PATTERN AVOIDANCE IN CAYLEY PERMUTATIONS

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ABSTRACT

PATTERN AVOIDANCE IN CAYLEY PERMUTATIONS

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Any permutation of n may be written in one-line notation as a sequence of entries representing the result of applying the permutation to the sequence $12\cdots n$. If p and q are two permutations, then p is said to contain q as a pattern if some subsequence of the entries of p has the same relative order as all of the entries of q. If p does not contain a pattern q, then p is said to avoid q. One of the first notable results in the field of permutation patterns was obtained by MacMahon in 1915 when he proved that the ubiquitous Catalan numbers count the 123-avoiding permutations. The study of permutation patterns began receiving focused attention following Knuth's introduction of stack-sorting in 1968. Knuth proved that a permutation can be sorted by a stack if and only if it avoids the pattern 231 and that the Catalan numbers also enumerate the stack-sortable permutations. In the subsequent years, the notion of pattern avoidance has been extended to numerous combinatorial objects, including multiset permutations, set partitions, ordered set partitions, compositions, and modified ascent sequences. In this thesis, we study pattern avoidance in the context of Cayley permutations, which were introduced by Mor and Fraenkel in 1983. A Cayley permutation is a finite sequence of positive integers that include at least one copy of each integer between one and its maximum value. When possible we will take a combinatorial species-first approach to enumerating Cayley permutations that avoid patterns of length two, pairs of patterns of length two, patterns of length three, and pairs of patterns of length three with the goal of providing species, exponential generating functions, and counting formulas. We also briefly study pattern avoidance in a special class of Cayley permutations known as primitive Cayley permutations. Throughout the thesis, we include several conjectures and open problems. The majority of the results of this thesis are new and were obtained in collaboration with G. Cerbai, A. Claesson, and my advisor D.C. Ernst.

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Chapter 1

Introduction to combinatorial species

The goal of this thesis is to analyze certain collections of Cayley permutations. We will often use combinatorial species to derive exponential generating functions for these subsets, allowing us to enumerate them and establish bijections involving the corresponding structures.

In this chapter, we mimic the development in [1] and [4]. We utilize two different types of species, namely $\mathbb{B}$-species and $\mathbb{L}$-species. We will first introduce the more general $\mathbb{B}$-species.

1.1 $\mathbb{B}$-species

We will look at an example involving graphs to build intuition. First, recall that an isomorphism of simple graphs (U, E) and (V, E') is a bijection $\sigma : U \to V$ that preserves adjacency relations. This means $\{x, y\} \in E$ if and only if $\{\sigma(x), \sigma(y)\} \in E'$.

Example 1.1. Figure 1.1 depicts a simple graph with vertex set $U = \{a, b, c, d, e\}$.

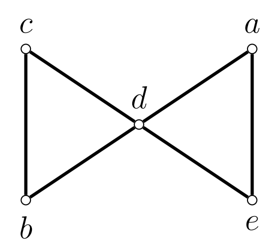

Figure 1.1: An example of a simple graph.

In general, a *species* defines both a class of (labeled) combinatorial objects and how those objects are impacted by relabeling. This mechanism of relabeling is called the *transport of structure*.

Given a finite set U, define $G[U]$ to be the set of simple graphs on U. For a bijection $\sigma : U \to V$, the transport of structure is $G[\sigma] : G[U] \to G[V]$ given by $G[\sigma](U, E) = (V, \sigma(E))$.

Example 1.2. Let $U = \{a, b, c, d, e\}$, $V = \{1, 2, 3, 4, 5\}$, and define $\sigma = \left(\begin{smallmatrix} a & b & c & d & e \\ 3 & 1 & 4 & 5 & 2 \end{smallmatrix}\right)$. Then Figure 1.2 illustrates the behavior of the corresponding transport of structure on the graph from Example 1.1.

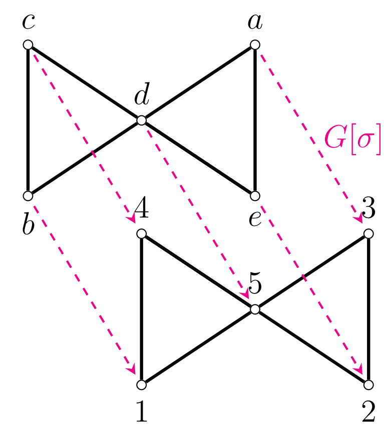

Figure 1.2: Transport of structure for a graph.

In terms of transport of structure, two graphs $S \in G[U]$ and $T \in G[V]$ are *isomorphic* if there exists a bijection $\sigma : U \to V$ such that $G[\sigma](S) = T$. It can be verified that for all bijections $\sigma : U \to V$ and $\tau : V \to W$ between vertex sets, the following hold:

- $G[\sigma \circ \tau] = G[\sigma] \circ G[\tau]$;
- $G[\mathrm{id}_U] = \mathrm{id}_{G[U]}$,

where id_U is the identity map on U and $\mathrm{id}_{G[U]}$ is the identity map on $G[U]$.

A $\mathbb{B}$-*species* (or simply *species*) F is a rule that produces

(i) for each finite set U, a finite set $F[U]$;

(ii) for each bijection $\sigma : U \to V$, a function $F[\sigma] : F[U] \to F[V]$,

where $F[\sigma \circ \tau] = F[\sigma] \circ F[\tau]$ for all bijections $\sigma : U \to V$, $\tau : V \to W$, and $F[\mathrm{id}_U] = \mathrm{id}_{F[U]}$ for the identity map $\mathrm{id}_U : U \to U$. These are visually represented in Figure 1.3.

An element $s \in F[U]$ is called an F-*structure* on U and the function $F[\sigma]$ is called the *transport of F-structures along σ*, or simply *transport of structure* if the context is clear. Two F-structures $s \in F[U]$ and $t \in F[V]$ are *isomorphic*, denoted $s \sim t$, if there exists a bijection $\sigma : U \to V$ such that $F[\sigma](s) = t$.

In the language of category theory, a $\mathbb{B}$-species is a functor $F : \mathbb{B} \to \mathbb{B}$, where $\mathbb{B}$ is the category of sets with bijective functions as morphisms.

We have already seen that G is the species of finite simple graphs. Below we define several species that will be utilized throughout this thesis. For many of these, we will elaborate on the corresponding F-structure and transport of structure in upcoming examples.

$$
\begin{tikzcd}
U \arrow[r, "\sigma"] \arrow[d, "F"'] \arrow[rr, bend left, "\tau \circ \sigma"] & V \arrow[r, "\tau"] \arrow[d, "F"] & W \arrow[d, "F"] \\
F[U] \arrow[r, "F[\sigma]"] \arrow[rr, bend right, "F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]"'] & F[V] \arrow[r, "F[\tau]"] & F[W]
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}
U \arrow[r, "\mathrm{id}_U"] \arrow[d, "F"'] & U \arrow[d, "F"] \\
F[U] \arrow[r, "F[\mathrm{id}_U] = \mathrm{id}_{F[U]}"'] & F[U]
\end{tikzcd}
$$

(b)

Figure 1.3: Rules for transport of structure.

