STRUCTURE OF BRAID GRAPHS IN SIMPLY-LACED COXETER SYSTEMS

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ABSTRACT

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Any two reduced expressions for the same Coxeter group element are related by a sequence of commutation and braid moves. We say that two reduced expressions are braid equivalent if they are related via a sequence of braid moves, and the corresponding equivalence classes are called braid classes. Each braid class can be encoded in terms of a braid graph in a natural way. In this thesis, we study the structure of braid graphs in simply-laced Coxeter systems. In a recent paper, Awik et al. proved that every reduced expression has a unique factorization as a product of so-called links, which in turn induces a decomposition of the braid graph into a box product of the braid graphs for each link factor. For a special class of links, called Fibonacci links, they showed that the corresponding braid graph is isomorphic to a Fibonacci cube graph. In this thesis, we prove that every Fibonacci cube occurs as the braid graph for a link in any simply-laced triangle-free Coxeter System whose corresponding braid graph contains the Coxeter graph of the Coxeter system of type $\widetilde{D}_4$ as a subgraph.

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Chapter 1

Preliminaries

We begin by discussing the background on Coxeter systems as well as their underlying fundamental structure.

A *Coxeter system* is a pair (W, S) consisting of a finite set S of generating involutions and a group W, called a *Coxeter group*, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \rangle,$$

where e is the identity element, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s) \geq 2$ for $s \neq t$. It is worth noting that the elements of S are distinct as group elements and $m(s,t)$ is the order of st [10]. We refer to $m(s,t)$ as the *bond strength* between s and t.

Since s and t are elements of order 2, the relation $(st)^{m(s,t)} = e$ can be rewritten as

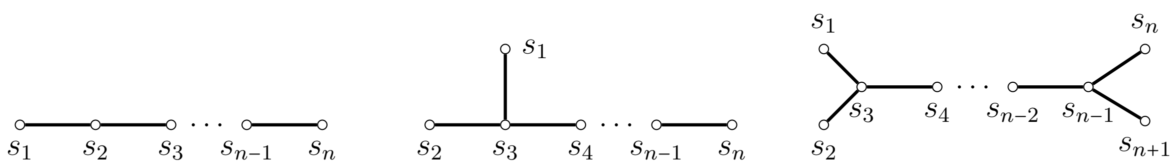
$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)} \tag{1.1}$$

with $m(s,t) \geq 2$ factors. When $m(s,t) = 2$, $st = ts$ is called a *commutation relation* and when $m(s,t) \geq 3$, the relation in (1.1) is called a *braid relation*. The replacement

$$\underbrace{sts\cdots}_{m(s,t)} \longmapsto \underbrace{tst\cdots}_{m(s,t)}$$

will be referred to as a *commutation move* if $m(s,t) = 2$ and a *braid move* if $m(s,t) \geq 3$. If $m(s,t) \leq 3$ for all $s, t \in S$, then (W, S) is called a *simply-laced Coxeter system*.

We can represent a Coxeter system (W, S) with a *Coxeter graph* $\boldsymbol{\gamma}$ having vertex set S and edges $\{s,t\}$ for each $m(s,t) \geq 3$. Each edge $\{s,t\}$ is labeled with its corresponding bond strength. Since $m(s,t) = 3$ occurs frequently, we omit this label. Note that s and t are not connected by a single edge in the graph if and only if $m(s,t) = 2$. Given a Coxeter graph $\boldsymbol{\gamma}$, we can easily reconstruct the corresponding Coxeter system. If (W, S) is a Coxeter system with corresponding Coxeter graph $\boldsymbol{\gamma}$, we may denote the Coxeter group as $W(\boldsymbol{\gamma})$ and the generating set as $S(\boldsymbol{\gamma})$. If $\boldsymbol{\gamma}$ does not contain any three cycles, then we say $\boldsymbol{\gamma}$ and (W, S) are *triangle-free*.



(a) A_n (b) D_n (c) $\widetilde{D}_n$

Figure 1.1: Examples of common simply-laced Coxeter graphs.

Example 1.1. The Coxeter graphs given in Figure 1.1 correspond to three common simply-laced triangle-free Coxeter systems. For each, we summarize the defining relations for the corresponding Coxeter systems.

(a) The Coxeter system of type A_n is given by the Coxeter graph in Figure 1.1(a). In this case, $W(A_n)$ is generated by $S(A_n) = \{s_1, s_2, \ldots, s_n\}$ and has defining relations

- $s_i s_i = e$ for all i;
- $s_i s_j = s_j s_i$ when $|i - j| > 1$;
- $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group S_{n+1} under the mapping that sends s_i to the adjacent transposition $(i, i + 1)$.

(b) The Coxeter system of type D_n is given by the graph in Figure 1.1(b). The Coxeter group $W(D_n)$ has generating set $S(D_n) = \{s_1, s_2, \ldots, s_n\}$ and defining relations

- $s_i^2 = e$ for all i;
- $s_i s_j = s_j s_i$ if $|i - j| > 1$ and $i, j \neq 1$;
- $s_i s_j = s_j s_i$ if $i = 1$ and $j \neq 3$;
- $s_1 s_3 s_1 = s_3 s_1 s_3$ and $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$.

The Coxeter group $W(D_n)$ is isomorphic to the index two subgroup of the group of signed permutations on n letters having an even number of sign changes.

(c) The Coxeter system of type $\widetilde{D}_n$ is given by the graph in Figure 1.1(c). The Coxeter group $W(\widetilde{D}_n)$ has generating set $S(\widetilde{D}_n) = \{s_1, s_2, \ldots, s_n, s_{n+1}\}$ and defining relations

- $s_i^2 = e$ for all i;
- $s_i s_j = s_j s_i$ if $|i-j| > 1$ and $i, j \neq 1$;
- $s_i s_j = s_j s_i$ if $i \in \{1, n+1\}$ and $j \in \{2, n\}$;
- $s_i s_j s_i = s_j s_i s_j$ if $|i-j| = 1$
- $s_i s_j s_i = s_j s_i s_j$ if $i = 1$ and $j = 3$ or $i = n-1$ and $j = n+1$.

Note that while $W(A_n)$ and $W(D_n)$ are finite groups, $W(\widetilde{D}_n)$ is infinite [10].

Given a Coxeter system (W, S), let S^* denote the free monoid on the alphabet S. An element $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is called a *word* while a *subword* of w is a consecutive word of the form $s_{x_i} s_{x_{i+1}} \cdots s_{x_{j-1}} s_{x_j}$ for $1 \leq i \leq j \leq m$. If a word $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is equal to w when considered as a group element of W, we say that w is an *expression* for w. If m is minimal among all possible expressions for w, we say that w is a *reduced expression* for w, and we call $\ell(w) := m$ the *length* of w. Note that any subword of a reduced expression is also reduced. The set of all reduced expressions for $w \in W$ will be denoted by $\mathcal{R}(w)$. A product of group elements $w_1 w_2 \cdots w_r$ with $w_i \in W$ is called *reduced* if $\ell(w_1 w_2 \cdots w_r) = \sum_{i=1}^r \ell(w_i)$.

The following theorem, called Matsumoto's Theorem [8], illuminates how reduced expressions for a given group element are related.

Proposition 1.2 (Matsumoto's Theorem)**.** In a Coxeter system (W, S), any two reduced expressions for the same group element differ by a sequence of commutations and braid moves.

For $w \in W$ define the *support* of w, denoted $\operatorname{supp}(w)$, to be the set of generators that appear in any reduced expression for w. Note that Matsumoto's Theorem guarantees that $\operatorname{supp}(w)$ is well defined. For sake of brevity, we will often write i in place of $s_i \in S$ throughout most of this thesis.

Example 1.3. Let $\mathsf{w} = 543242$ be an expression for some $w \in W(A_5)$. Then we have

$$\begin{aligned} 543242 &= 543422 && \text{(via the commutativity of } s_2 \text{ and } s_4\text{)} \\ &= 5434 && \text{(because the order of } s_2 \text{ is 2)} \\ &= 5343 && \text{(via the braid relation } s_4 s_3 s_4 = s_3 s_4 s_3\text{).} \end{aligned}$$

This shows that the original expression w is not reduced, i.e, $\mathsf{w} \notin \mathcal{R}(w)$. However, it turns out 5343 and 5434 are reduced, so $\ell(w) = 4$ and $\operatorname{supp}(w) = \{3, 4, 5\}$.

According to [10], every finite Coxeter group contains a unique element of maximal length, which we often refer to as the *longest element* and denote by w_0. It is well known that the longest element in $W(A_n)$ is given in one-line notation by

$$w_0 = [n+1, n, \ldots, 2, 1]$$

and that $\ell(w_0) = \binom{n+1}{2}$. One possible reduced expression for w_0 is given by

$$s_1 \mid s_2 s_1 \mid s_3 s_2 s_1 \mid \cdots \mid s_n s_{n-1} \cdots s_3 s_2 s_1,$$

where we have inserted vertical bars to better help visualize this pattern. Indeed, since w_0 is reduced, it follows that each subword formed by these bars in w_0 are also reduced. A formula for the number of reduced expressions for $w_0 \in W(A_n)$ is given in [15].

In light of Matsumoto's Theorem, we can define a graph on the set of reduced expressions of a given element in a Coxeter group. For $w \in W$, define the *Matsumoto graph* $\mathcal{G}(w)$ to be the graph having vertex set equal to $\mathcal{R}(w)$, where two reduced expressions w_1 and w_2 are connected by an edge if and only if w_1 and w_2 are related via a single commutation or braid move. In order to distinguish between commutation and braid moves, an edge is colored red if it corresponds to a commutation move and colored green if it corresponds to a braid move. Matsumoto's Theorem implies that $\mathcal{G}(w)$ is connected. Bergeron, Ceballos, and Labbé [3] proved that for finite Coxeter groups, every cycle in $\mathcal{G}(w)$ has even length. In [9], Grinberg and Postinov extended this result to arbitrary Coxeter systems.

We now define two different equivalence relations on the set of reduced expressions for a given element of a Coxeter group. Let (W, S) be a Coxeter system of type γ and let $w \in W$. For $\mathsf{w}_1, \mathsf{w}_2 \in \mathcal{R}(w)$, $\mathsf{w}_1 \sim_c \mathsf{w}_2$ if we can obtain w_1 from w_2 by applying a single commutation move of the form $st \mapsto ts$, where $m(s,t) = 2$. We define the equivalence relation $\approx_c$ by taking the reflexive transitive closure of $\sim_c$. Each equivalence class under $\approx_c$ is called a *commutation class*, denoted $[\mathsf{w}]_c$ for $\mathsf{w} \in \mathcal{R}(w)$. Two reduced expressions are said to be *commutation equivalent* if they are in the same commutation class.

Similarly, we define $\mathsf{w}_1 \sim \mathsf{w}_2$ if we can obtain w_1 from w_2 by applying a single braid move. The equivalence relation $\approx$ is defined by taking the reflexive transitive closure of $\sim$. Each equivalence class under $\approx$ is called a *braid class*, denoted $[\mathsf{w}]$ for $\mathsf{w} \in \mathcal{R}(w)$. Two reduced expressions are said to be *braid equivalent* if they are in the same braid class.

Commutation classes have been studied extensively in the literature. In the case of Coxeter systems of type A_n, Elnitsky [5] showed that the set of commutation classes for a given permutation w is in one-to-one correspondence with the set of rhombic tilings of a certain polygon determined by w.

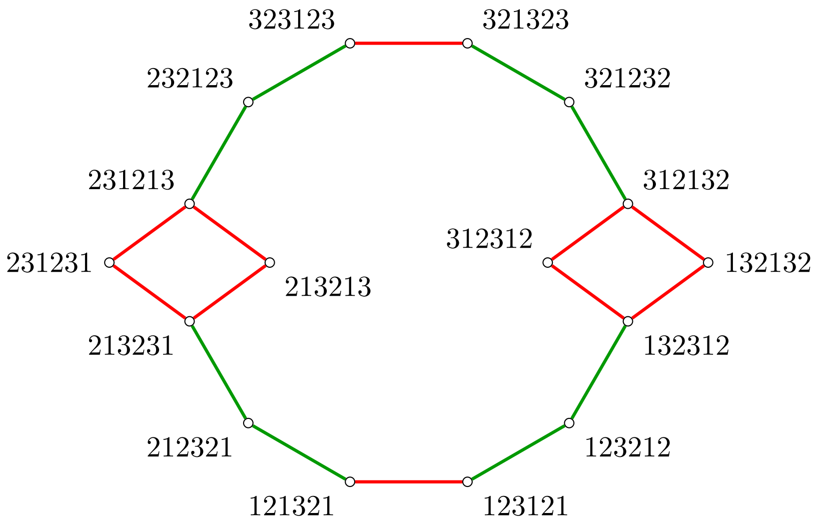


Figure 1.2: Matsumoto graph for the longest element in $W(A_3)$.

Example 1.4. The set of 16 reduced expressions for the longest element in the Coxeter system of type A_3 is partitioned into eight commutation classes and eight braid classes:

$[232123]_c = \{232123\}$
$[231213]_c = \{231213, 213213, 213231, 231231\}$
$[321323]_c = \{321323, 323123\}$
$[212321]_c = \{212321\}$
$[321232]_c = \{321232\}$
$[123123]_c = \{123121, 121321\}$
$[132312]_c = \{132312, 132132, 312132, 312312\}$
$[123212]_c = \{123212\}$

$[123121]_b = \{123121, 123212, 132312\}$
$[312312]_b = \{312312\}$
$[312132]_b = \{312132, 321232, 321323\}$
$[132132]_b = \{132132\}$
$[121321]_b = \{121321, 212321, 213231\}$
$[213213]_b = \{213213\}$
$[231213]_b = \{231213, 232123, 323123\}$
$[231231]_b = \{231231\}$

In general, it is not the case that the number of commutation classes for a fixed group element is equal to the number of braid classes. Notice that the four braid classes of size 3 correspond to the vertices in the green connected components of the Matsumoto graph given in Figure 1.2 while the singleton braid classes correspond to the four vertices that are not incident to any green edges.

