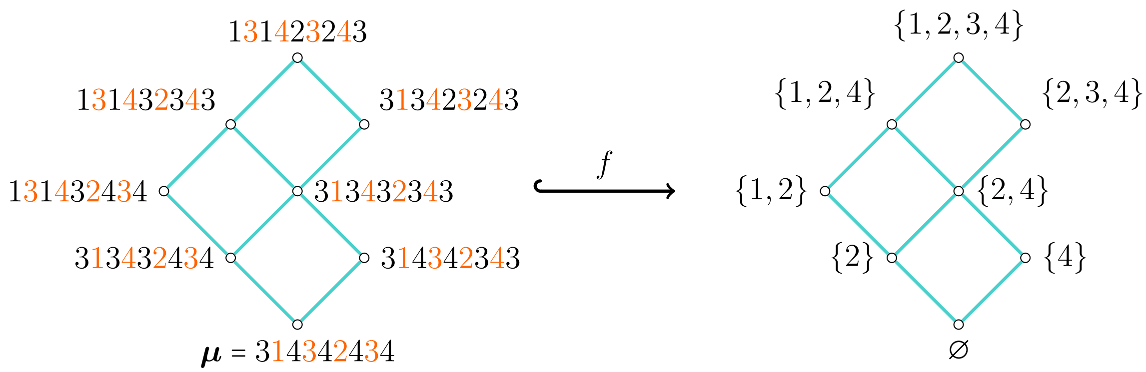


Figure 4.7: Example of bijection from $\mathcal{P}(\boldsymbol{\mu})$ to a ring of sets.

Chapter 5

Conclusion

In Chapter 1 we provided an overview of necessary terminology and results regarding simple connected graphs and partially ordered sets. A main focus of this chapter was on partial cubes and the related Djoković–Winkler relation, as well as median graphs and distributive lattices. In Chapter 2, we discussed Coxeter systems and braid graphs. Properties of reduced expressions and braid classes were introduced along with the notions of braid shadow, link, and braid chain. The fact that every reduced expression has a unique factorization in terms of links implies that every braid graph can be obtained from a box product of the braid graphs of the corresponding links. This factorization allowed us to focus the remainder of the thesis primarily on links, which then extends the results to any reduced expression.

Building off the results in [2], in Chapter 3 we provided explicit descriptions of the local structure of links in Coxeter systems of type Λ. In particular, we were able to precisely describe what links of rank 1 and 2 look like, as well as what the four leftmost and rightmost positions of a link look like. We also described the possible forms the the middle five positions of three overlapping braid shadows may take in a link. We concluded the chapter by describing the possible forms that the five middle positions of two overlapping braid shadows may take in a link. Taken together, the results of this chapter completely characterize the local structure of a link.

In our final chapter, we provided an alternate proof that braid graphs for links in Coxeter systems of type Λ are partial cubes. In order to do so, we proved two results (Theorems 4.1 and 4.2) concluding that any shortest path between two braid equivalent links uses the same collection of braid shadows where each braid shadow in the collection appears exactly once. One of the key results from [2] is that the signature uniquely determines a link. We proved that the differences in signature can be used to represent the distance between two braid equivalent links in a braid chain. Moreover, we applied the notion of semicubes from Chapter 1 and signatures of a link to conclude that the Djoković–Winkler relation is transitive for braid graphs, which results in the braid graph of a link being a partial cube. This allowed us to settle a conjecture from [2] stating that the isometric dimension of the braid graph for a link is equal to the rank of the link. We also showed that the interval between two braid equivalent links is equal to the set of braid equivalent expressions that share the entries of the signature that the two initial links share. Chapter 4 contained several conjectures and in most cases we outlined potential methods of attack.

We now summarize these conjectures here as a list of open problems. Suppose (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link of rank r.

- Conjecture 4.11: $\operatorname{diam}(\mathcal{B}(\boldsymbol{\alpha})) = \operatorname{rank}(\boldsymbol{\alpha})$.
- Conjecture 4.12: There exists a unique pair $\boldsymbol{\mu}, \boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$ such that $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are diametrical.
- Conjecture 4.16: For braid equivalent links $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\sigma}$, there exists $\boldsymbol{\mu} \in [\boldsymbol{\alpha}]$ such that $\operatorname{sig}(\boldsymbol{\mu}) = \operatorname{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})$.
- Conjecture 4.17: $\mathcal{B}(\boldsymbol{\alpha})$ is median.
- Conjecture 4.22: $\mathcal{B}(\boldsymbol{\alpha})$ is the underlying graph for the Hasse diagram of a distributive lattice.
- Conjecture 4.23: Let $\boldsymbol{\mu}$ to be the designated vertex of $\mathcal{P}(\boldsymbol{\mu})$. If $\operatorname{sig}_i(\boldsymbol{\alpha}) = \operatorname{sig}_i(\boldsymbol{\mu})$, then there exists $\boldsymbol{\beta} \in [\boldsymbol{\alpha}]$ such that $\boldsymbol{\alpha} \lessdot \boldsymbol{\beta}$. Moreover, $\mathcal{P}(\boldsymbol{\mu})$ has a unique maximum element and a unique minimal element, which are diametrical in $\mathcal{B}(\boldsymbol{\alpha})$.

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