(a) E: species of sets;

(b) E_+: species of nonempty sets;

(c) E_n: species of sets of cardinality n;

(d) E_{even}: species of sets of even cardinality;

(e) E_{odd}: species of sets of odd cardinality;

(f) X: species of singletons;

(g) 1: species of the characteristic of the empty set;

(h) L: species of linear orders;

(i) L_+: species of nonempty linear orders;

(j) S: species of permutations;

(k) Par: species of set partitions;

(l) Bal: species of ballots (i.e., ordered set partitions);

(m) Der: species of derangements.

Throughout this thesis, we define $[n] := \{1, 2, \ldots, n\}$ for ease of notation.

Example 1.3. We describe the E-structures and transport of structure for the species of sets E. The E-structures are defined by $E[U] := \{U\}$ and the transport of structure for a bijection $\sigma : U \to V$ is given by $E[\sigma](U) = V$. Note that $|E[U]| = 1$ for all finite sets U.

Example 1.4. We will address the L-structures and transport of structure for the species of linear orders L. Recall that a linear order of a set U with $|U| = n$ is a bijection $f : [n] \to U$, which we may represent using 1-line notation: $f(1) \cdots f(n)$. The L-structures are defined by $L[U] := \{f : [n] \to U \mid f \text{ bijection}\}$ such that $|U| = n$. The transport of structure for a bijection $\sigma : U \to V$ is given by $L[\sigma](f) = \sigma \circ f = \sigma(f(1)) \cdots \sigma(f(n))$ since $[n] \xrightarrow{f} U \xrightarrow{\sigma} V$ is a bijection. Note that there are $n!$ linear orders on U when $|U| = n$, which implies that $|L[U]| = n!$.

Example 1.5. We now describe the S-structures and transport of structure for the species of permutations S. The S-structures are defined by $S[U] := \{f : U \to U \mid f \text{ bijection}\}$ while the corresponding transport of structure for a bijection $\sigma : U \to V$ is given by $S[\sigma](f) = \sigma \circ f \circ \sigma^{-1}$ since $V \xrightarrow{\sigma^{-1}} U \xrightarrow{f} U \xrightarrow{\sigma} V$ is a bijection. This reflects the fact that conjugation preserves the cycle type of a permutation. Note that $|S[U]| = n!$ when $|U| = n$.

To aid in the enumeration of F-structures, we associate an exponential generating function, denoted by $F(x)$. For all finite sets U, the number of F-structures on U depends only on the number of elements of U instead of the specific elements in U. For ease of notation, we use $F[n] := F[[n]]$. The cardinalities $|F[U]|$ are characterized by the sequence of values, $f_n := |F[n]|$ for $n \geq 0$.

We define the *exponential generating function* of the species F to be the formal power series

$$F(x) := \sum_{n \geq 0} f_n \frac{x^n}{n!},$$

and the corresponding *ordinary generating function* is defined via

$$\hat{F}(x) := \sum_{n \geq 0} f_n x^n.$$

The exponential generating functions associated with some of the species introduced above are provided below. These results appear in [1].

Proposition 1.6.

(a) $E(x) = e^x$;

(b) $E_+(x) = e^x - 1$;

(c) $E_n(x) = \dfrac{x^n}{n!}$;

(d) $E_{\text{even}}(x) = \cosh(x)$;

(e) $E_{\text{odd}} = \sinh(x)$;

(f) $X(x) = x$;

(g) $1(x) = 1$;

(h) $L(x) = \dfrac{1}{1-x}$;

(i) $L_+(x) = \dfrac{x}{1-x}$;

(j) $S(x) = \dfrac{1}{1-x}$;

(k) $\text{Par}(x) = e^{e^x - 1}$;

(l) $\text{Bal}(x) = \dfrac{1}{2-e^x}$;

(m) $\text{Der}(x) = \dfrac{e^{-x}}{1-x}$.

We now look at examples of species together with their corresponding exponential generating functions.

Example 1.7. Recall the species L (linear orders) and S (permutations) from Examples 1.4 and 1.5. Despite L and S being different species, it should come as no surprise that they have the same exponential generating function. Since $|L[n]| = n! = |S[n]|$, we have

$$S(x) = L(x) = \sum_{n\geq 0} |L[n]| \frac{x^n}{n!} = \sum_{n\geq 0} n! \frac{x^n}{n!} = \sum_{n\geq 0} x^n = \frac{1}{1-x}.$$

The previous example illustrates that we may have $F(x) = G(x)$ despite F and G being different species.

Example 1.8. We define the 1-structures for the species 1, the characteristic of the empty set. The 1-structures are given by

$$1[U] := \begin{cases} \{U\}, & U = \emptyset \\ \emptyset, & \text{otherwise.} \end{cases}$$

It follows that $1(x) = 1$.

$$
\begin{tikzcd}
F[U] \arrow[r, "\alpha_U"] \arrow[d, "F[\sigma]"'] & G[U] \arrow[d, "G[\sigma]"] \\
F[V] \arrow[r, "\alpha_V"'] & G[V]
\end{tikzcd}
$$

Figure 1.4: Commutative diagram for isomorphic species.

Example 1.9. The E_+-structures for the species E_+ of nonempty sets are given by

$$E_+[U] := \begin{cases} \{U\}, & U \neq \emptyset \\ \emptyset, & U = \emptyset. \end{cases}$$

It follows that $E_+(x) = \displaystyle\sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1$.

A natural question to ask is when two species are considered the same combinatorially. Species F and G are called *equipotent*, denoted $F \equiv G$, if and only if there is the same number of F-structures as G-structures for each $|U| = n$. The following obvious result appears in [1].

Proposition 1.10. For $\mathbb{B}$-species F and G, $F \equiv G$ if and only if $F(x) = G(x)$.

Species F and G are *isomorphic*, written $F \cong G$, if there is a family of bijections $\alpha_U : F[U] \to G[U]$ that satisfies: for any bijection $\sigma : U \to V$ between two finite sets, the diagram in Figure 1.4 commutes. That is, $\alpha_V \circ F[\sigma] = G[\sigma] \circ \alpha_U$. In the language of category theory, $F \cong G$ if and only if there exists a natural isomorphism between the functors F and G. For ease of notation, we write $F = G$ as species when $F \cong G$. This is standard in the literature. The next result appears in [1].

Proposition 1.11. For $\mathbb{B}$-species F and G, if $F = G$, then $F(x) = G(x)$.

It is important to note that the converse of the previous proposition is not true.

Example 1.12. Recall Examples 1.4 and 1.5 with the species L (linear orders) and S (permutations). These two species are not isomorphic even though they are equipotent and their exponential generating functions are equal.

One can build new species with operations on previously known species. The first operation we will explore is addition. For species F and G, an $(F+G)$*-structure* is either an F-structure or G-structure. That is, $(F+G)[U] := F[U] \bigsqcup G[U]$ and for all bijections $\sigma : U \to V$,

$$(F+G)[\sigma](s) := \begin{cases} F[\sigma](s), & s \in F[U] \\ G[\sigma](s), & s \in G[U]. \end{cases}$$

It turns out that $(F+G)(x) = F(x) + G(x)$.