Meng [14] studied the number of commutation classes and their relationships via braid moves, and Bédard [2] developed recursive formulas for the number of reduced expressions in each commutation class. Determining the number of commutation classes for the longest element in $W(A_n)$ remains an open problem. To our knowledge, this problem was first introduced in 1992 by Knuth in Section 9 of [13] using different terminology. A more general version of the problem appears in Section 5.2 of [11]. In the paragraph following the proof of Proposition 4.4 of [16], Tenner explicitly states the open problem in terms of commutation classes. The best known bound for the number of commutation classes for the longest element in $W(A_n)$ appears in [6]. Even less is known about the number of commutation classes of the longest elements in other finite Coxeter groups.

Braid classes have appeared in the work of Bergeron, Ceballos, and Labbé [3] while Zollinger [18] provided formulas for the size of braid classes for permutations in the case of Coxeter systems of type A_n. Fishel et al. [7] provided upper and lower bounds on the number of reduced expressions for a fixed permutation in $W(A_n)$ by studying the commutation classes and braid classes together. However, unlike commutation classes, braid classes have received very little attention. Many natural questions regarding braid classes remain open, even for type A_n. The remainder of this thesis will focus exclusively on braid classes in simply-laced Coxeter systems with an aim of describing their combinatorial architecture. For brevity, we will write $[\mathsf{w}]$ in place of $[\mathsf{w}]_b$.

The relationship among the reduced expressions in a fixed braid class can be encoded using the maximal green connected components of the corresponding Matsumoto graph. Let w be a reduced expression for $w \in W$. We define the *braid graph* of w, denoted $\mathcal{B}(\mathsf{w})$, to be the graph with vertex set equal to $[\mathsf{w}]$, where $\mathsf{w}_1, \mathsf{w}_2 \in [\mathsf{w}]$ are connected by an edge if and only if w_1 and w_2 are related via a single braid move. Note that we are defining braid graphs with respect to a fixed reduced expression (or equivalence class) as opposed to the corresponding group element. The latter are the graphs that arise from contracting the edges corresponding to braid moves in the Matsumoto graph. Very little is known about the overall structure of braid graphs for reduced expressions.

Example 1.5. Below we describe three different braid classes and illustrate their corresponding braid graphs.

(a) Consider the Coxeter system of type A_4. The braid class for the reduced expression 1213243 consists of the following reduced expressions:

$$\mathsf{w}_1 := 1213243,\ \mathsf{w}_2 := 2123243,\ \mathsf{w}_3 := 2132343,\ \mathsf{w}_4 := 2132434.$$

(b) In the Coxeter system of type D_4, the expression 343132343 is reduced. Its braid class consists of the following reduced expressions:

$$\mathsf{w}'_1 := 1213243565,\ \mathsf{w}'_2 := 2123243565,\ \mathsf{w}'_3 := 2132343565,\ \mathsf{w}'_4 := 2132434565.$$
$$\mathsf{w}'_5 := 1213243656, \mathsf{w}'_6 := 2123243656,\ \mathsf{w}'_7 := 2132343656,\ \mathsf{w}'_8 := 2132434656.$$

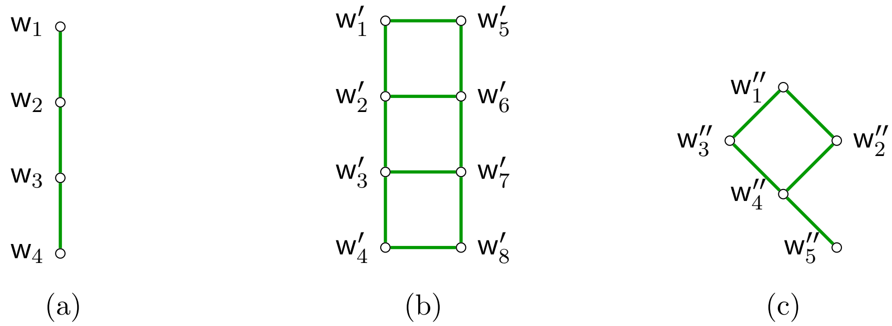


Figure 1.3: Braid graphs generated by various reduced expressions.

(c) Now, consider the Coxeter system of type D_4. The expression 2321434 is reduced and its braid class consists of the following reduced expressions:

$$\mathsf{w}''_1 := 2321434,\ \mathsf{w}''_2 := 3231434,\ \mathsf{w}''_3 := 2321343,\ \mathsf{w}''_4 := 3231343,\ \mathsf{w}''_5 := 3213143.$$

The braid graphs $\mathcal{B}(\mathsf{w}_1)$, $\mathcal{B}(\mathsf{w}'_1)$, and $\mathcal{B}(\mathsf{w}''_1)$ are depicted in Figures 1.3(a), 1.3(b), and 1.3(c), respectively.

We see that for any Coxeter system (W, S), we can have many expressions for a given $w \in W$, many of which are not reduced (see Example 1.2.4). Indeed, for any $w \in W$, we can utilize the defining relations for the given group to convert a non-reduced expression into a reduced expression [10]. However, as our expressions get larger, this becomes an inefficient way to test whether a given reduced expression is reduced or not. A more efficient method is needed.

Humphreys gives an explanation of how the definition of Coxeter systems are motivated by finite groups generated by reflections, as well as affine Weyl groups, both of which have geometric representations (see [10, Chapters 1 and 5] for details). Interestingly, Coxeter systems admit a sort of "weak" geometric representation via groups generated by "reflections," which have many uses.

Every finite Coxeter system (W, S) can be realized as a finite reflection group acting on real Euclidean space, where each $s \in S$ corresponds to reflection across a hyperplane passing through the origin. For an arbitrary Coxeter system (W, S) we generalize the finite case as follows. Begin with a vector space V_W over $\mathbb{R}$ having basis $\{\alpha_s \mid s \in S\}$ in bijection with S and then impose a geometry on V_W using a bilinear form defined via

$$B(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m(s,t)}\right),$$

where $s, t \in S$. Note that for the purposes of this thesis, we restrict ourselves to only simply-laced Coxeter systems, i.e., when $m(s,t) \leq 3$ for all $s, t \in S$. Thus, we can describe all possible values of $B(\alpha_s, \alpha_t)$:

$$B(\alpha_s, \alpha_t) = \begin{cases} -\cos(\pi) = 1, & \text{if } m(s,t) = 1 \\ -\cos(\frac{\pi}{2}) = 0, & \text{if } m(s,t) = 2 \\ -\cos(\frac{\pi}{3}) = -\frac{1}{2}, & \text{if } m(s,t) = 3. \end{cases}$$

Note that in the case $s = t$, we have $m(s,t) = 1$. Now, let H_s denote the subspace of V_W that is orthogonal to α_s. For each $s \in S$, we define the reflection $\sigma_s : V \to V$ via

$$\sigma_s(\lambda) = \lambda - 2B(\alpha_s, \lambda)\alpha_s.$$

Note that for $t \in S$, we have

$$\sigma_s(\alpha_t) = \alpha_t + 2\cos\left(\frac{\pi}{m(s,t)}\right)\alpha_s.$$

It follows that $\sigma_s(\alpha_s) = -\alpha_s$. Restricting this action to simply-laced Coxeter systems, we would obtain the following:

$$\sigma_s(\alpha_t) = \begin{cases} \alpha_t - 2\alpha_s, & \text{if } m(s,t) = 1 \\ \alpha_t, & \text{if } m(s,t) = 2 \\ \alpha_t + \alpha_s, & \text{if } m(s,t) = 3. \end{cases}$$

Indeed, this definition implies that if $m(s,t) = 3$ for $s, t \in S$, then $\sigma_s(\alpha_t) = \sigma_t(\alpha_s)$. It turns out that for $s, t \in S$, $\sigma_s\sigma_t$ is a rotation through an angle of $\frac{2\pi}{m(s,t)}$, where σ_s fixes the subspace H_s pointwise and σ_s has order 2 in $GL(V_W)$. The next result appears in [10].

Proposition 1.6. If (W, S) is a Coxeter system, then there exists a unique monomorphism $\sigma : W \to GL(V_w)$ sending s to σ_s, where the group image $\sigma(w)$ preserves the form B on V_W.

The map σ is often referred to as the *geometric representation* of W. For convenience sake, we write $w(\alpha)$ in place of $\sigma(w)(\alpha)$, where $w \in W$. Moreover, we will write $i(\alpha)$ in place of $s_i(\alpha)$ when the simple generators have a numeric label.

Example 1.7. Consider the simply-laced Coxeter system of type D_4, whose Coxeter graph is determined by Figure 1.1(b). Observe that $3, 4 \in S(D_4)$ and $m(3,4) = 3$. Hence, $3(\alpha_4) = \alpha_4 + \alpha_3 = \alpha_3 + \alpha_4 = 4(\alpha_3)$. Moreover, $1, 2 \in S(D_4)$ and $m(1,2) = 2$, and so $1(\alpha_2) = \alpha_2$, while $2(\alpha_1) = \alpha_1$.

Definition 1.8. Let (W, S) be a Coxeter system and define $\Phi := \{w(\alpha_s) \mid s \in S, w \in W\}$. The set Φ is called the *root system* for (W, S), where the elements of Φ are called *roots*, and if $s \in S$, then α_s is referred to as a *simple root*.

Roots in Φ are vectors and notice that each root is a unit vector since w preserves the bilinear form B on V_W. If $\alpha \in \Phi$, we can write α uniquely as

$$\alpha = \sum_{s \in S} c_s \alpha_s,$$

where each $c_s \in \mathbb{R}$, which is a linear combination of simple roots.

It turns out that we can partition Φ into two sets composed of *positive roots* and *negative roots*, respectively. *Positive roots* are all $\alpha \in \Phi$ such that $c_s \geq 0$ for all $s \in S$. In particular, each α_s is a positive root. *Negative roots* are all $\alpha \in \Phi$ such that $c_s \leq 0$ for all $s \in S$. The set of positive roots is denoted by Φ^+, while the set of negative roots is denoted by Φ^-. It is known that $\Phi = \Phi^+ \bigcup \Phi^-$, where the union is disjoint, and $\Phi = -\Phi$ [10].

Definition 1.9. Let (W, S) be a Coxeter system. Let $w \in W$ and $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_m}$ be an expression of w, not necessarily reduced. We define the *root sequence* of w via

$$\theta(\mathsf{w}) := (\alpha_{x_m}, x_m(\alpha_{x_{m-1}}), x_m x_{m-1}(\alpha_{x_{m-2}}), \ldots, x_m x_{m-1} x_{m-2} \cdots x_2(\alpha_{x_1})).$$

Note that $\theta(\mathsf{w})$ consists of m entries from the corresponding root system Φ. Notice that we are inputting roots of a given word by reversing its order and sequentially feeding it roots from right to left, where each time we are increasing the size of our word by one generator. See Example 1.11. The next proposition appeared in [4].

Proposition 1.10. Let (W, S) be a Coxeter system with root system Φ. Let $w \in W$ and $s \in S$.

1. If $\mathsf{w}_1, \mathsf{w}_2 \in \mathcal{R}(w)$ such that $\theta(\mathsf{w}_1) = \theta(\mathsf{w}_2)$, then $\mathsf{w}_1 = \mathsf{w}_2$.

2. $\mathsf{w} \in \mathcal{R}(w)$ if and only if each entry of $\theta(\mathsf{w})$ is a positive root.

Proposition 1.10(1) implies that the root sequences of a given reduced expression is unique to that expression. The second item of Proposition 1.10 is of particular importance to us. It states that an expression is reduced if and only if its corresponding root sequence consists of entirely positive roots. Let us consider an example.

Example 1.11. Consider the simply-laced Coxeter system of type D_4 whose Coxeter graph was shown in Figure 1.1(b). We will verify that the expression 3432313 is reduced by computing its root sequence $\theta(3432313)$, which is described below:

- α_3,

- $3(\alpha_1) = \alpha_1 + \alpha_3$,

- $31(\alpha_3) = 3(\alpha_3 + \alpha_1) = -\alpha_3 + \alpha_1 + \alpha_3 = \alpha_1$,
- $313(\alpha_2) = 31(\alpha_2 + \alpha_3) = 3(\alpha_2 + \alpha_3 + \alpha_1) = \alpha_2 + \alpha_3 - \alpha_3 + \alpha_1 + \alpha_3 = \alpha_1 + \alpha_2 + \alpha_3$,
- $3132(\alpha_3) = 313(\alpha_3 + \alpha_2) = 31(-\alpha_3 + \alpha_2 + \alpha_3) = 31(\alpha_2) = 3(\alpha_2) = \alpha_2 + \alpha_3$,
- $31323(\alpha_4) = 3132(\alpha_4 + \alpha_3) = 313(\alpha_4 + \alpha_3 + \alpha_2) = 31(\alpha_4 + \alpha_3 + -\alpha_3 + \alpha_2 + \alpha_3) = 31(\alpha_2 + \alpha_3 + \alpha_4) = 3(\alpha_2 + \alpha_3 + \alpha_1 + \alpha_4) = \alpha_2 + \alpha_3 - \alpha_3 + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$,
- $313234(\alpha_3) = 31323(\alpha_3 + \alpha_4) = 3132(-\alpha_3 + \alpha_4 + \alpha_3) = 3132(\alpha_4) = 313(\alpha_4) = 31(\alpha_4 + \alpha_3) = 3(\alpha_4 + \alpha_3 + \alpha_1) = \alpha_1 + \alpha_3 + \alpha_4$.