We now look at an example of the addition of known species.

Example 1.13. It follows from Examples 1.8 and 1.9 that the species of sets is $E = 1 + E_+$ since every set is either an empty set or a nonempty set. In terms of the exponential generating functions, we have

$$E(x) = e^x = 1 + \sum_{n\geq 1} \frac{x^n}{n!} = 1(x) + E_+(x),$$

as expected.

Next we define the multiplication of species and provide an example. For species F and G, we define an $(F \cdot G)$-*structure* on a set U to be a pair (s,t) such that s is an F-structure on a subset $U_1 \in U$ and t is a G-structure on $U_2 = U \backslash U_1$. Formally,

$$(F \cdot G)[U] := \bigsqcup_{(U_1, U_2)} F[U_1] \times G[U_2]$$

with $U = U_1 \sqcup U_2$, and the transport of structure is defined by

$$(F \cdot G)[\sigma](s,t) := (F[\sigma_1](s), G[\sigma_2](t)),$$

where $\sigma_1 = \sigma|_{U_1}$ and $\sigma_2 = \sigma|_{U_2}$. It is also true that $(F \cdot G)(x) = F(x) \cdot G(x)$. While the addition and multiplication of species are associative and commutative up to isomorphism, $F \cdot G$ is not equal to $G \cdot F$, in general.

We now look at an example.

Example 1.14. For the singleton species $X = E_1$, we have

$$X[U] := \begin{cases} \{U\}, & |U| = 1 \\ \emptyset, & \text{otherwise.} \end{cases}$$

It follows that $X(x) = E_1(x) = x$. The claim is that for the species of linear orders, we have $L = 1 + X \cdot L$. Combinatorially, a linear order is either empty or consists of a first element followed by its remaining elements. In terms of exponential generating functions, we have

$$1 + x \cdot L(x) = 1 + \frac{x}{1-x} = \frac{1-x+x}{1-x} = \frac{1}{1-x} = L(x).$$

Now we define the composition of species. An $(F \circ G)$-*structure* is a generalized partition in which each block of a partition carries a G-structure and blocks are structured by F. Formally, if F and G are two species such that $G[\emptyset] = \emptyset$, we define

$$(F \circ G)[U] := \bigsqcup_{\beta = \{\beta_1, \ldots, \beta_k\}} F[\beta] \times G[\beta_1] \times \cdots G[\beta_k],$$

where $\beta = \{\beta_1, \ldots, \beta_k\}$ is a partition of U. The details on the transport of structure can be found in Chapter 1 of [1]. It is also true that $(F \circ G)(x) = F(G(x))$.

Example 1.15. We will now look at the species of ballots. We will encounter ballots throughout this thesis and make connections to our topic of interest, Cayley permutations, in Chapter 2. A *ballot* on a finite set U is an ordered set partition of U, denoted by $(B_1, B_2, \ldots, B_k)$, where each B_i is a nonempty subset of U, $B_i \cap B_j \neq \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{k} B_i = U$. Each B_i is referred to as a *block*. The collection of ballots on U is denoted by $\mathrm{Bal}[U]$. The species Bal has structures $\mathrm{Bal}[U]$ and for a bijection $\sigma : U \to V$, the transport of structure $\mathrm{Bal}[\sigma] : \mathrm{Bal}[U] \to \mathrm{Bal}[V]$ is given by

$$\mathrm{Bal}[\sigma](B_1, \ldots, B_k) = (\sigma(B_1), \ldots, \sigma(B_k)).$$

If $U = [n]$, we write $\mathrm{Bal}_n := \mathrm{Bal}[n]$. Since every ordered set partition is a linear order of nonempty sets, it follows that $\mathrm{Bal} = L(E_+)$. We see that

$$\mathrm{Bal}(x) = L(E_+(x)) = \frac{1}{1 - (e^x - 1)} = \frac{1}{2 - e^x}.$$

Next, we look at derivatives of species. Recall that for a species F,

$$F(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}.$$

This implies that

$$\begin{aligned} \frac{d}{dx}[F(x)] &= \sum_{n \geq 0} n \cdot f_n \frac{x^{n-1}}{n!} \\ &= \sum_{n \geq 1} f_n \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n \geq 0} f_{n+1} \frac{x^n}{n!}. \end{aligned}$$

For $G(x) = \displaystyle\sum_{n \geq 0} g_n \frac{x^n}{n!}$, we have $G(x) = F'(x)$ if and only if $g_n = f_{n+1}$ for $n \geq 0$. This motivates the following. The *derivative* of a species F, denoted F', is defined via

$$F'[U] := F[U \sqcup \{\star\}]$$

and for a bijective map $\sigma : U \to V$, we define $F'[\sigma] := F[\tau]$, where

$$\tau[x] = \begin{cases} \sigma(x), & x \in U \\ \star, & x = \star. \end{cases}$$

It turns out that in terms of exponential generating functions, we have $F'(x) = \frac{d}{dx}[F(x)]$.

Example 1.16. To illustrate the derivative of a species, we look at the derivative of the species of linear orders. We claim that $L' = L^2$. Combinatorially, each L'-structure is simply an L-structure preceding $\star$ followed by another L-structure. That is, the derivative of a linear order just separates the linear order into two linearly ordered components. More concretely, consider $U = [6]$ and $w = 41 \star 5263 \in L'[U]$. This would transfer to $L^2[U]$ as an ordered pair (s,t) where $s = 41$ and $t = 5263$. We see that

$$L'(x) = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2} = L^2(x).$$

The last operation we will define for $\mathbb{B}$-species is *pointing*. For a species F, we define the species $F^\bullet$, called *F-pointed*, via

$$F^\bullet[U] := F[U] \times U.$$

That is, an *$F^\bullet$-structure* on U is a pair (s,u), where s is an F-structure on U and $u \in U$ is a distinguished element that we can think of as being "pointed at". The operations of pointing and derivation are related by

$$F^\bullet = X \cdot F',$$

where the distinguished element in the $F^\bullet$-structure is replaced by $\star$. Further, we have $|F^\bullet[n]| = n|F[n]|$, which implies that

$$F^\bullet(x) = \sum_{n\geq 0} n \cdot f_n \frac{x^n}{n!} = x\frac{d}{dx}[F(x)] = x \cdot F'(x).$$

For more details on the transport of structure for $F^\bullet$, see Chapter 1 of [1]. We will see an example of this operation later in this thesis.

Above we stated how each operation on species translated to the corresponding exponential generating functions. We collect these results from [1] below.

Proposition 1.17. If F and G are species, then

(a) $(F+G)(x) = F(x) + G(x)$;

(b) $(F \cdot G)(x) = F(x) \cdot G(x)$;

(c) $(F \circ G)(x) = F(G(x))$;

(d) $F'(x) = \frac{d}{dx}[F(x)]$;

(e) $F^\bullet(x) = x\frac{d}{dx}[F(x)]$.

1.2 $\mathbb{L}$-species

Now we look at $\mathbb{L}$-species, a specialization of $\mathbb{B}$-species. The key difference is that for $\mathbb{L}$-species, the underlying sets are totally ordered. Recall that a finite *totally ordered set* is a pair $l = (U, \preceq)$, where U is a finite set and $\preceq$ is a total order on U. In other words, $\preceq$ is an L-structure on U. We write $u \prec v$ if $u \preceq v$ and $u \neq v$.

The (*ordinary*) *sum* of two totally ordered sets $l_1 = (U_1, \preceq_1)$ and $l_2 = (U_2, \preceq_2)$ is the unordered set $U = U_1 \sqcup U_2$ that results from taking the disjoint union of U_1 and U_2. The *ordinal sum* of l_1 and l_2 is the totally ordered set $l = (U, \preceq)$, denoted by $l = l_1 \oplus l_2$, where

$$u \prec_l v \iff \begin{cases} u \prec_1 v, \text{when } u, v \in U_1, \\ u \in U_1 \text{ and } v \in U_2, \\ u \prec_2 v, \text{when } u, v \in U_2. \end{cases}$$

In other words, l respects l_1 and l_2 and all elements of l_1 are smaller than the elements of l_2. The totally ordered set obtained by adding a new minimum element to l is denoted by $1 \oplus l$.

A function $f : l_1 \to l_2$ between two totally ordered sets l_1 and l_2 is (*strict*) *order preserving* if $u \prec_{l_1} v$ implies $f(u) \prec_{l_2} f(v)$ for all u and v in l_1. Note that every order preserving map is injective but not necessarily surjective. If $f : l_1 \to l_2$ is an order preserving bijection, then f is the unique order preserving bijection between l_1 and l_2.

An $\mathbb{L}$-species is a rule F that associates

(i) for each finite totally ordered set l, a finite set $F[l]$;

(ii) for each order preserving bijection $\sigma : l_1 \to l_2$, a function

$$F[\sigma] : F[l_1] \to F[l_2],$$

where $F[\beta \circ \sigma] = F[\beta] \circ F[\sigma]$ for all order preserving bijections $\beta : U \to V$, $\sigma : V \to W$ and $F[\mathrm{id}_l] = \mathrm{id}_{F[l]}$ for the identity map $\mathrm{id}_U : U \to U$.

An element $s \in F[l]$ is said to be an *F-structure* on l, and the function $F[\sigma]$ is the *transport of F-structures* along σ.

Two $\mathbb{L}$-species F and G are *isomorphic* if there is a family of bijections

$$\alpha_l : F[l] \to G[l],$$

for each totally ordered set l that commutes with the transports of structures. This means that for any order preserving bijection $\sigma : l_1 \to l_2$, one should have

$$G[\sigma] \circ \alpha_{l_1} = \alpha_{l_2} \circ F[\sigma].$$

As with $\mathbb{B}$-species, it is standard to write $F = G$ to indicate that F and G are isomorphic as $\mathbb{L}$-species.

Any $\mathbb{B}$-species F produces an $\mathbb{L}$-species, also denoted by F. This species is defined by setting

$$F[U, \preceq] = F[U],$$

for any totally ordered set $l = (U, \preceq)$, where the transport of structure is obtained by restriction to order preserving bijections.

For an $\mathbb{L}$-species F, the associated exponential generating function is defined by

$$F(x) := \sum_{n\geq 0} f_n \frac{x^n}{n!},$$

where $f_n = |F[n]|$ and the corresponding ordinary generating function is defined via

$$\hat{F}(x) := \sum_{n\geq 0} f_n x^n.$$

Note that if F is a $\mathbb{B}$-species, the F-structures that we obtain are identical regardless of whether we interpret F as a $\mathbb{B}$-species or $\mathbb{L}$-species. This implies the exponential generating function is the same regardless of whether F is a $\mathbb{B}$-species or $\mathbb{L}$-species.

In contrast to $\mathbb{B}$-species, two $\mathbb{L}$-species are isomorphic if and only if their exponential generating functions agree as shown in [1].

Proposition 1.18. For $\mathbb{L}$-species, $F = G$ if and only if $F(x) = G(x)$.

All of the $\mathbb{B}$-species defined earlier have the same name when interpreted as $\mathbb{L}$-species. Moreover, it is possible for two nonisomorphic $\mathbb{B}$-species to become isomorphic when looked at as $\mathbb{L}$-species.

Example 1.19. As $\mathbb{B}$-species, L and S are not isomorphic but rather equipotent. However, as $\mathbb{L}$-species, $L = S$. In this case, $f \in S[n]$ naturally becomes the word $f(1)f(2)\cdots f(n) \in L[n]$.

Operations on $\mathbb{B}$-species can be extended to $\mathbb{L}$-species. One can verify the operations on $\mathbb{L}$-species satisfy the properties of associativity, commutativity, and linearity up to isomorphism, in analogy with formal power series. New operations such as integration, ordinal product, and convolution also become possible. For $\mathbb{L}$-species F and G and a finite totally ordered set $l = (U, \preceq)$, we define the following:

- *Sum* $F + G$:
$$(F + G)[l] = F[l] \sqcup G[l].$$
- *Product* $F \cdot G$:
$$(F \cdot G)[l] = \sum_{l_1 + l_2 = l} F[l_1] \times G[l_2].$$

- *Derivative* $F' = \frac{d}{dx}F(X)$:

$$F'[l] = F[1 \oplus l].$$

- *Integral* $\int_0^X F(T)dT$:

$$\left(\int F\right)[l] = \begin{cases} \emptyset, & l = \emptyset \\ F[l \backslash \{\min l\}], & l \neq \emptyset. \end{cases}$$

- *Ordinal product* $F \odot G$:

$$(F \odot G)[l] = \sum_{l = l_1 \oplus l_2} F[l_1] \times G[l_2].$$

- *Convolution* $F * G$:

$$F * G = F \odot X \odot G.$$

In contrast to the product structure, an ordinal product structure $(F \odot G)[l]$ is obtained by splitting l into an initial segment l_1 and a terminating segment l_2, where l_1 has an F-structure and l_2 has a G-structure.

The convolution product is commutative. Recall that for two continuous functions f and g, the convolution of f and g is defined via

$$(f * g)(x) = \int_0^x f(x-t)g(t)dt.$$

We will utilize the operation of convolution later.

In light of Propositions 1.17 and 1.18, we can solve differential equations to find an $\mathbb{L}$-species or its corresponding exponential generating function. This is not possible with $\mathbb{B}$-species. We will utilize this approach in Chapter 3. The following result appears in [1].

Proposition 1.20. If F and G are $\mathbb{L}$-species, then

(a) $(F + G)(x) = F(x) + G(x)$;

(b) $(F \cdot G)(x) = F(x)G(x)$;

(c) $(F \circ G)(x) = F(G(x))$, where $G(0) = 0$;

(d) $F'(x) = \frac{d}{dx}F(x)$;

(e) $\left(\int_0^X F(T)dT\right)(x) = \int_o^x F(t)dt$;

(f) $(F * G)(x) = F(x) * G(x)$.

The following result is sometimes referred to as the Leibniz Rule.