Note that each entry of $\theta(3132343)$ is highlighted in blue. Since each root in the root sequence is positive, the expression is reduced by Proposition 1.10.

We can use this same proposition to show that a given word is not reduced. Consider the expression 3132313 for some element in $W(D_4)$. We obtain the following root sequence:

- α_3,
- $3(\alpha_1) = \alpha_1 + \alpha_3$,
- $31(\alpha_3) = 3(\alpha_3 + \alpha_1) = -\alpha_3 + \alpha_1 + \alpha_3 = \alpha_1$,
- $313(\alpha_2) = 31(\alpha_2 + \alpha_3) = 3(\alpha_2 + \alpha_3 + \alpha_1) = \alpha_2 + \alpha_3 - \alpha_3 + \alpha_1 + \alpha_3 = \alpha_1 + \alpha_2 + \alpha_3$,
- $3132(\alpha_3) = 313(\alpha_3 + \alpha_2) = 31(-\alpha_3 + \alpha_2 + \alpha_3) = 31(\alpha_2) = 3(\alpha_2) = \alpha_2 + \alpha_3$,
- $31323(\alpha_1) = 3132(\alpha_1 + \alpha_3) = 313(\alpha_1 + \alpha_3 + \alpha_2) = 31(\alpha_1 + \alpha_3 - \alpha_3 + \alpha_2 + \alpha_3) = 31(\alpha_2 + \alpha_3 + \alpha_1) = 3(\alpha_2 + \alpha_3 + \alpha_1 - \alpha_1) = \alpha_2 + \alpha_3 - \alpha_3 = \alpha_2$,
- $313231(\alpha_3) = 31323(\alpha_3 + \alpha_1) = 3132(-\alpha_3 + \alpha_3 + \alpha_1) = 3132(\alpha_1) = 313(\alpha_1) = 31(\alpha_1 + \alpha_3) = 3(-\alpha_1 + \alpha_3 + \alpha_1) = 3(\alpha_3) = -\alpha_3$.

Observe that the last entry of $\theta(3132313)$ is a negative root, highlighted in red, which implies that 3132313 is not reduced by Proposition 1.10. A reduced expression for the corresponding group element would be 13231, which can be verified via its roots sequence.

Indeed, the root sequence of a given expression provides us with a powerful tool for determining whether that expression is reduced in any Coxeter system.

Chapter 2

Structure of braid classes for simply-laced Coxeter systems

An effort to understand the structure of the braid graphs led to the discovery of some interesting results that characterize the structure of the corresponding braid classes in certain simply-laced Coxeter systems. The goal of this chapter is to summarize results in a recent paper by Awik et al. [1]. Throughout this chapter, we assume that (W, S) is a simply-laced Coxeter system. It is worth pointing out that many of the results presented here do not hold in arbitrary Coxeter systems.

If $i, j \in \mathbb{N}$ with $i \leq j$, then we define the interval $[\![i, j]\!] := \{i, i+1, \dots, j-1, j\}$.

Definition 2.1. Suppose $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression for $w \in W$. We define the *local support* of w over $[\![i, j]\!]$ via

$$\mathrm{supp}_{[\![i,j]\!]}(\mathsf{w}) := \{s_{x_k} \mid k \in [\![i, j]\!]\}.$$

The *global support* of the braid class $[\mathsf{w}]$ over $[\![i, j]\!]$ is defined by

$$\mathrm{supp}_{[\![i,j]\!]}([\mathsf{w}]) := \bigcup_{\mathsf{w}_1 \in [\mathsf{w}]} \mathrm{supp}_{[\![i,j]\!]}(\mathsf{w}_1).$$

In other words, $\mathrm{supp}_{[\![i,j]\!]}(\mathsf{w})$ is the set consisting of the generators that appear in positions $i, i+1, \dots, j$ of w while $\mathrm{supp}_{[\![i,j]\!]}([\mathsf{w}])$ is the set of generators that appear in positions $i, i+1, \dots, j$ of any reduced expression braid equivalent to w. Note that in the case of the degenerate interval $[\![i, i]\!]$, we will use the notation $\mathrm{supp}_i(\mathsf{w})$ and $\mathrm{supp}_i([\mathsf{w}])$ and we will simply write $\mathrm{supp}(\mathsf{w})$ for the set of generators that appear in w. The following definition first appeared in [1].

Definition 2.2. Suppose that (W, S) is a simply-laced Coxeter system. If $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression for $w \in W$, then the interval $[\![i, i+2]\!]$ is a *braid shadow* for w if and only if $s_{x_i} = s_{x_{i+2}}$ and $m(s_{x_i}, s_{x_{i+1}}) = 3$. The collection of braid shadows for w is denoted by $\mathcal{S}(\mathsf{w})$ and the set of braid shadows for the braid class $[\mathsf{w}]$ is given by

$$\mathcal{S}([\mathsf{w}]) := \bigcup_{\mathsf{w}_1 \in [\mathsf{w}]} \mathcal{S}(\mathsf{w}_1).$$

The cardinality of $\mathcal{S}([\mathsf{w}])$ is called the *rank* of w, which we denote by $\operatorname{rank}(\mathsf{w})$.

In other words, a braid shadow is the location in a reduced expression where we have an opportunity to apply a braid move. Note that a given reduced expression may have many braid shadows or possibly none at all. The sets $\mathcal{S}(\mathsf{w})$ and $\mathcal{S}([\mathsf{w}])$ capture the locations where braid moves can be performed in w in any reduced expression braid equivalent to w, respectively. If $[\![i, i+2]\!]$ is a braid shadow for $[\mathsf{w}]$, then we may refer to position $i+1$ in any reduced expression in $[\mathsf{w}]$ as the *center* of the braid shadow.

Example 2.3. Consider the reduced expressions given in Example 1.5. By inspection, we see that:

(a) $\mathcal{S}(\mathsf{w}_1) = \{[\![1,3]\!]\}$ and $\mathcal{S}([\mathsf{w}_1]) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!]\}$.

(b) $\mathcal{S}(\mathsf{w}_1') = \{[\![1,3]\!], [\![8,10]\!]\}$ and $\mathcal{S}([\mathsf{w}_1']) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!], [\![8,10]\!]\}$.

(c) $\mathcal{S}(\mathsf{w}_1'') = \{[\![1,3]\!], [\![5,7]\!]\}$ and $\mathcal{S}([\mathsf{w}_1'']) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!]\}$.

In [1], the authors prove that braid shadows for braid equivalent reduced expression may only overlap by a single position.

Proposition 2.4. Suppose (W,S) is a simply-laced Coxeter system. If w is a reduced expression for $w \in W$ with $[\![i, i+2]\!] \in \mathcal{S}([\mathsf{w}])$, then $[\![i+1, i+3]\!] \notin \mathcal{S}([\mathsf{w}])$.

This proposition motivates the following definition, which will be vital in upcoming discussions.

Definition 2.5. Suppose (W,S) is a simply-laced Coxeter system. Let $\mathsf{w} = s_{x_1}s_{x_2}\cdots s_{x_m}$ be a reduced expression for $w \in W$. We say that w is a *link* if and only if either $m = 1$ or m is odd and $\mathcal{S}([\mathsf{w}]) = \{[\![1,3]\!], [\![3,5]\!], \ldots, [\![m-4, m-2]\!], [\![m-2, m]\!]\}$. If w is a link, then the corresponding braid class $[\mathsf{w}]$ is called a *braid chain*.

Loosely speaking, w is a link if there is a sequence of overlapping braid moves that "cover" the positions $1, 2, \ldots, m$. Note that if w is a link, then the rank of w is k if an only if w consists of $2k+1$ letters. Notice that the center of every braid shadow is an even index.

Example 2.6. Consider the reduced expressions given in Example 1.5. Since $\mathcal{S}([\mathsf{w}_1]) = \{[\![1,3]\!],[\![3,5]\!],[\![5,7]\!]\}$, w_1 is a link and $[\mathsf{w}_1]$ is a braid chain. On the other hand, since $\mathcal{S}([\mathsf{w}_1']) = \{[\![1,3]\!],[\![3,5]\!],[\![5,7]\!],[\![8,10]\!]\}$, it follows that w_1' is not a link and hence $[\mathsf{w}_1']$ is not a braid chain. However, it turns out that the subwords 1213243 and 565 of w_1' are links in their own right. Lastly, since $\mathcal{S}([\mathsf{w}_1'']) = \{[\![1,3]\!],[\![3,5]\!],[\![5,7]\!]\}$, w_1'' is a link and $[\mathsf{w}_1'']$ is a braid chain.

Definition 2.7. Let w_1 be a reduced expression for $w \in W$ with $\ell(w) \geq 1$. We say that w_2 is a *link factor* of w_1 provided that

1. w_2 is a subword of w_1,
2. w_2 is a link, and
3. for every subword w_3 of w_1, if w_2 is a subword of w_3 and w_3 is a link, then $\mathsf{w}_2 = \mathsf{w}_3$.

It follows immediately from Definition 2.7 that every reduced expression w for a nonidentity group element can be written uniquely as a product of link factors, say $\mathsf{w}_1\mathsf{w}_2\cdots\mathsf{w}_k$, where each w_i is a link factor of w. We refer to this product as the *link factorization* of w. For emphasis, we will often denote such a factorization via $\mathsf{w} = \mathsf{w}_1 \mid \mathsf{w}_2 \mid \cdots \mid \mathsf{w}_k$. The following proposition, which appears in [1], is an immediate consequence of the definitions.

Proposition 2.8. Suppose (W,S) is a simply-laced Coxeter system. If w is a reduced expression for $w \in W$ with link factorization $\mathsf{w}_1 \mid \mathsf{w}_2 \mid \cdots \mid \mathsf{w}_k$, then

$$[\mathsf{w}] = \left\{\mathsf{w}_1' \mid \mathsf{w}_2' \mid \cdots \mid \mathsf{w}_k' : \mathsf{w}_i' \in [\mathsf{w}_i], i = 1,2,\ldots,k\right\}.$$

Moreover, the cardinality of the braid class for w is given by

$$\operatorname{card}([\mathsf{w}]) = \prod_{i=1}^{k} \operatorname{card}([\mathsf{w}_i]),$$

and rank of w is given by

$$\operatorname{rank}(\mathsf{w}) = \sum_{i=1}^{k} \operatorname{rank}(\mathsf{w}_i).$$

We can utilize the following definition to decompose a braid graph according to the link factorizations of the corresponding braid class.

Definition 2.9. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are arbitrary graphs ordered pairwise by their vertex set and edge set, respectively, then the *box product* of the two graphs, denoted $G_1 \square G_2$, is the graph with vertex set $V_1 \times V_2$ and there is an edge from (x_1, y_1) to (x_2, y_2) if and only if either

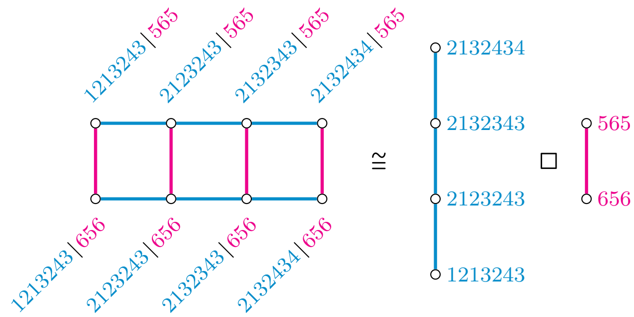


Figure 2.1: The braid graph for the reduced expression from Example 2.11 and its decomposition.

1. $x_1 = x_2$ and there is an edge from y_1 to y_2 in G_2, or
2. $y_1 = y_2$ and there is an edge from x_1 to x_2 in G_1.

Proposition 2.8 implies that the braid graph for any reduced expression for a nonidentity group element can be decomposed as the box product of the braid graphs for the corresponding factors in the link factorization. Note that the decomposition is unique if one respects the ordering of the link factors.

Corollary 2.10. Suppose (W, S) is a simply-laced Coxeter system. If w is a reduced expression for $w \in W$ with link factorization $\mathsf{w}_1 \mid \mathsf{w}_2 \mid \cdots \mid \mathsf{w}_k$, then

$$B(\mathsf{w}) \cong B(\mathsf{w}_1) \square B(\mathsf{w}_2) \square \cdots \square B(\mathsf{w}_k).$$

In other words, we can isolate each individual link factor in a given link factorization and piece together their respective braid graphs.

Example 2.11. Consider the reduced expression $\mathsf{w} = 1213243565$ for some $w \in W(A_6)$. The link factorization for w is $1213243 \mid 565$. The decomposition $B(\mathsf{w}) \cong B(1213243) \square B(565)$ guaranteed by Corollary 2.10 is shown in Figure 2.1. Note that we have utilized colors to help distinguish the link factors.

Example 2.12. Consider the Coxeter system of type D_7 determined by the Coxeter graph in Figure 1.1(b). The reduced expression 31323437547634313223 has link factorization

$$3132343 \mid 7 \mid 5 \mid 4 \mid 7 \mid 6 \mid 3431323.$$

The braid graphs for the first and last factors are isomorphic to the braid graph in Figure 1.3(c). The braid graph for each singleton factor consists of a single vertex. The braid graph for the entire reduced expression and its decomposition are shown in Figure 2.2.