Proposition 1.21. For $\mathbb{L}$-species F and G,

$$\frac{d}{dx}[F(x) * G(x)] = F(0) \cdot G(x) + F'(x) * G(x).$$

Chapter 2

Pattern avoidance in Cayley permutations

In this chapter, we will introduce pattern avoidance in the context of Cayley permutations and ballots.

2.1 Cayley permutations

As first introduced in [10], a Cayley permutation is a word that consists of positive integers that include at least one copy of each integer between one and its maximum value. More formally, a *Cayley permutation* on a finite set U is a function $p : U \to [n]$ such that $|U| = n$ and $\mathrm{Rng}(p) = [k]$ for some $k \leq n$. The collection of Cayley permutations from U to $[n]$ is denoted by $\mathrm{Cay}[U]$. If $U = [n]$, we write $\mathrm{Cay}_n := \mathrm{Cay}[n]$. For $p \in \mathrm{Cay}_n$, we utilize one-line notation and write $p = p_1 p_2 ... p_n$, where $p_i := p(i)$. In this case, we say p is of length n. We also define $\mathrm{cay}_n := |\mathrm{Cay}_n|$.

Example 2.1. We look at Cayley permutations of lengths 1, 2, and 3. We have $\mathrm{Cay}_1 = \{1\}$, $\mathrm{Cay}_2 = \{11, 12, 21\}$, and

$$\mathrm{Cay}_3 = \{111, 112, 121, 122, 123, 132, 211, 212, 213, 221, 231, 312, 321\}.$$

We define Cay to be the $\mathbb{B}$-species with structures

$$\mathrm{Cay}[U] := \{f : U \to [n] \mid \mathrm{Rng}(f) = [k], k \in [n]\},$$

together with the transport of structure along a bijection $\sigma : U \to V$ defined via

$$\mathrm{Cay}[\sigma](p) = p \circ \sigma^{-1}.$$

In Proposition 2.5, we will prove that Cayley permutations and ballots are isomorphic as $\mathbb{B}$-species. For a finite set U with $|U| = n$, we define $\alpha_U : \mathrm{Cay}[U] \to \mathrm{Bal}[U]$ via

$$\alpha_U(p) = (p^{-1}(\{1\}), \ldots, p^{-1}(\{k\})),$$

where $\mathrm{Rng}(p) = [k]$ and $k \leq n$. This map is clearly reversible, and hence a bijection. We immediately get the following.

Proposition 2.2. For $n \geq 0$, $\mathrm{bal}_n = \mathrm{cay}_n$.

Example 2.3. For the Cayley permutation $p = 31211245$ in Cay_8, we have the corresponding ballot $(\{2,4,5\},\{3,6\},\{1\},\{7\},\{8\})$ in Bal_8. For the ballot $(\{2\},\{5,6,7\},\{1,3\},\{4,8\},\{9\})$ in Bal_9 we have the corresponding Cayley permutation 313422245 in Cay_9.

It is well known that bal_n is equal to the nth Fubini number, which appears as entry A000670 in the Online Encyclopedia of Integer Sequences (OEIS) [11]. The nth Fubini number is given by $\sum_{k=0}^{n} k!\left\{{n \atop k}\right\}$, where $\left\{{n \atop k}\right\}$ is a Stirling number of the second kind. Hence we get the following theorem.

Proposition 2.4. For $n \geq 0, \mathrm{cay}_n = \displaystyle\sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}$.

For a function f with $A \subseteq \mathrm{Dom}(f)$, we define $f(A) := \{f(a) \mid a \in A\}$. This notation is used in the proof below.

Proposition 2.5. As $\mathbb{B}$-species, $\mathrm{Cay} = \mathrm{Bal}$.

Proof. Let $\sigma : U \to V$ be a bijection for finite sets U and V. We aim to show that Figure 1.4 commutes for species Cay and Bal. Let $p \in \mathrm{Cay}[U]$ with $\mathrm{Rng}(p) = [k]$. We see that

$$\begin{aligned}
\alpha_V \circ \mathrm{Cay}[\sigma](p) &= \alpha_V(\mathrm{Cay}[\sigma](p)) \\
&= \alpha_V(p \circ \sigma^{-1}) \\
&= ((p \circ \sigma^{-1})^{-1}(\{1\}), \ldots, (p \circ \sigma^{-1})^{-1}(\{k\})) \\
&= ((\sigma \circ p^{-1})(\{1\}), \ldots, (\sigma \circ p^{-1})(\{k\})) \\
&= (\sigma(p^{-1}(\{1\})), \ldots, \sigma(p^{-1}(\{k\})))
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Bal}[\sigma] \circ \alpha_U(p) &= \mathrm{Bal}[\sigma](\alpha_U(p)) \\
&= \mathrm{Bal}[\sigma](p^{-1}(\{1\}), \ldots, p^{-1}(\{k\})) \\
&= (\sigma(p^{-1}(\{1\})), \ldots, \sigma(p^{-1}(\{k\}))).
\end{aligned}$$

□

We immediately get the following from Example 1.15 and Proposition 2.5.

Corollary 2.6. $\mathrm{Cay}(x) = \dfrac{1}{2 - e^x}$.

There are known expressions for the ordinary generating function for the Fubini numbers given in terms of continued fractions, but we have omitted those results here.

2.2 Pattern avoidance

We now introduce the notions of pattern containment and pattern avoidance. Consider the Cayley permutations $p = p_1p_2 \cdots p_n \in \mathrm{Cay}_n$ and $q = q_1q_2 \cdots q_k \in \mathrm{Cay}_k$. In this case, we say that p contains q if there exists a subsequence $p_{i_1}p_{i_2} \cdots p_{i_k}$ with $i_1 \leq i_2 \leq \cdots \leq i_k$ that is order isomorphic to q (i.e., $p_{i_a} \leq p_{i_b}$ if and only if $q_a \leq q_b$). If p does not contain a pattern q, then we say p *avoids* q.

If P is a finite set such that each $q \in P$ is a Cayley permutation in Cay_k for some k (not necessarily all the same k), then we say $p \in \mathrm{Cay}_n$ *avoids* P if p avoids every $q \in P$. In this case, we say that P is a *set of patterns*. Note that a set of patterns is always assumed to be finite. We define $\mathrm{Cay}_n(P)$ to be the set of Cayley permutations of length n that avoid the set of patterns P.

The remainder of this thesis focuses on pattern avoidance in Cayley permutations. In order to sensibly define pattern avoidance, we needed to restrict the domain of Cayley permutations to some $[n]$. One consequence of this is that all species for collections of Cayley permutations avoiding a set of patterns are $\mathbb{L}$-species since the underlying sets are ordered. Unless we state otherwise, all species in the remainder of this thesis are assumed to be $\mathbb{L}$-species.

We define $\mathrm{Cay}(P)$ to be the $\mathbb{L}$-species of Cayley permutations avoiding a set of patterns P. The corresponding structures are

$$\mathrm{Cay}_n(P) = \mathrm{Cay}(P)[n] = \{p : [n] \to [n] \mid \mathrm{Rng}(f) = [k], k \in [n], p \text{ avoids } P\}.$$

The transport of structure is inherited from Cay. As expected, we define $\mathrm{cay}_n(P) := |\,\mathrm{Cay}_n(P)|$. We also let $\mathrm{Cay}(P)(x)$ and $\hat{\mathrm{C}}\mathrm{ay}(P)(x)$ denote the corresponding exponential and ordinary generating functions, respectively.

Example 2.7. Consider $\mathrm{Cay}_5(112)$. Computer calculations show that there are 360 Cayley permutations of length 5 that are 112-avoiding. Two of these are 11111 and 13221. However, 12324 is not in $\mathrm{Cay}_5(112)$ since the pattern 112 occurs in positions 2, 4, 5.

Consider $\mathrm{Cay}_5(111, 112)$. There are 309 Cayley permutations of length 5 that avoid both 111 and 112. Two of these are 13321 and 32211. However, 21232 is not in $\mathrm{Cay}_5(111, 112)$ since it contains the pattern 111 in positions 1, 3, 5, and contains the pattern 112 in positions 1, 3, 4.

Because there exists a bijection between Cayley permutations and ballots, we can describe what it means for a ballot to avoid a set of patterns. For a set of patterns P, we define $\mathrm{Bal}_n(P)$ to be the image of $\mathrm{Cay}_n(P)$ in Bal_n under the natural bijection between Cayley permutations and ballots described earlier.

Example 2.8. Consider $\mathrm{Cay}_5(112)$ and $\mathrm{Bal}_5(112)$. We have the following examples:

- $11111 \in \mathrm{Cay}_5(112)$ corresponds to the ballot $(\{1, 2, 3, 4, 5\}) \in \mathrm{Bal}_5(112)$;

- $13221 \in \mathrm{Cay}_5(112)$ corresponds to the ballot $(\{1,5\},\{3,4\},\{2\}) \in \mathrm{Bal}_5(112)$;
- $42143 \in \mathrm{Cay}_5(112)$ corresponds to the ballot $(\{3\},\{2\},\{5\},\{1,4\}) \in \mathrm{Bal}_5(112)$.

For 112-avoiding ballots, given $k \in [n]$, each block preceding the block containing k contains at most one value from $\{1,\ldots,k-1\}$, otherwise the corresponding Cayley permutation would contain the pattern 112. In particular, in a 112-avoiding ballot, all blocks preceding the block containing the maximum element must be singletons.

We naturally use the notation $\mathrm{bal}_n(P), \mathrm{Bal}(P)(x)$, and $\hat{\mathrm{B}}\mathrm{al}(P)(x)$ to have the expected meaning. Given the definition of $\mathrm{Bal}_n(P)$ and the isomorphism between species Cay and Bal, the next result is immediate.

Proposition 2.9. For a set of patterns P:

(a) $\mathrm{Cay}(P) = \mathrm{Bal}(P)$ (as $\mathbb{L}$-species);

(b) $\mathrm{cay}_n(P) = \mathrm{bal}_n(P)$;

(c) $\mathrm{Cay}(P)(x) = \mathrm{Bal}(P)(x)$;

(d) $\hat{\mathrm{C}}\mathrm{ay}(P)(x) = \hat{\mathrm{B}}\mathrm{al}(P)(x)$.

We now extend the idea of Wilf equivalence in [2] to Cayley permutations. For sets of patterns P and Q, we say that $\mathrm{Cay}_n(P)$ and $\mathrm{Cay}_n(Q)$ are *Wilf equivalent* if $\mathrm{cay}_n(P) = \mathrm{cay}_n(Q)$. If $\mathrm{Cay}_n(P)$ and $\mathrm{Cay}_n(Q)$ are Wilf equivalent, we write $\mathrm{Cay}_n(P) \sim \mathrm{Cay}_n(Q)$, or simply $P \sim Q$. In the special case that $P = \{p\}$ and $Q = \{q\}$, we write $p \sim q$. Certainly, Wilf equivalence is an equivalence relation. The corresponding equivalence classes are called *Wilf classes*. We define $[P]_w$ to be the Wilf class of P.

We define the following maps:

- The *reverse map* $r : \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathrm{Cay}_n \to \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathrm{Cay}_n$ via
$$r(p_1 \cdots p_n) = p_n \cdots p_1.$$
- The *complement map* $c : \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathrm{Cay}_n \to \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathrm{Cay}_n$ via
$$c(p_1 \cdots p_n) = (m+1-p_1)\cdots(m+1-p_n),$$
where $m := \max\{p_i\}$.
- The *reverse-complement map* $c \circ r : \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathrm{Cay}_n \to \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathrm{Cay}_n$ via
$$c \circ r(p_1 \cdots p_n) = (m+1-p_n)\cdots(m+1-p_1),$$
where $m := \max\{p_i\}$. Note that $r \circ c = c \circ r$.

Example 2.10. Given the Cayley permutation $132411 \in \mathrm{Cay}_6$, we have:

$$\begin{tikzcd} 132411 \arrow[r, "r"] & 114231 \arrow[r, "c"] & 441324 \arrow[r, "r"] & 423144 \arrow[lll, bend left, "c"] \end{tikzcd}$$

The reverse and complement maps are bijections on Cay_n, and therefore $c \circ r$ is also a bijection on Cay_n. In addition, for any set of patterns P, the maps $r : \mathrm{Cay}_n(P) \to \mathrm{Cay}_n(r(P)), c : \mathrm{Cay}_n(P) \to \mathrm{Cay}_n(c(P))$, and $r \circ c : \mathrm{Cay}_n(P) \to \mathrm{Cay}_n(r \circ c(P))$ are bijections. This implies that for a set of patterns P, the sets $\mathrm{Cay}_n(P), \mathrm{Cay}_n(c(P)), \mathrm{Cay}_n(r(P))$, and $\mathrm{Cay}_n(r \circ c(P))$ are all Wilf equivalent.

Proposition 2.11. For a set of patterns P, $P \sim r(P) \sim c(P) \sim r \circ c(P)$.

Given a set of patterns P, $\{P, r(P), c(P), r \circ c(P)\}$ is called the *symmetry class* of P. We denote the symmetry class of P as $[P]_s$. Note that for a set of patterns P, $[P]_s \subseteq [P]_w$ by Proposition 2.11. In fact, every Wilf class of a set of patterns is the union of symmetry classes. We omit the proof of the next result as it is an easy computation.

Proposition 2.12. For single patterns of lengths 2 and 3, we have the following symmetry classes:

(a) $[11]_s = \{11\}$;

(b) $[12]_s = \{12, 21\}$;

(c) $[111]_s = \{111\}$;

(d) $[123]_s = \{123, 321\}$;

(e) $[132]_s = \{132, 312, 213, 231\}$;

(f) $[112]_s = \{112, 221, 122, 211\}$;

(g) $[121]_s = \{121, 212\}$.

It follows from [8], together with Proposition 1.1 in [9], that we get the following Wilf classes.

Proposition 2.13. For patterns of length 2 and length 3, we have the following distinct Wilf classes:

(a) $[11]_w = \{11\}$;

(b) $[12]_w = \{12, 21\}$;

(c) $[123]_w = [123]_w \cup [132]_w = \{123, 321, 132, 312, 213, 231\}$;

(d) $[112]_w = [112]_w \cup [121]_w = \{112, 221, 122, 211, 121, 212\}$.