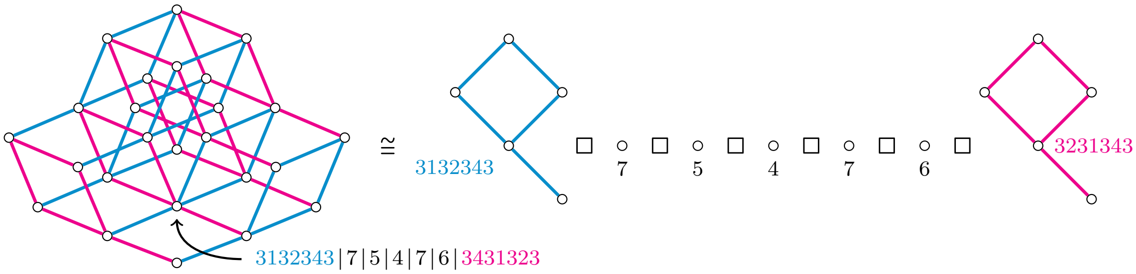


Figure 2.2: The braid graph for the reduced expression from Example 2.12 and its decomposition.

Proposition 2.8 provides us with a powerful tool for understanding the fundamental structure of braid graphs in simply-laced Coxeter system. Corollary 2.10 naturally begs the question: What are the potential braid graph structures for braid links? Moreover, is there a Coxeter system in which all possible braid graphs for links occur? We look to answer these questions in upcoming sections.

According to the next proposition from [1], the support of a braid shadow is constant across an entire braid class in simply-laced triangle-free Coxeter systems.

Proposition 2.13. Suppose (W,S) is a simply-laced triangle-free Coxeter system. If w_1 and w_2 are two braid equivalent reduced expressions for $w \in W$ with $\ell(w) \geq 3$, then for all $[\![i,i+2]\!] \in \mathcal{S}(\mathsf{w}_1) \cap \mathcal{S}(\mathsf{w}_2)$, $\operatorname{supp}_{[\![i,i+2]\!]}(\mathsf{w}_1) = \operatorname{supp}_{[\![i,i+2]\!]}(\mathsf{w}_2)$.

Example 2.14. In the simply-laced triangle-free Coxeter system of type A_3 given in Example 1.4, we partitioned all 16 reduced expressions into eight commutation classes and eight braid classes. Indeed, we see that for any braid class if two braid equivalent expressions share a braid shadow, the support is the same.

$$[123121] = \{123\underline{121}, 1\overline{23\underline{2}}\underline{12}, 1\overline{323}12\}$$

$$[312312] = \{312312\}$$

$$[312132] = \{3\underline{121}32, 3\underline{21\overline{2}}\overline{32}, 321\overline{323}\}$$

$$[132132] = \{132132\}$$

$$[121321] = \{\underline{121}321, \underline{21\overline{2}}\overline{32}1, 21\overline{323}1\}$$

$$[213213] = \{213213\}$$

$$[231213] = \{23\overline{121}3, \underline{23\overline{2}}\overline{12}3, \underline{323}123\}$$

$$[231231] = \{231231\}$$

Note that we provide overlines and underlines to represent braid shadow locations in each braid class. In any case we see that for any two braid equivalent expressions, if they have braid shadows that occur in the same position, the support is the same. This example also demonstrates Proposition 2.4, i.e, if two braid shadows intersect, they intersect by one position.

Indeed, the previous result is false without the assumption that the Coxeter system is triangle free. A counterexample is given below.

Example 2.15. Consider the Coxeter system of type $\widetilde{A}_2$, which is determined by the Coxeter graph in Figure 2.3. Note that the Coxeter graph has bond strength 3 for all pairs of vertices. The expression $\mathsf{w}_1 = 1213121$ is reduced, and it is easy to see that $\mathsf{w}_2 = 2123212 \in [\mathsf{w}_1]$. However, $\text{supp}_{[\![3,5]\!]}(\mathsf{w}_1) = \{1,3\}$ while $\text{supp}_{[\![3,5]\!]}(\mathsf{w}_2) = \{2,3\}$. This shows that Proposition 2.13 is false when the Coxeter graph has a three-cycle.

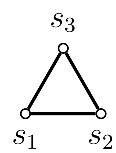


Figure 2.3: Coxeter graph of $\widetilde{A}_2$.

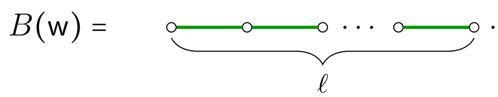
When a reduced expression has a braid shadow, the collection of generators that may appear at the center of the braid shadow in any braid equivalent reduced expression is completely determined by the support of that braid shadow. The next result from [1] makes this more precise.

Proposition 2.16. If (W,S) is a simply-laced triangle-free Coxeter system and w is a reduced expression for $w \in W$, then $[\![i,i+2]\!] \in \mathcal{S}(\mathsf{w})$ if and only if $[\![i,i+2]\!] \in \mathcal{S}([\mathsf{w}])$ and $\text{supp}_{[\![i,i+2]\!]}(\mathsf{w}) = \text{supp}_{i+1}([\mathsf{w}])$.

The previous proposition allows us to assume that $\text{supp}_{[\![i+1]\!]}([\mathsf{w}]) = \{s,t\}$ whenever $[\![i,i+2]\!] \in \mathcal{S}(\mathsf{w})$ with $\text{supp}_{[\![i,i+2]\!]}(\mathsf{w}) = \{s,t\}$.

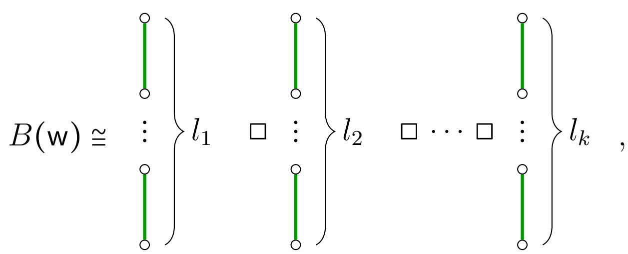
Braid classes in simply-laced Coxeter systems seem to have a reasonable structure that translates to nice classifications of their respective braid graphs. We now wish to classify links and braid graphs in the simply-laced Coxeter system of type A_n. Our notions of link and link factorization generalize Zollinger's definitions of string and maximal string decomposition, respectively, for Coxeter systems of type A_n that appear in [18]. The next result is a reformulation of Lemma 1 from [18] in terms of braid graphs.

Proposition 2.17. If w is a link in the Coxeter system of type A_n consisting of $2\ell - 1$ letters, then



Recall that Corollary 2.10 says that the braid graph for any reduced expression for a nonidentity group element can be decomposed as the box product of the braid graphs for the corresponding factors in the link factorization. In light of Proposition 2.17, we can be more explicit for Coxeter systems of type A_n, as the next result indicates. This result is also a reformulation of Corollary 6 in [18] and can be thought of as a classification of braid graphs for reduced expressions in Coxeter systems of type A_n.

Proposition 2.18. If w is a reduced expression for $w \in W(A_n)$ with link factorization $\mathsf{w}_1 \mid \mathsf{w}_2 \mid \cdots \mid \mathsf{w}_k$ such that each link has $2l_i - 1$ letters, then



where the ith factor in the decomposition is a path with l_i vertices.

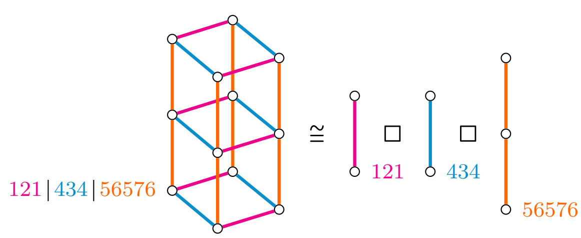


Figure 2.4: The decomposition of the braid graph for the reduced expression in Example 2.19.

Example 2.19. The word $\mathsf{w} = 12143456576$ is a reduced expression for some element in $W(A_7)$. The link factorization for w is $121 \mid 434 \mid 56576$. The resulting braid graph and its decomposition into a product of paths is shown in Figure 2.4.

In a future paper, we plan to provide a characterization of the braid graphs for links, and hence all reduced expressions, in Coxeter systems of type D_n. Indeed, we see from our examples that the structure of braid graphs are determined by applying a braid move in a respective braid shadow or not. We can interpret this as "turning" a braid shadow on or off.

Chapter 3

Braid graphs as subgraphs of hypergraphs

This chapter is a further summary of results from [1]. For a positive integer m we will denote the set of binary strings of length m by $\{0,1\}^m$. That is,

$$\{0,1\}^m := \{a_1a_2\cdots a_m \mid a_k \in \{0,1\}\}.$$

Define the *hypercube graph* of dimension m, denoted by Q_m, to be the graph whose vertices are elements of the $\{0,1\}^m$ with two binary strings connected by an edge exactly when they differ by a single digit.

If G is a graph, let $V(G)$ denote the vertex set of G. An *embedding* of a simple graph G into a simple graph H is an injection $f : V(G) \to V(H)$ with the property that u and v are adjacent vertices in G if and only if $f(u)$ and $f(v)$ are adjacent in H. If in addition, $f(u)$ and $f(v)$ adjacent in H implies u and v are adjacent in G, then f is called an *induced embedding*. If f is an induced embedding, then G is isomorphic to the subgraph of H induced by the image of f.

Assume (W,S) is a simply-laced triangle-free Coxeter system and suppose w is a reduced expression for some $w \in W$ with link factorization $\mathsf{w}_1 \mid \mathsf{w}_2 \mid \cdots \mid \mathsf{w}_k$. One goal of this chapter is to establish an induced embedding of $B(\mathsf{w})$ into $Q_{\mathrm{rank}(\mathsf{w})}$. In particular, for simply-laced triangle-free Coxeter systems, the braid graph for every reduced expression is isomorphic to an induced subgraph of a hypercube graph whose number of vertices are determined by the number of braid shadows. In light of Proposition 2.16, we are able to define the following map.

Definition 3.1. Assume that (W,S) is a simply-laced triangle-free Coxeter system and let w be a link of rank m. For each $k = 1,2,\ldots,m$, assume that $\mathrm{supp}_{2k}([\mathsf{w}]) = \{s_{2k}, t_{2k}\}$, where $s_{2k} := \mathrm{supp}_{2k}(\mathsf{w})$. Define $\Psi_{\mathsf{w}} : [\mathsf{w}] \to \{0,1\}^m$ via $\Psi_{\mathsf{w}}(\mathsf{w}_1) = a_1a_2\cdots a_m$ when $m \geq 1$ and

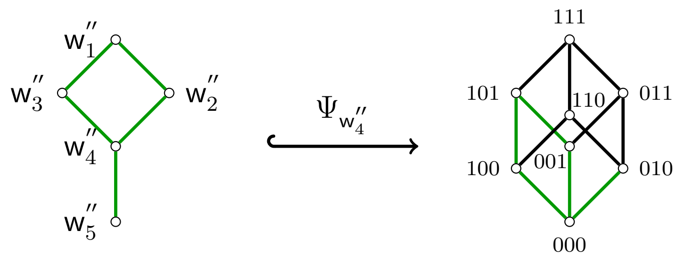


Figure 3.1: The induced embedding of $B(\mathsf{w}_1'')$ into Q_3 as in Example 3.3.

$\mathsf{w}_1 \in \mathcal{R}(\mathsf{w})$, where

$$a_k = \begin{cases} 0, & \text{if } \mathrm{supp}_{2k}(\mathsf{w}_1) = s_{2k} \\ 1, & \text{if } \mathrm{supp}_{2k}(\mathsf{w}_1) = t_{2k}. \end{cases}$$

In the case that $m = 0$, the unique element (consisting of a single letter) in the braid class is mapped to the empty binary string.

It is worth pointing out that the definition of the map Ψ_{w} depends on the choice of link. Choosing a different representative of $[\mathsf{w}]$ will necessarily result in a different mapping. The next result appears in [1].

Proposition 3.2. If (W, S) is a simply-laced triangle-free Coxeter system and w is a link of rank m, then the map Ψ_{w} given in Definition 3.1 is an induced embedding of $B(\mathsf{w})$ into Q_m.

Example 3.3. Consider the reduced expressions $\mathsf{w}_1'', \dots, \mathsf{w}_5''$ given in Example 1.5(c). The braid graph $B(\mathsf{w}_1'')$ was shown in Figure 1.3(c). In Example 2.6 we showed that w_1'' is a link of rank 3. Proposition 3.2 guarantees that there are at least five distinct induced embeddings of $B(\mathsf{w}_1'')$ into Q_3, one for each of w_1'', w_2'', w_3'', w_4'', and w_5''. One possible induced embedding, using w_4'', is shown in Figure 3.1.

In light of Corollary 2.10, we can construct an induced embedding for any reduced expression in simply-laced triangle-free Coxeter systems by applying Proposition 3.2 to each link factor and then concatenating the resulting binary strings. This yields the following result, which appears in [1].

Proposition 3.4. If (W, S) is a simply-laced triangle-free Coxeter system and w is a reduced expression for some nonidentity $w \in W$, then there is an induced embedding of $B(\mathsf{w})$ into $Q_{\mathrm{rank}(\mathsf{w})}$.

Since the number of vertices in $Q_{\mathrm{rank}(\mathsf{w})}$ is $2^{\mathrm{rank}(\mathsf{w})}$, we obtain the following corollary.

Corollary 3.5. If (W,S) is a simply-laced triangle-free Coxeter system and w is a reduced expression for some nonidentity $w \in W$, then $\operatorname{card}([\mathsf{w}]) \leq 2^{\operatorname{rank}(\mathsf{w})}$. Moreover, if

$$\mathsf{w} = \mathsf{w}_1 \mid \mathsf{w}_2 \mid \cdots \mid \mathsf{w}_k,$$

is a braid link factorization, then the bound is achieved precisely when $\operatorname{rank}(\mathsf{w}_i) \leq 1$ for all i.