We will avoid relying on the previous result when possible and verify the Wilf equivalence directly.

We describe all symmetry classes for pairs of patterns of length 2 and several pairs of patterns of length 2 in the next result. Note that there are $\binom{12}{2} = 66$ pairs of distinct patterns of length 3. It turns out that there are 25 distinct symmetry classes for pairs of distinct patterns of length 3. Again, we omit the proof.

Proposition 2.14. We have the following symmetry classes:

(a) $[11, 12]_s = \{\{11, 12\}, \{11, 21\}\}$;

(b) $[12, 21]_s = \{\{12, 21\}\}$;

(c) $[112, 221]_s = \{\{112, 221\}, \{211, 122\}\}$;

(d) $[112, 211]_s = \{\{112, 211\}, \{221, 122\}\}$;

(e) $[112, 212]_s = \{\{112, 212\}, \{211, 212\}, \{221, 121\}, \{122, 121\}\}$;

(f) $[221, 212]_s = \{\{221, 212\}, \{122, 212\}, \{112, 121\}, \{211, 121\}\}$;

(g) $[112, 122]_s = \{\{112, 122\}, \{211, 221\}\}$;

(h) $[112, 123]_s = \{\{112, 123\}, \{211, 321\}, \{221, 321\}, \{122, 123\}\}$;

(i) $[112, 213]_s = \{\{112, 213\}, \{211, 312\}, \{221, 231\}, \{122, 132\}\}$;

(j) $[121, 132]_s = \{\{121, 132\}, \{212, 312\}\}$;

(k) $[121, 231]_s = \{\{121, 231\}, \{212, 213\}, \{121, 132\}, \{212, 312\}\}$;

(l) $[123, 132]_s = \{\{123, 132\}, \{321, 231\}, \{321, 312\}, \{123, 213\}\}$;

(m) $[312, 132]_s = \{\{312, 132\}, \{213, 231\}\}$.

We will prove that the pairs of patterns in (a) and (b) of Proposition 2.14 are Wilf equivalent in Propositions 3.14–3.17.

Chapter 3

Enumeration of pattern-avoiding Cayley permutations

In this chapter, we will investigate Cayley permutations avoiding 1^k ($k \geq 2$), patterns of length 2, pairs of patterns of length 2, patterns of length 3, and pairs of patterns of length 3.

3.1 Patterns of type 1^k

In this section, we explore Cayley permutations that avoid the pattern

$$1^k := \underbrace{11 \cdots 1}_{k}$$

for some $k \geq 2$. We will rely on the isomorphism between $\mathrm{Cay}(1^k)$ and $\mathrm{Bal}(1^k)$.

Proposition 3.1. For $k \geq 2$, $\mathrm{Cay}(1^k) = L(E_1 + E_2 + \cdots + E_{k-1})$.

Proof. We will verify the result in terms of ballots. Note that a 1^k-avoiding ballot has blocks of sizes at most $k-1$. That is, blocks may be any size between 1 and $k-1$. It follows that $\mathrm{Bal}(1^k) = L(E_1 + E_2 + \cdots + E_{k-1})$, and so $\mathrm{Cay}(1^k) = L(E_1 + E_2 + \cdots + E_{k-1})$. □

Proposition 3.2. $\mathrm{Cay}(1^k)(x) = \dfrac{1}{1 - \left(\sum_{i=1}^{k-1} \frac{x^i}{i!}\right)}$.

Proof. By Propositions 1.6 and 1.18, we get

$$\begin{aligned}
\mathrm{Cay}(1^k)(x) &= L(E_1 + \cdots + E_{k-1})(x) \\
&= L(E_1(x) + \cdots + E_{k-1}(x)) \\
&= \frac{1}{1 - \left(\sum_{i=1}^{k-1} \frac{x^i}{i!}\right)}.
\end{aligned}$$

□

Our next goal is to provide an enumeration for $\mathrm{Cay}(1^k)$. We need an intermediate result. Given a ballot $B = (B_1, \ldots, B_l) \in \mathrm{Bal}_n$, define the *shadow* of B to be $(|B_1|, \ldots, |B_l|)$, which is a composition of n (i.e., an ordered list of nonnegative integers where the sum of the entries is n).

Proposition 3.3. The number of ballots in Bal_n with shadow $(b_1, \ldots, b_l)$ is given by

$$\frac{n!}{1^{j_1}(2!)^{j_2}(3!)^{j_3}\cdots((k-1)!)^{j_{k-1}}},$$

where j_i is the number of occurrences of i among $b_1, \ldots, b_l$.

Proof. The number of ballots in Bal_n with shadow $(b_1, \ldots, b_l)$ is

$$\binom{n}{b_1}\binom{n-b_1}{b_2}\cdots\binom{n-b_1-\cdots-b_{l-1}}{b_l} = \binom{n}{b_1, \ldots, b_l} = \frac{n!}{1^{j_1}(2!)^{j_2}(3!)^{j_3}\cdots((k-1)!)^{j_{k-1}}},$$

where j_i is the number of occurrences of i among $b_1, \ldots, b_l$. ◻

Proposition 3.4. For $k \geq 2$, we have

$$\mathrm{cay}_n(1^k) = \sum_{n=\sum_{i=1}^{k-1} ij_i} \binom{n-j_2-2j_3-\cdots-(k-2)j_{k-1}}{j_1, j_2, \ldots j_{k-1}} \frac{n!}{1^{j_1}(2!)^{j_2}(3!)^{j_3}\cdots((k-1)!)^{j_{k-1}}}.$$

Proof. For a fixed collection of multiplicities $j_1, \ldots, j_{k-1}$ for blocks of size $1, \ldots, k-1$, respectively, we can form a ballot in $\mathrm{Bal}_n(1^k)$ by first choosing an ordering of the block sizes (i.e., shadow) and then distribute $[n]$ across the blocks. The first step can be done in

$$\binom{n-j_2-2j_3-\cdots-(k-2)j_{k-1}}{j_1, j_2, \ldots j_{k-1}}$$

many ways since $n = j_1 + 2j_2 + \cdots + (k-1)j_{k-1}$. The second step can be accomplished in

$$\frac{n!}{1^{j_1}(2!)^{j_2}(3!)^{j_3}\cdots((k-1)!)^{j_{k-1}}}$$

many ways by Proposition 3.3. Summing across all possible cases of block sizes and their multiplicity, we get

$$\mathrm{cay}_n(1^k) = \sum_{n=\sum_{i=1}^{k-1} ij_i} \binom{n-j_2-2j_3-\cdots-(k-2)j_{k-1}}{j_1, j_2, \ldots j_{k-1}} \frac{n!}{1^{j_1}(2!)^{j_2}(3!)^{j_3}\cdots((k-1)!)^{j_{k-1}}}.$$

◻

Below we have an example.