If (W,S) is a simply-laced Coxeter system and $w \in W$ such that $\ell(w) = n$, then the maximum number of braid shadows that any reduced expression for w can have is $\lfloor \frac{n-1}{2} \rfloor$. For simply-laced Coxeter systems, it is clear that as the length increases, so too does the number of possible braid shadows. For finite Coxeter groups, the length function is bounded above by the length of the longest element. It turns out that all of the finite simply-laced Coxeter systems are triangle free [10]. For each of these groups, we can utilize Corollary 3.5 to obtain an upper bound on the cardinality of any braid class in that group.

Since the length of the longest element in the Coxeter system of type A_n is $\frac{n(n+1)}{2}$, the maximum number of braid shadows that a reduced expression for an element in $W(A_n)$ could have is $\lfloor \frac{n^2+n-2}{4} \rfloor$. Indeed, this maximum is attained for at least $n = 1,2,3$. Nonetheless, Corollary 3.5 implies that if w is a reduced expression for some nonidentity element in $W(A_n)$, then $\operatorname{card}([\mathsf{w}]) \leq 2^{\lfloor \frac{n^2+n-2}{4} \rfloor}$. For example, if $n = 3$, then the cardinality of every braid class in $W(A_3)$ must be less than or equal to 4. However, the actual maximum size of a braid class in $W(A_3)$ is 3. As n increases, the bound in Corollary 3.5 deteriorates. In [18], Zollinger establishes sharp upper bounds on the cardinality of a braid class for a fixed length across all Coxeter systems of type A_n.

In the case of type D_n, the longest element has length $n^2 - n$ [10], and so the maximum number of braid shadows that a reduced expression for an element in $W(D_n)$ could have is $\lfloor \frac{n^2-n-1}{2} \rfloor$. Thus, if w is a reduced expression for some nonidentity element in $W(D_n)$, then $\operatorname{card}([\mathsf{w}]) \leq 2^{\lfloor \frac{n^2-n-1}{2} \rfloor}$.

We can view any connected graph G as a metric space by defining the distance between $u, v \in V(G)$ via

$$d_G(u,v) := \text{length of any minimal path between } u \text{ and } v.$$

Note that in the case of the hypercube graph Q_n, this distance agrees with the Hamming distance. An *isometric embedding* of a graph G into a graph H is a function $f : V(G) \to V(H)$ with the property that $d_G(u,v) = d_H(f(u), f(v))$ for all $u, v \in V(G)$. In this case, G is isometrically isomorphic to an induced subgraph of H. Since two vertices are adjacent if and only if the distance between them is one, every isometric embedding is also a graph embedding (i.e., f is also injective). Note that every isometric embedding is an induced embedding.

In [1], the authors conjecture that the embedding given in Proposition 3.4 is actually an isometric embedding and we will illustrate this claim for a special class of links that are closely related to the Fibonacci numbers. If true in general, this would imply that every braid graph in a simply-laced triangle-free Coxeter system is a partial cube. A partial cube is a graph that is isometric to a subgraph of a hypercube.

Definition 3.6. Suppose (W, S) is a simply-laced triangle-free Coxeter system. If $\boldsymbol{\varphi}$ is a link with the property that $\mathcal{S}([\boldsymbol{\varphi}]) = \mathcal{S}(\boldsymbol{\varphi})$, then we will refer to $\boldsymbol{\varphi}$ as a *Fibonacci link* and the corresponding braid class $[\boldsymbol{\varphi}]$ will be referred to as a *Fibonacci chain.*

That is, a Fibonacci link is a special type of link with the property that every braid shadow for its corresponding braid class appears in the link. The following proposition from [1] describes a standard representation for Fibonacci links that will be of great use in the succeeding chapter of this thesis.

Proposition 3.7. Suppose (W, S) is a simply-laced triangle-free Coxeter system of type $\boldsymbol{\gamma}$. Then every Fibonacci link of rank m can be written in the form

$$\boldsymbol{\varphi} = st_{x_1}st_{x_2}\cdots st_{x_{m-1}}st_{x_m}s$$

for some $s, t_{x_1}, t_{x_2}, \ldots, t_{x_m} \in S$ with $m(s, t_{x_i}) = 3$ for all $i = 1, 2 \ldots, m$ and $m(t_{x_i}, t_{x_j}) = 2$ for $i \neq j$.

Notice that the structure of $\boldsymbol{\varphi}$ in the previous proposition implies that the Coxeter graph induced by $\{s, t_{x_1}, t_{x_2}, \ldots, t_{x_m}\}$ must be a star graph with s connected to each of $t_{x_1}, t_{x_2}, \ldots, t_{x_m}$. We now construct several examples and nonexamples of Fibonacci links in some simply-laced triangle-free Coxeter systems.

Example 3.8. Not every link is a Fibonacci link. For instance, in the Coxeter system of type A_4, the reduced expression 1213243 is a link, but not a Fibonacci link, since $[\![3, 5]\!]$ and $[\![5, 7]\!]$ are not braid shadows. In the Coxeter system of type D_5, the reduced expression 4534313234313 is a link, but it is not a Fibonacci link since $[\![1, 3]\!]$ is not a braid shadow for the expression.

Example 3.9. In the Coxeter system of type A_n, there are no Fibonacci links of rank 3 or larger due to the characterization in Proposition 2.17 together with Proposition 3.7. The only Fibonacci links we can build in Coxeter systems of type A_n are of the forms $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_{i-1} s_i s_{i+1} s_i$. It is more difficult to write down all the Fibonacci links in the Coxeter system of type D_n. The following reduced expressions are Fibonacci links of ranks 1 through 5 in type D_4:

343, 34313, 3431323, 343132343, 34313234313.

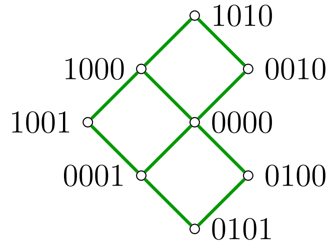


Figure 3.2: Fibonacci cube $\mathcal{F}_4$.

In fact, it is not possible to find a Fibonacci link of rank greater than 5 in type D_n. This fact is obvious in type D_4 since the longest element has length 12, but remains true even for large n. The longest possible Fibonacci link in type D_n is of the form $3a3b3c3a3b$, where $a, b, c \in \{1, 2, 4\}$. Notice that Proposition 3.7 implies that every link in type D_4 is braid equivalent to a Fibonacci link since the Coxeter graph for D_4 is a star graph.

Each Fibonacci link carries with it rich recursive properties that manifest in the structure of the braid class and the corresponding braid graph. The next result from [1] tells us that Fibonacci chains contain a Fibonacci number of links. As usual, we define the Fibonacci numbers recursively by $F_{n+1} = F_n + F_{n-1}$ with the initial condition $F_1 = F_2 = 1$.

Proposition 3.10. Suppose (W, S) is a simply-laced triangle-free Coxeter system. If $\boldsymbol{\varphi}$ is a Fibonacci link of rank $m \geq 0$, then $\operatorname{card}([\boldsymbol{\varphi}]) = F_{m+2}$.

It turns out that the braid graph for a Fibonacci link of rank m is isomorphic to a well-known induced subgraph of Q_m having a Fibonacci number of vertices. We define the *Fibonacci cube* of order m as the subgraph $\mathcal{F}_m$ of Q_m induced by set of vertices

$$X_m = \{a_1 a_2 \cdots a_m \in \{0,1\}^m : a_i \cdot a_{i+1} = 0,\ 1 \leq i \leq m-1\}.$$

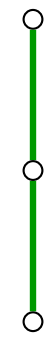
That is, X_m is the collection of length m binary strings that do not contain the consecutive substring 11. Fibonacci cubes have been studied extensively in the literature. According to [12], each Fibonacci cube is a partial cube and $|X_m| = F_{m+2}$.

Example 3.11. The Fibonacci cube $\mathcal{F}_4$ is given in Figure 3.2. We see that $\mathcal{F}_4$ has vertex set $X_4 = \{0000, 1000, 0010, 0100, 1010, 1010, 0101, 0001\}$, and so $|X_4| = 8 = F_6$.

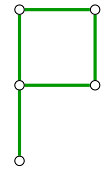
The next proposition is the main result of [1].

Proposition 3.12. Suppose (W, S) is a simply-laced triangle-free Coxeter system. If $\boldsymbol{\varphi}$ is a Fibonacci link of rank m, then $B(\boldsymbol{\varphi})$ is isomorphic to the Fibonacci cube $\mathcal{F}_m$.

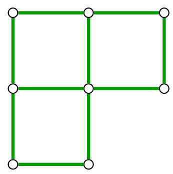
(a) $B(343) \cong \mathcal{F}_1$



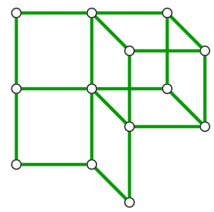
(b) $B(34313) \cong \mathcal{F}_2$



(c) $B(3431323) \cong \mathcal{F}_3$



(d) $B(343132343) \cong \mathcal{F}_4$



(e) $B(34313234313) \cong \mathcal{F}_5$

Figure 3.3: Braids graphs for several Fibonacci links in the Coxeter system of type D_4.

Example 3.13. Consider the Fibonacci links in type D_4 from Example 3.9:

$$343, 34313, 3431323, 343132343, 34313234313.$$

According to Theorem 3.12, the corresponding braid graphs are Fibonacci cubes. Each braid graph is depicted in Figure 3.3. In each graph, the Fibonacci link always corresponds to the vertex of highest degree.

Indeed, Fibonacci links and their corresponding Fibonacci cubes provide a detailed description for the structure of special types of braid links. In the next chapter, we will prove that every Fibonacci cube occurs as a braid graph when the given Coxeter graph has the Coxeter graph of type $\widetilde{D}_4$ as a subgraph.

Chapter 4

Existence of Fibonacci cubes as braid graphs

We will now expand on the relationship between Fibonacci cubes and braid graphs in simply-laced Coxeter systems. As seen in the previous chapter, every braid graph for a Fibonacci link in a simply-laced triangle-free Coxeter system is isomorphic to a Fibonacci cube.

Recall from Example 3.13 that the Fibonacci cubes $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$, and $\mathcal{F}_5$ are realized as braid graphs for Fibonacci links in the Coxeter system of type D_4. We obtained these five Fibonacci graphs from the Fibonacci links:

$$343, 34313, 3431323, 343132343, 34313234313.$$

However, these are not the only Fibonacci links that yield these graphs. For example, 313 and 3134323 are also Fibonacci links in $W(D_4)$ whose braid graphs are isomorphic to $\mathcal{F}_1$ and $\mathcal{F}_3$, respectively. It is only natural to ask whether the Fibonacci cubes $\mathcal{F}_n$ for $n \geq 6$ also occur as a braid graph for a Fibonacci link in some Coxeter system.

In this chapter, we will show that every Fibonacci cube is realized as the braid graph for a Fibonacci link in any simply-laced triangle-free Coxeter system having type $\widetilde{D}_4$ as a subgraph of its Coxeter graph. This addresses an open question posed at the end of [1].

Definition 4.1. The Coxeter system of type $\widetilde{D}_4$ is given by the Coxeter graph in Figure 4.1. Using our labeling, $W(\widetilde{D}_4)$ has generating set $S(\widetilde{D}_4) = \{x, a, b, c, d\}$ and defining relations:

1. $i^2 = e$ for all $i \in \{x, a, b, c, d\}$;
2. $ij = ji$ if $i \neq j$ and $i, j \in \{a, b, c, d\}$;
3. $iji = jij$ if $i = x$ and $j \in \{a, b, c, d\}$.

Note that we have seen the general Coxeter graph of type $\widetilde{D}_n$ in Figure 1.1(c). Moreover, the Coxeter graph of the system of type D_4, given in Figure 1.1(b), is a subgraph of the Coxeter graph of type $\widetilde{D}_4$, as seen in Figure 4.1 up to relabeling. It is worth noting that the Coxeter graph of type $\widetilde{D}_4$ can be thought of as the Coxeter graph of type D_4 with an additional generator. Unlike $W(D_4)$, $W(\widetilde{D}_4)$ is infinite. This implies that there are reduced expressions of arbitrary length, which provides the opportunity for possibly infinitely many Fibonacci links. Additionally, a given braid shadow performs the same function regardless of the support of it, i.e., if two Fibonacci links have the same rank, their respective braid graphs will be isomorphic.

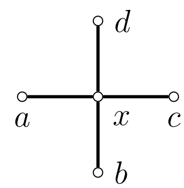


Figure 4.1: Coxeter graph for type $\widetilde{D}_4$.

Now, by Proposition 3.7, we know that we can express any Fibonacci link, $\boldsymbol{\varphi}$, of rank m in the Coxeter system of type $\widetilde{D}_4$ in the form:

$$\boldsymbol{\varphi} = xt_{x_1}xt_{x_2}xt_{x_3}x\cdots t_{x_{m-1}}xt_{x_m}x,$$

where $t_{x_i} \in \{a, b, c, d\}$, $m(x, t_{x_i}) = 3$, and $m(t_{x_i}, t_{x_j}) = 2$, for all $i \neq j$. As described earlier, we can potentially express Fibonacci links in multiple forms, as long as the expressions have the same rank and have isomorphic braid graphs.

We will show that the following expression is a reduced expression for a Fibonacci link of rank m:

$$\boldsymbol{\varphi}_m := xt_{x_m}xt_{x_{m-1}}\cdots xt_{x_3}xt_{x_2}xt_{x_1}x,$$

where

$$t_{x_i} = \begin{cases} d & \text{if } i \equiv 1 \pmod 4 \\ c & \text{if } i \equiv 2 \pmod 4 \\ b & \text{if } i \equiv 3 \pmod 4 \\ a & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

In other words, we cycle through generators d, c, b, a from left to right. As an example, consider the following expressions:

- $\boldsymbol{\varphi}_1 = xdx$

- $\varphi_2 = xcxdx$
- $\varphi_3 = xbxcxdx$
- $\varphi_4 = xaxbxcxdx$
- $\varphi_5 = xdxaxbxcxdx$

It turns out that each of these expressions is reduced (see Example 4.3). However, while these Fibonacci links have the same rank as those given in Example 3.9 and Example 3.13, they are different reduced expressions. Regardless, the braid graph for each is isomorphic to $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$, and $\mathcal{F}_5$, respectively. Also, notice that as we increase the rank of each potential Fibonacci link from m to $m+1$, we are appending xt_{m+1} to the left of the expression.

We claim that we can obtain Fibonacci links in type $\widetilde{D}_4$ with rank greater than 5. When considering potential Fibonacci links of rank higher than 5, we are interested in whether the expression in question is reduced or not. If the expression is reduced, then the definition of a Fibonacci link would be satisfied. We will verify potentially higher-ranked Fibonacci links via their root sequences. The following special vector is key to simplifying our computations:

$$\delta := \alpha_a + \alpha_b + \alpha_c + \alpha_d + 2\alpha_x.$$

It turns out that $\delta \in \Phi$ for the Coxeter system of type $\widetilde{D}_4$, but this fact is not needed in what follows. The next result tells us that δ is fixed by all $w \in W(\widetilde{D}_4)$.

Proposition 4.2. For all $w \in W(\widetilde{D}_4)$, $w(\delta) = \delta$.

Proof. Suppose $w \in W(\widetilde{D}_4)$. We will first show that $a(\delta) = b(\delta) = c(\delta) = d(\delta) = x(\delta) = \delta$. Observe that:

- $a(\delta) = a(\alpha_a+\alpha_b+\alpha_c+\alpha_d+2\alpha_x) = -\alpha_a+\alpha_b+\alpha_c+\alpha_d+2\alpha_x+2\alpha_a = \alpha_a+\alpha_b+\alpha_c+\alpha_d+2\alpha_x = \delta$
- $b(\delta) = b(\alpha_a+\alpha_b+\alpha_c+\alpha_d+2\alpha_x) = \alpha_a-\alpha_b+\alpha_c+\alpha_d+2\alpha_x+2\alpha_b = \alpha_a+\alpha_b+\alpha_c+\alpha_d+2\alpha_x = \delta$
- $c(\delta) = c(\alpha_a+\alpha_b+\alpha_c+\alpha_d+2\alpha_x) = \alpha_a+\alpha_b-\alpha_c+\alpha_d+2\alpha_x+2\alpha_c = \alpha_a+\alpha_b+\alpha_c+\alpha_d+2\alpha_x = \delta$
- $d(\delta) = d(\alpha_a+\alpha_b+\alpha_c+\alpha_d+2\alpha_x) = \alpha_a+\alpha_b+\alpha_c-\alpha_d+2\alpha_x+2\alpha_d = \alpha_a+\alpha_b+\alpha_c+\alpha_d+2\alpha_x = \delta$
- $x(\delta) = x(\alpha_a + \alpha_b + \alpha_c + \alpha_d + 2\alpha_x) = \alpha_a + \alpha_x + \alpha_b + \alpha_x + \alpha_c + \alpha_x + \alpha_d + \alpha_x - 2\alpha_x = \alpha_a + \alpha_b + \alpha_c + \alpha_d - 2\alpha_x + 4\alpha_x = \alpha_a + \alpha_b + \alpha_c + \alpha_d + 2\alpha_x = \delta$.

Therefore, since every $w \in W(\widetilde{D}_4)$ is a word constructed from the generating set $S(\widetilde{D}_4) = \{a, b, c, d, x\}$, it follows that for all $w \in W(\widetilde{D}_4)$, $w(\delta) = \delta$. □

Now, let us consider an example of a potential Fibonacci link of rank 6 via the representation established earlier.

Example 4.3. Consider the expression $\boldsymbol{\varphi}_6 = xcxdxaxbxcxdx$ for some group element in $W(\widetilde{D}_4)$. Then by Definition 1.9, we obtain the following root sequence:

- α_x
- $x(\alpha_d) = \alpha_d + \alpha_x$
- $xd(\alpha_x) = \alpha_d$
- $xdx(\alpha_c) = \alpha_c + \alpha_d + \alpha_x$
- $xdxc(\alpha_x) = \alpha_c + \alpha_x$
- $xdxcx(\alpha_b) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x$
- $xdxcxb(\alpha_x) = \alpha_b + \alpha_d + \alpha_x$
- $xdxcxbx(\alpha_a) = \alpha_d + \alpha_x + \delta$
- $xdxcxbxa(\alpha_x) = \alpha_a + \alpha_c + \alpha_d + 2\alpha_x$
- $xdxcxbxax(\alpha_d) = \alpha_c + \alpha_d + \alpha_x + \delta$
- $xdxcxbxaxd(\alpha_x) = \alpha_b + \alpha_c + \alpha_d + \alpha_x$
- $xdxcxbxaxdx(\alpha_c) = \alpha_b + \alpha_c + \alpha_d$
- $xdxcxbxaxdxc(\alpha_x) = \alpha_x + \delta$.

Thus, by Proposition 1.10, since every entry in the root sequence of the expression $xcxdxaxbxcxdx$ is positive (highlighted in blue), the expression is reduced. It follows that $\boldsymbol{\varphi}_6$ is a Fibonacci link of rank 6. Note that since $\boldsymbol{\varphi}_6$ is reduced, each $\boldsymbol{\varphi}_m$ is also reduced for $1 \le m \le 5$, as described earlier.

We can recursively calculate root sequences of potentially higher-ranked Fibonacci links. The root sequence for $\boldsymbol{\varphi}_k$ appears as the initial sequence of every $\boldsymbol{\varphi}_m$ with $k \le m$. As we move from $\boldsymbol{\varphi}_m$ to $\boldsymbol{\varphi}_{m+1}$, we append two additional entries to the root sequence.

In order for us to proceed, we calculate the root sequence for $\boldsymbol{\varphi}_{24}$. Note that this root sequence also captures the root sequence for $\boldsymbol{\varphi}_i$, where $1 \le i \le 23$. Also, notice that the first 13 lines match the computations above.

1. α_x
2. $x(\alpha_d) = \alpha_d + \alpha_x$
3. $xd(\alpha_x) = \alpha_d$
4. $xdx(\alpha_c) = \alpha_c + \alpha_d + \alpha_x$
5. $xdxc(\alpha_x) = \alpha_c + \alpha_x$
6. $xdxcx(\alpha_b) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x$

7. $xdxcxb(\alpha_x) = \alpha_b + \alpha_d + \alpha_x$
8. $xdxcxbx(\alpha_a) = \alpha_d + \alpha_x + \delta$
9. $xdxcxbxa(\alpha_x) = \alpha_a + \alpha_c + \alpha_d + 2\alpha_x$
10. $xdxcxbxax(\alpha_d) = \alpha_c + \alpha_d + \alpha_x + \delta$
11. $xdxcxbxaxd(\alpha_x) = \alpha_b + \alpha_c + \alpha_d + \alpha_x$
12. $xdxcxbxaxdx(\alpha_c) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + \delta$
13. $xdxcxbxaxdxc(\alpha_x) = \alpha_x + \delta$
14. $xdxcxbxaxdxcx(\alpha_b) = \alpha_d + \alpha_x + 2\delta$
15. $xdxcxbxaxdxcxb(\alpha_x) = \alpha_d + \delta$
16. $xdxcxbxaxdxcxbx(\alpha_a) = \alpha_c + \alpha_d + \alpha_x + 2\delta$
17. $xdxcxbxaxdxcxbxa(\alpha_x) = \alpha_c + \alpha_x + \delta$
18. $xdxcxbxaxdxcxbxax(\alpha_d) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + 2\delta$
19. $xdxcxbxaxdxcxbxaxd(\alpha_x) = \alpha_b + \alpha_d + \alpha_x + \delta$
20. $xdxcxbxaxdxcxbxaxdx(\alpha_c) = \alpha_d + \alpha_x + 3\delta$
21. $xdxcxbxaxdxcxbxaxdxc(\alpha_x) = \alpha_a + \alpha_c + \alpha_d + 2\alpha_x + \delta$
22. $xdxcxbxaxdxcxbxaxdxcx(\alpha_b) = \alpha_c + \alpha_d + \alpha_x + 3\delta$
23. $xdxcxbxaxdxcxbxaxdxcxb(\alpha_x) = \alpha_b + \alpha_c + \alpha_d + \alpha_x + \delta$
24. $xdxcxbxaxdxcxbxaxdxcxbx(\alpha_a) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + 3\delta$
25. $xdxcxbxaxdxcxbxaxdxcxbxa(\alpha_x) = \alpha_x + 2\delta.$
26. $xdxcxbxaxdxcxbxaxdxcxbxax(\alpha_d) = \alpha_d + \alpha_x + 4\delta$
27. $xdxcxbxaxdxcxbxaxdxcxbxaxd(\alpha_x) = \alpha_d + 2\delta$
28. $xdxcxbxaxdxcxbxaxdxcxbxaxdx(\alpha_c) = \alpha_c + \alpha_d + \alpha_x + 4\delta$
29. $xdxcxbxaxdxcxbxaxdxcxbxaxdxc(\alpha_x) = \alpha_c + \alpha_x + 2\delta$

30. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcx(\alpha_b) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + 4\delta$

31. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxb(\alpha_x) = \alpha_b + \alpha_d + \alpha_x + 2\delta$

32. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbx(\alpha_a) = \alpha_d + \alpha_x + 5\delta$

33. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxa(\alpha_x) = \alpha_a + \alpha_c + \alpha_d + 2\alpha_x + 2\delta$

34. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxax(\alpha_d) = \alpha_c + \alpha_d + \alpha_x + 5\delta$

35. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxd(\alpha_x) = \alpha_b + \alpha_c + \alpha_d + \alpha_x + 2\delta$

36. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdx(\alpha_c) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + 5\delta$

37. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxc(\alpha_x) = \alpha_x + 3\delta$

38. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcx(\alpha_b) = \alpha_d + \alpha_x + 6\delta$

39. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxb(\alpha_x) = \alpha_d + 3\delta$

40. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxbx(\alpha_a) = \alpha_c + \alpha_d + \alpha_x + 6\delta$

41. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxa(\alpha_x) = \alpha_c + \alpha_x + 3\delta$

42. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxax(\alpha_d) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + 6\delta$

43. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxd(\alpha_x) = \alpha_b + \alpha_d + \alpha_x + 3\delta$

44. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdx(\alpha_c) = \alpha_d + \alpha_x + 7\delta$

45. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxc(\alpha_x) = \alpha_a + \alpha_c + \alpha_d + 2\alpha_x + 3\delta$

46. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcx(\alpha_b) = \alpha_c + \alpha_d + \alpha_x + 7\delta$

47. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxb(\alpha_x) = \alpha_b + \alpha_c + \alpha_d + \alpha_x + 3\delta$

48. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxbx(\alpha_a) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + 7\delta$

49. $xdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxaxdxcxbxa(\alpha_x) = \alpha_x + 4\delta$

These computations were determined by hand and then verified with a Python program, which was written with the help of Hugh Denoncourt and run in CoCalc (`cocalc.com`). Since each entry is positive, by Proposition 1.10, $\boldsymbol{\varphi}_{24}$ is in fact a Fibonacci link of rank 24. Hence, in the Coxeter system of type $\widetilde{D}_4$, we know that Fibonacci links exist up to at least rank 24 of the form $\boldsymbol{\varphi}_i$, where $1 \leq i \leq 24$. Therefore, by Theorem 3.12, we know that we have Fibonacci links whose respective braid graphs are isomorphic to Fibonacci cubes up to $\mathcal{F}_{24}$. We list braid graphs for Fibonacci links up to rank 6 in Figure 3.3.

Proposition 4.4. For all $i \geq 1$, $\boldsymbol{\varphi}_i$ is reduced.

Proof. We first note that by previous computations, $\boldsymbol{\varphi}_i$ is reduced for all $1 \leq i \leq 24$. We will consider the form $i = 12k$. Now, define

$$\widehat{\boldsymbol{\varphi}}_{12} \coloneqq xaxbxcxdxaxbxcxdxaxbxcxd,$$

and

$$\widehat{\boldsymbol{\varphi}}_{24} \coloneqq xaxbxcxdxaxbxcxdxaxbxcxdxaxbxcxdxaxbxcxdxaxbxcxd.$$

Note that $\widehat{\boldsymbol{\varphi}}_{12}$ can be interpreted the Fibonacci link of rank 12, $\boldsymbol{\varphi}_{12}$, with the most right generator removed, i.e., $\widehat{\boldsymbol{\varphi}}_{12}x = \boldsymbol{\varphi}_{12}$. Similarly, we have $\widehat{\boldsymbol{\varphi}}_{24}x = \boldsymbol{\varphi}_{24}$. Next, we utilize our Python code to compute $(\widehat{\boldsymbol{\varphi}}_{12})^{-1}$ and $((\widehat{\boldsymbol{\varphi}}_{12})^2)^{-1}$ acting on our simple roots, α_i, where $i \in \{a, b, c, d, x\}$. We obtain:

- $(\widehat{\boldsymbol{\varphi}}_{12})^{-1}(\alpha_a) = -3\alpha_a - 4\alpha_b - 4\alpha_c - 4\alpha_d - 8\alpha_x = \alpha_a - 4\delta$
- $(\widehat{\boldsymbol{\varphi}}_{12})^{-1}(\alpha_b) = -2\alpha_a - \alpha_b - 2\alpha_c - 2\alpha_d - 4\alpha_x = \alpha_b - 2\delta$
- $(\widehat{\boldsymbol{\varphi}}_{12})^{-1}(\alpha_c) = \alpha_c$
- $(\widehat{\boldsymbol{\varphi}}_{12})^{-1}(\alpha_d) = 2\alpha_a + 2\alpha_b + 2\alpha_c + 3\alpha_d + 4\alpha_x = \alpha_d + 2\delta$
- $(\widehat{\boldsymbol{\varphi}}_{12})^{-1}(\alpha_x) = 2\alpha_a + 2\alpha_b + 2\alpha_c + 2\alpha_d + 5\alpha_x = \alpha_x + 2\delta$
- $(\widehat{\boldsymbol{\varphi}}_{24})^{-1}(\alpha_a) = -7\alpha_a - 8\alpha_b - 8\alpha_c - 8\alpha_d - 16\alpha_x = \alpha_a - 8\delta$
- $(\widehat{\boldsymbol{\varphi}}_{24})^{-1}(\alpha_b) = -4\alpha_a - 3\alpha_b - 4\alpha_c - 4\alpha_d - 8\alpha_x = \alpha_b - 4\delta$
- $(\widehat{\boldsymbol{\varphi}}_{24})^{-1}(\alpha_c) = \alpha_c$
- $(\widehat{\boldsymbol{\varphi}}_{24})^{-1}(\alpha_d) = 4\alpha_a + 4\alpha_b + 4\alpha_c + 5\alpha_d + 8\alpha_x = \alpha_d + 4\delta$
- $(\widehat{\boldsymbol{\varphi}}_{24})^{-1}(\alpha_x) = 4\alpha_a + 4\alpha_b + 4\alpha_c + 4\alpha_d + 9\alpha_x = \alpha_x + 4\delta$

As expected, $(\widehat{\boldsymbol{\varphi}}_{12})^{-1}(\alpha_a)$, $(\widehat{\boldsymbol{\varphi}}_{12})^{-1}(\alpha_b)$, $(\widehat{\boldsymbol{\varphi}}_{24})^{-1}(\alpha_a)$, and $(\widehat{\boldsymbol{\varphi}}_{24})^{-1}(\alpha_b)$ are negative roots, while the others are positive.

Now, let us compute higher values of $\boldsymbol{\varphi}_i$. We need 24 intermediate results to handle different size prefixes. For each $k \geq 1$, note that $\boldsymbol{\varphi}_{12k} = (\widehat{\boldsymbol{\varphi}}_{12})^k x$ and $((\widehat{\boldsymbol{\varphi}}_{12})^2)^{-1} = (\widehat{\boldsymbol{\varphi}}_{24})^{-1}$. We compute the tail of the root sequence for $\boldsymbol{\varphi}_i = \boldsymbol{\varphi}_{12(k+1)}$ by handling cases of when k is odd versus when k is even. For both cases, we will provide details on a few non-trivial size prefixes, and then provide a list of values. Along the way, we will utilize the values of $(\widehat{\boldsymbol{\varphi}}_{12})^{-1}$ and $((\widehat{\boldsymbol{\varphi}}_{12})^2)^{-1}$ acting on simple roots given above.

First, assume k is odd. Then $k = 2n + 1$ for some $n \in \mathbb{Z}_{\geq 0}$. We claim that we obtain the following tail of the root sequence for $\boldsymbol{\varphi}_i = \widehat{\boldsymbol{\varphi}}_{12(k+1)}$. We list the odd cases of the 24 intermediate prefix sizes below.

1. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k(\alpha_x) = \alpha_x + (4m+2)\delta \equiv \alpha_x \pmod{\delta}$
2. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k x(\alpha_d) = \alpha_d + \alpha_x + (8m+4)\delta \equiv \alpha_d + \alpha_x \pmod{\delta}$
3. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xd(\alpha_x) = \alpha_d + (4m+2)\delta \equiv \alpha_d \pmod{\delta}$
4. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdx(\alpha_c) = \alpha_c + \alpha_d + \alpha_x + (8m+4)\delta \equiv \alpha_c + \alpha_d + \alpha_x \pmod{\delta}$
5. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxc(\alpha_x) = \alpha_c + \alpha_x + (4m+2)\delta \equiv \alpha_c + \alpha_x \pmod{\delta}$
6. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcx(\alpha_b) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + (8m+4)\delta \equiv \alpha_b + \alpha_c + \alpha_d + 2\alpha_x \pmod{\delta}$
7. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxb(\alpha_x) = \alpha_b + \alpha_d + \alpha_x + (4m+2)\delta \equiv \alpha_b + \alpha_d + \alpha_x \pmod{\delta}$
8. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxbx(\alpha_a) = \alpha_d + \alpha_x + (8m+5)\delta \equiv \alpha_d + \alpha_x \pmod{\delta}$
9. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxbxa(\alpha_x) = \alpha_a + \alpha_c + \alpha_d + 2\alpha_x + (4m+2)\delta \equiv \alpha_a + \alpha_c + \alpha_d + 2\alpha_x \pmod{\delta}$
10. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxbxax(\alpha_d) = \alpha_c + \alpha_d + \alpha_x + (8m+5)\delta \equiv \alpha_c + \alpha_d + \alpha_x \pmod{\delta}$
11. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxbxaxd(\alpha_x) = \alpha_b + \alpha_c + \alpha_d + \alpha_x + (4m+2)\delta \equiv \alpha_b + \alpha_c + \alpha_d + \alpha_x \pmod{\delta}$
12. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxbxaxdx(\alpha_c) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + (8m+5)\delta \equiv \alpha_b + \alpha_c + \alpha_d + 2\alpha_x \pmod{\delta}$
13. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxbxaxdxc(\alpha_x) = \alpha_x + (4m+3)\delta \equiv \alpha_x \pmod{\delta}$.
14. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxbxaxdxcx(\alpha_b) = \alpha_d + \alpha_x + (8m+6)\delta \equiv \alpha_d + \alpha_x \pmod{\delta}$
15. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxbxaxdxcxb(\alpha_x) = \alpha_d + (4m+3)\delta \equiv \alpha_d \pmod{\delta}$
16. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxbxaxdxcxbx(\alpha_a) = \alpha_c + \alpha_d + \alpha_x + (8m+6)\delta \equiv \alpha_c + \alpha_d + \alpha_x \pmod{\delta}$
17. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxbxaxdxcxbxa(\alpha_x) = \alpha_c + \alpha_d + (4m+3)\delta \equiv \alpha_c + \alpha_d \pmod{\delta}$
18. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxbxaxdxcxbxax(\alpha_d) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + (8m+6)\delta \equiv \alpha_b + \alpha_c + \alpha_d + 2\alpha_x \pmod{\delta}$
19. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxbxaxdxcxbxaxd(\alpha_x) = \alpha_b + \alpha_d + \alpha_x + (4m+3)\delta \equiv \alpha_b + \alpha_d + \alpha_x \pmod{\delta}$
20. $((\widehat{\boldsymbol{\varphi}}_{12})^{-1})^k xdxcxbxaxdxcxbxaxdx(\alpha_c) = \alpha_d + \alpha_x + (8m+7)\delta \equiv \alpha_d + \alpha_x \pmod{\delta}$

21. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbxaxdxc(\alpha_x) = \alpha_a + \alpha_c + \alpha_d + 2\alpha_x + (4m+3)\delta \equiv \alpha_a + \alpha_c + \alpha_d + 2\alpha_x \pmod{\delta}$

22. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbxaxdxcx(\alpha_b) = \alpha_c + \alpha_d + \alpha_x + (8m+7)\delta \equiv \alpha_c + \alpha_d + \alpha_x \pmod{\delta}$

23. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbxaxdxcxb(\alpha_x) = \alpha_b + \alpha_c + \alpha_d + \alpha_x + (4m+3)\delta \equiv \alpha_b + \alpha_c + \alpha_d + \alpha_x \pmod{\delta}$

24. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbxaxdxcxbx(\alpha_a) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + (8m+7)\delta \equiv \alpha_b + \alpha_c + \alpha_d + 2\alpha_x \pmod{\delta}$

Each computation follows from our earlier work. We include details for lines 2 and 3 below, i.e., prefixes of size 1 and 2. We see that

$$\begin{aligned}
((\widehat{\varphi}_{12})^{-1})^k x(\alpha_d) &= ((\widehat{\varphi}_{12})^{-1})^{2n+1} x(\alpha_d) \\
&= ((\widehat{\varphi}_{12})^{-1})^{2n}((\widehat{\varphi}_{12})^{-1})^1 x(\alpha_d) \\
&= ((\widehat{\varphi}_{24})^{-1})^{n}((\widehat{\varphi}_{12})^{-1}) x(\alpha_d) \\
&= ((\widehat{\varphi}_{24})^{-1})^{n}((\widehat{\varphi}_{12})^{-1})(\alpha_d + \alpha_x) \\
&= ((\widehat{\varphi}_{24})^{-1})^{n}(\alpha_d + 2\delta + \alpha_x + 2\delta) \\
&= ((\widehat{\varphi}_{24})^{-1})^{n}(\alpha_d + \alpha_x + 4\delta) \\
&= \alpha_d + 4n\delta + \alpha_x + 4n\delta + 4\delta \\
&= \alpha_d + \alpha_x + (8n+4)\delta \\
&\equiv \alpha_d + \alpha_x \pmod{\delta}
\end{aligned}$$

while

$$\begin{aligned}
((\widehat{\varphi}_{12})^{-1})^k xd(\alpha_x) &= ((\widehat{\varphi}_{12})^{-1})^{2n+1} xd(\alpha_x) \\
&= ((\widehat{\varphi}_{12})^{-1})^{2n}((\widehat{\varphi}_{12})^{-1})^1 xd(\alpha_x) \\
&= ((\widehat{\varphi}_{24})^{-1})^{n}((\widehat{\varphi}_{12})^{-1}) xd(\alpha_x) \\
&= ((\widehat{\varphi}_{24})^{-1})^{n}((\widehat{\varphi}_{12})^{-1}) x(\alpha_d + \alpha_x) \\
&= ((\widehat{\varphi}_{24})^{-1})^{n}((\widehat{\varphi}_{12})^{-1})(\alpha_d) \\
&= ((\widehat{\varphi}_{24})^{-1})^{n}(\alpha_d + 2\delta) \\
&= ((\widehat{\varphi}_{24})^{-1})^{n}(\alpha_d) + ((\widehat{\varphi}_{24})^{-1})^{n}(2\delta) \\
&= \alpha_d + 4n\delta + 2\delta \\
&= \alpha_d + (4n+2)\delta \\
&\equiv \alpha_d \pmod{\delta}.
\end{aligned}$$

Now, assume k is even. Then $k = 2n$ for some $n \in \mathbb{Z}_{\geq 0}$. We claim that we obtain the following tail of the root sequence for $\boldsymbol{\varphi}_i = \boldsymbol{\varphi}_{12(k+1)}$. We list the even cases of the 24 intermediate prefix sizes below.

1. $((\widehat{\varphi}_{12})^{-1})^k(\alpha_x) = \alpha_x + 2m\delta \equiv \alpha_x \pmod{\delta}$
2. $((\widehat{\varphi}_{12})^{-1})^k x(\alpha_d) = \alpha_d + \alpha_x + 8m\delta \equiv \alpha_d + \alpha_x \pmod{\delta}$
3. $((\widehat{\varphi}_{12})^{-1})^k xd(\alpha_x) = \alpha_d + 4m\delta \equiv \alpha_d \pmod{\delta}$
4. $((\widehat{\varphi}_{12})^{-1})^k xdx(\alpha_c) = \alpha_c + \alpha_d + \alpha_x + 8m\delta \equiv \alpha_c + \alpha_d + \alpha_x \pmod{\delta}$
5. $((\widehat{\varphi}_{12})^{-1})^k xdxc(\alpha_x) = \alpha_c + \alpha_x + 4m\delta \equiv \alpha_c + \alpha_x \pmod{\delta}$
6. $((\widehat{\varphi}_{12})^{-1})^k xdxcx(\alpha_b) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + 8m\delta \equiv \alpha_b + \alpha_c + \alpha_d + 2\alpha_x \pmod{\delta}$
7. $((\widehat{\varphi}_{12})^{-1})^k xdxcxb(\alpha_x) = \alpha_b + \alpha_d + \alpha_x + 4m\delta \equiv \alpha_b + \alpha_d + \alpha_x \pmod{\delta}$
8. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbx(\alpha_a) = \alpha_d + \alpha_x + (8m+1)\delta \equiv \alpha_d + \alpha_x \pmod{\delta}$
9. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxa(\alpha_x) = \alpha_a + \alpha_c + \alpha_d + 2\alpha_x + 4m\delta \equiv \alpha_a + \alpha_c + \alpha_d + 2\alpha_x \pmod{\delta}$
10. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxax(\alpha_d) = \alpha_c + \alpha_d + \alpha_x + (8m+1)\delta \equiv \alpha_c + \alpha_d + \alpha_x \pmod{\delta}$
11. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxd(\alpha_x) = \alpha_b + \alpha_c + \alpha_d + \alpha_x + 4m\delta \equiv \alpha_b + \alpha_c + \alpha_d + \alpha_x \pmod{\delta}$
12. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdx(\alpha_c) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + (8m+1)\delta \equiv \alpha_b + \alpha_c + \alpha_d + 2\alpha_x \pmod{\delta}$
13. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxc(\alpha_x) = \alpha_x + (4m+1)\delta \equiv \alpha_x \pmod{\delta}$.
14. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcx(\alpha_b) = \alpha_d + \alpha_x + (8m+2)\delta \equiv \alpha_d + \alpha_x \pmod{\delta}$
15. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxb(\alpha_x) = \alpha_d + (4m+1)\delta \equiv \alpha_d \pmod{\delta}$
16. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbx(\alpha_a) = \alpha_c + \alpha_d + \alpha_x + (8m+2)\delta \equiv \alpha_c + \alpha_d + \alpha_x \pmod{\delta}$
17. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbxa(\alpha_x) = \alpha_c + \alpha_d + (4m+1)\delta \equiv \alpha_c + \alpha_d \pmod{\delta}$
18. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbxax(\alpha_d) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + (8m+2)\delta \equiv \alpha_b + \alpha_c + \alpha_d + 2\alpha_x \pmod{\delta}$
19. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbxaxd(\alpha_x) = \alpha_b + \alpha_d + \alpha_x + (4m+1)\delta \equiv \alpha_b + \alpha_d + \alpha_x \pmod{\delta}$
20. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbxaxdx(\alpha_c) = \alpha_d + \alpha_x + (8m+3)\delta \equiv \alpha_d + \alpha_x \pmod{\delta}$

21. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbxaxdxc(\alpha_x) = \alpha_a + \alpha_c + \alpha_d + 2\alpha_x + (4m+1)\delta \equiv \alpha_a + \alpha_c + \alpha_d + 2\alpha_x \pmod{\delta}$

22. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbxaxdxcx(\alpha_b) = \alpha_c + \alpha_d + \alpha_x + (8m+3)\delta \equiv \alpha_c + \alpha_d + \alpha_x \pmod{\delta}$

23. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbxaxdxcxb(\alpha_x) = \alpha_b + \alpha_c + \alpha_d + \alpha_x + (4m+1)\delta \equiv \alpha_b + \alpha_c + \alpha_d + \alpha_x \pmod{\delta}$

24. $((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbxaxdxcxbx(\alpha_a) = \alpha_b + \alpha_c + \alpha_d + 2\alpha_x + (8m+3)\delta$

As with the odd case, we include details for lines 2 and 3 below, which correspond to prefix of sizes 1 and 2, respectively. Observe that

$$\begin{aligned}
((\widehat{\varphi}_{12})^{-1})^k x(\alpha_d) &= ((\widehat{\varphi}_{12})^{-1})^{2n} x(\alpha_d) \\
&= ((\widehat{\varphi}_{24})^{-1})^n x(\alpha_d) \\
&= ((\widehat{\varphi}_{24})^{-1})^n (\alpha_d + \alpha_x) \\
&= ((\widehat{\varphi}_{24})^{-1})^n (\alpha_d) + ((\widehat{\varphi}_{24})^{-1})^n (\alpha_x) \\
&= \alpha_d + 4n\delta + \alpha_x + 4n\delta \\
&= \alpha_d + \alpha_x + 8n\delta \\
&\equiv \alpha_d + \alpha_x \pmod{\delta}
\end{aligned}$$

while,

$$\begin{aligned}
((\widehat{\varphi}_{12})^{-1})^k xd(\alpha_x) &= ((\widehat{\varphi}_{12})^{-1})^{2n} xd(\alpha_x) \\
&= ((\widehat{\varphi}_{24})^{-1})^n x(\alpha_d + \alpha_x) \\
&= ((\widehat{\varphi}_{24})^{-1})^n (\alpha_d) \\
&= \alpha_d + 4n\delta \\
&\equiv \alpha_d \pmod{\delta}.
\end{aligned}$$

Note that if we were to consider a prefix of size 25, we would have

$$((\widehat{\varphi}_{12})^{-1})^k xdxcxbxaxdxcxbxaxdxcxbxa(\alpha_x) = ((\widehat{\varphi}_{12})^{-1})^k (\widehat{\varphi}_{12})^{-1} \equiv \alpha_x \pmod{\delta}.$$

This case would match our computation on line 1 but with a change to the coefficient on δ. The rest of the root sequence behaves similarly. We see that in either case of when k is odd or even, all roots are positive. Thus, by Proposition 1.10, φ_{12k} is reduced for any k. Therefore, since every φ_i is a consecutive subexpression of some φ_{12k}, it follows that φ_i is reduced for all $i \geq 1$.

□

We identify a pattern in the root sequence for $((\widehat{\varphi}_{12})^{-1})^{k+1}$ modulo δ. This pattern is represented by the 6 distinct colors in each list of when k is odd or even. Every time our prefix size increases by 6, we repeat root values modulo δ in two forms. If we are feeding an expression α_x, represented by the *blue*, *black*, and *orange* colors, we repeat the same root values modulo δ. If we are feeding an expression α_i, where $i \in \{a, b, c, d\}$, then the expressions alternate between two distinct root values, represented by the colors *red*, *green*, and *purple*. For example, starting with a prefix of size 0 (highlighted in *red*), every time we increase our prefix by 6, we alternate between the root values α_x and $\alpha_b + \alpha_d + \alpha_x$ modulo δ. However, if we consider the initial prefix of size 1, every time we increase our prefix by 6, we repeat the root value $\alpha_d + \alpha_x$ modulo δ. Regardless of whether k is odd or even, we have the same pattern.

Since $\boldsymbol{\varphi}_i$ is reduced for all $i \geq 1$, $\boldsymbol{\varphi}_i$ is a Fibonacci link of rank i such that $i \geq 1$, i.e., in the Coxeter system of type $\widetilde{D}_4$, Fibonacci links exist for any rank size. The previous result together with Proposition 3.7 immediately implies the following.

Corollary 4.5. In the simply-laced triangle-free Coxeter system of type $\widetilde{D}_4$, there exists a Fibonacci link of rank m for all $m \geq 1$.

Suppose (W, S) is a simply-laced triangle-free Coxeter system whose Coxeter graph has the Coxeter system of type $\widetilde{D}_4$ as a subgraph. Since each $\boldsymbol{\varphi}_m$ is reduced, after relabeling generators, the corresponding expression is reduced in W. This yields the following, which is the main result of this thesis.

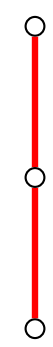
Theorem 4.6. If (W, S) is a simply-laced Coxeter system whose Coxeter graph has a vertex of degree 4, then for each $m \geq 1$, there exists a Fibonacci link of rank m whose braid graph is isomorphic to the Fibonacci cube $\mathcal{F}_m$.

Proposition 3.7 provides us with a characterization of the braid graphs for Fibonacci links in simply-laced triangle-free Coxeter systems, while Theorem 4.6 indicates a class of Coxeter systems for which every possible Fibonacci cube is realized as a braid graph for some Fibonacci link. Moreover, since we know that there exists an isomorphism between the braid graph of a Fibonacci link of rank m and $\mathcal{F}_m$, as we increase rank from a Fibonacci link of rank m to rank $m+1$, it turns out that the corresponding braid graph is composed of the braid graphs of a Fibonacci link of rank m and $m-1$, by "gluing" the braid graphs together, using the recursive structure of Fibonacci cubes. Earlier, we explained how we go from a Fibonacci link of rank m, denoted $\boldsymbol{\varphi}_m$, to a rank of $m+1$ via appending xt_{m+1} to the left of $\boldsymbol{\varphi}_m$. Appending xt_{m+1} functions as gluing the braid graphs of $\boldsymbol{\varphi}_{m-1}$ and $\boldsymbol{\varphi}_m$ to obtain the braid graph of $\boldsymbol{\varphi}_{m+1}$. Figure 4.2 gives examples of this gluing process for Fibonacci links up to rank 6. We use the symbol $\oplus$ to represent the gluing of the two previous braid graphs. We also represent how these pieces are glued together via dashed lines.

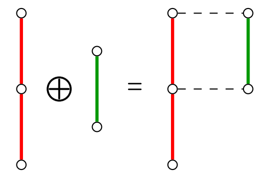
The Fibonacci cube is not the only interesting graph that arises as the braid graph for a reduced expression. For instance, the *matchable Lucas cubes* introduced in [17] can be found in the Coxeter system of type D_5 as seen in the following example. The matchable Lucas cubes are very similar to the Fibonacci cubes, for example, the number of vertices in the nth matchable Lucas cube is equal to the nth number L_n in the sequence of Lucas numbers defined by $L_n = L_{n-1} + L_{n-2}$ with initial conditions $L_0 = 2$ and $L_1 = 1$.

Example 4.7. Consider the reduced expression 4534313234313 for $w \in W(D_5)$ from Example 3.8. The braid graph, depicted in Figure 4.3(c), is isomorphic to the 6th matchable Lucas cube as introduced in [17]. In fact, it is not hard to see that the braid graph is a "gluing together" (depicted with dashed edges) of the braid graphs for the reduced expressions 45343132343 and 453431323. The braid graph for 45343132343 is shown in Figure 4.3(b) and is isomorphic to the 5th matachable Lucas cube, while the braid graph for 453431323 in Figure 4.3(a) is isomorphic to the 4th matachable Lucas cube. Moreover, observe that 4534313234313 has 34313234313 as a consecutive subword, which is a Fibonacci link. We can think of the original reduced expression as a "type A extension" of a Fibonacci link.

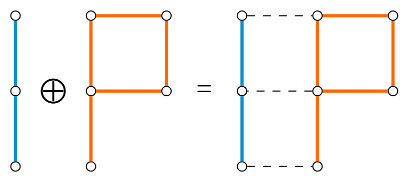
(a) $B(xdx) \cong \mathcal{F}_1$



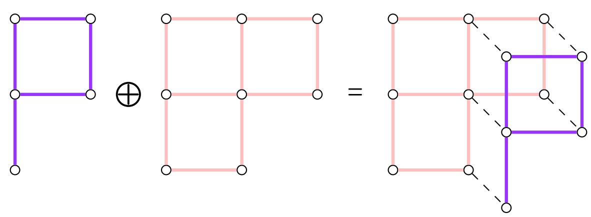
(b) $B(xdx) \cong \mathcal{F}_2$



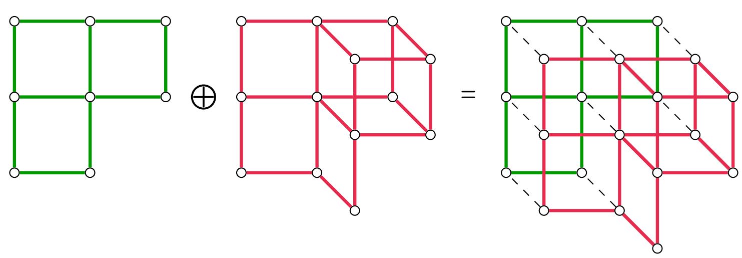
(c) $B(xbxcxdx) \cong \mathcal{F}_3$



(d) $B(xaxbxcxdx) \cong \mathcal{F}_4$

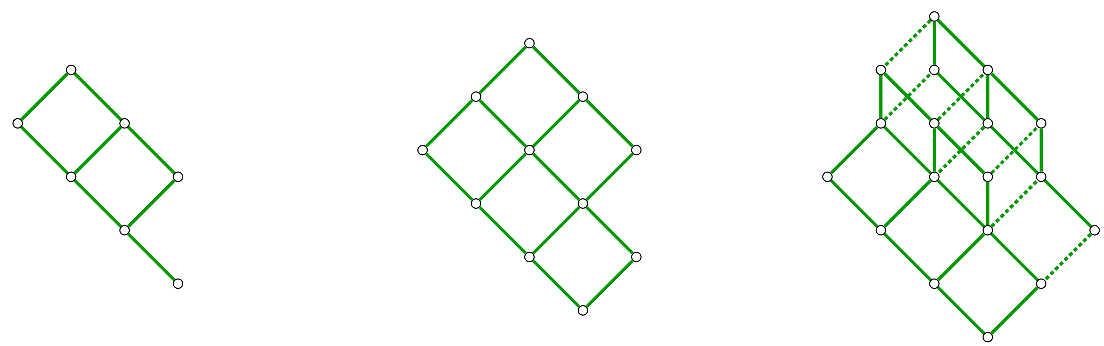


(e) $B(xdxaxbxcxdx) \cong \mathcal{F}_5$



(f) $B(xcxdxaxbxcxdx) \cong \mathcal{F}_6$

Figure 4.2: Braids graphs for several Fibonacci links in the Coxeter system of type $\widetilde{D}_4$.



(a) $B(453431323)$ (b) $B(45343132343)$ (c) $B(4534313234313)$

Figure 4.3: Braid graphs for the reduced expressions in Example 4.7.

Chapter 5

Conclusion

The relationship among the reduced expressions in a fixed braid class can be encoded in a graph. We define the braid graph for a reduced expression to be the graph with vertex set equal to the corresponding braid class, where two vertices are connected by an edge if and only if the corresponding reduced expressions are related via a single braid move. In [1], the authors proved that in simply-laced Coxeter systems every reduced expression has a unique factorization into links, which then yields a decomposition of the corresponding braid graph into a box product of the braid graphs for each link factor. Moreover, the authors proved that for a special class of links, called Fibonacci links, the corresponding braid graph is isomorphic to a Fibonacci cube graph. In this thesis, we proved that for any simply-laced Coxeter system whose corresponding Coxeter graph has a vertex of degree 4, there exist infinitely many Fibonacci links. In particular, we proved that for each $m \geq 1$, there exists a Fibonacci link of rank m whose braid graph is isomorphic to the Fibonacci cube $\mathcal{F}_m$.

We conclude by presenting a few open problems:

- Characterize Fibonacci links in simply-laced Coxeter systems in which each vertex of the Coxeter graph has degree less than or equal to 3.
- Characterize reduced expressions whose braid graphs are realized as generalized Lucas cubes in simply-laced Coxeter systems.
- Provide a classification of braid graphs for links in arbitrary simply-laced Coxeter systems (e.g., type $\widetilde{A}_n$).
- Generalize the notions of braid shadow, link, and braid chain to arbitrary Coxeter systems and obtain analogous results to the simply-laced case.

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