Example 3.5. For $\mathrm{Bal}_n(111)$, we can have blocks of at most size 2. Each ballot in $\mathrm{Bal}_n(111)$ has j_1 many blocks of size 1 and j_2 many blocks of size 2, where $n = j_1 + 2j_2$. For $n = 3$, the possible combinations of j_1 and j_2 are $j_1 = 3, j_2 = 0$ and $j_1 = 1, j_2 = 1$. Applying Proposition 3.4, we see that

$$\begin{aligned}\sum_{3=j_1+2j_2} \frac{3!}{1^{j_1}(2!)^{j_2}}\binom{3-j_2}{j_2} &= \underbrace{\frac{3!}{1^3(2!)^0}\binom{3-0}{0}}_{j_1=3,\ j_2=0} + \underbrace{\frac{3!}{1^1(2!)^1}\binom{3-1}{1}}_{j_1=1,\ j_2=1}\\ &= 6\cdot 1 + 3\cdot 2\\ &= 12.\end{aligned}$$

Indeed, we see that there are 12 Cayley permutations of length 3 avoiding 111:

$$\mathrm{Cay}_3(111) = \{112, 121, 122, 123, 132, 211, 212, 213, 221, 231, 312, 321\}.$$

3.2 Patterns of length 2

In this section, we will enumerate all Cayley permutations that avoid patterns of length two and pairs of patterns of length two. In all cases, we will provide species, exponential generating functions, ordinary generating functions, and counting formulas. When possible, we provide species first. As a reminder, because we are studying pattern avoidance, we assume all species are $\mathbb{L}$-species unless otherwise specified. First, we look at single patterns of length 2.

Proposition 3.6. $\mathrm{Cay}(11) = L$.

Proof. By Proposition 3.1, $\mathrm{Cay}(11) = L(E_1) = L$. □

Note that $\mathrm{Cay}_n(11)$ is the set of ordinary permutations on $[n]$, which we interpret as linear orders. In terms of ballots, each element in $\mathrm{Bal}_n(11)$ is an ordered set partition consisting of singleton blocks, which implies that $\mathrm{Cay}(11) = L$ as $\mathbb{B}$-species, as well.

By Propositions 1.18 and 3.6, we immediately get the following. Alternatively, we can apply Proposition 3.2.

Proposition 3.7. $\mathrm{Cay}(11)(x) = L(x) = \dfrac{1}{1-x}$.

We obtain the following result as $\mathrm{Cay}_n(11)$ is the set of ordinary permutations on $[n]$.

Proposition 3.8. For $n \geq 0$, $\mathrm{cay}_n(11) = n!$.

Note that there is no "nice" representation of the ordinary generating function for the sequence $n!$. However, we include a continued fraction representation, which appears as Theorem 3B in [6].

Proposition 3.9. We have

$$\hat{\mathrm{C}}\mathrm{ay}(11)(x) = \cfrac{1}{1 - x - \cfrac{1^2x^2}{1-3x-\cfrac{2^2x^2}{1-5x-\cfrac{3^2x^2}{\ddots}}}},$$

where odd numbers are the coefficients of x and squares are the coefficients of x^2.

We now derive the counting formula for Cay(21) using a bijection to a certain collection of stars and bars.

Proposition 3.10. For $n \geq 0$,

$$\mathrm{cay}_n(21) = \begin{cases} 1, & n = 0 \\ 2^{n-1}, & n \geq 1. \end{cases}$$

Proof. Certainly, $\mathrm{cay}_0(21) = 1$.

Now, suppose $n \geq 1$. Since each Cayley permutation in $\mathrm{Cay}_n(21)$ must contain an occurrence of each value from 1 up to its maximum value and is weakly increasing, each element in $\mathrm{Cay}_n(21)$ is of the form

$$\underbrace{1\cdots1}_{k_1}\underbrace{2\cdots2}_{k_2}\cdots\underbrace{m\cdots m}_{k_m},$$

where m is the maximum value and there are $k_i \geq 1$ occurrences of value i. We can uniquely encode each Cayley permutation in $\mathrm{Cay}_n(21)$ into a particular stars and bars model as follows:

$$\underbrace{1\cdots1}_{k_1}\underbrace{2\cdots2}_{k_2}\cdots\underbrace{m\cdots m}_{k_m} \to \underbrace{\star\cdots\star}_{k_1}|\underbrace{\star\cdots\star}_{k_2}|\cdots|\underbrace{\star\cdots\star}_{k_m}.$$

The image of this map is the collection of sequences of stars and bars with n stars and between 0 and $n-1$ bars such that every pair of consecutive bars is separated by at least one star. We can think of each bar as indicating an increase by 1 in the corresponding Cayley permutation. This mapping is reversible and therefore a bijection. Because there are $n-1$ gaps between n stars and in each gap we can place a bar or not, it must be the case that $\mathrm{cay}_n(21) = 2^{n-1}$ for $n \geq 1$. $\square$

The exponential generating function for weakly increasing finite sequences with no missing values is likely known, but we could not find a reference, so we include a proof below.

Proposition 3.11. $\mathrm{Cay}(21)(x) = \dfrac{e^{2x}+1}{2} = \cosh(x)e^x$.

Proof. Consider the geometric sequence given by 2^n for $n \geq 0$. The exponential generating function for this sequence is known to be e^x. It follows from Propositions 1.18, 1.20(e), and 3.10 that the exponential generating function for Cay(21) is given by

$$\begin{aligned} 1 + \int_0^x e^{2t}dt &= \frac{e^{2x}+1}{2} \\ &= \frac{e^x + e^{-x}}{2} e^x \\ &= \cosh(x) e^x. \end{aligned}$$

$\square$

Proposition 3.12. $\mathrm{Cay}(21) = E_{\text{even}} \cdot E$.

Proof. Recall from Chapter 1 that the species associated with the exponential generating function $\cosh(x)$ is E_{even}, the species of even subsets, whereas the species associated with the exponential generating function e^x is E. By Propositions 1.18 and 3.11, it follows that $\mathrm{Cay}(21) = E_{\text{even}} \cdot E$. $\square$

Presently, we do not have a combinatorial interpretation of the previous result.

Proposition 3.13. $\hat{\mathrm{C}}\mathrm{ay}(21)(x) = \dfrac{1-x}{1-2x}$.

Proof. Recall that $\sum_{n\geq 0} 2^n x^n = \dfrac{1}{1-2x}$ (geometric series). This implies that

$$\begin{aligned} \frac{x}{1-2x} &= x \cdot \frac{1}{1-2x} \\ &= x \cdot \sum_{n\geq 0} 2^n x^n \\ &= \sum_{n\geq 0} 2^n x^{n+1} \\ &= \sum_{n\geq 1} 2^{n-1} x^n \\ &= 1\cdot x^1 + 2\cdot x^2 + 4\cdot x^3 + 8\cdot x^4 + \cdots . \end{aligned}$$

Adding a leading 1 gives us

$$1 + \sum_{n\geq 1} 2^{n-1}x^n = 1 + \frac{x}{1-2x} = \frac{1-x}{1-2x}.$$

$\square$

Each Cayley permutation in $\mathrm{Cay}_n(21)$ is weakly increasing while each Cayley permutation in $\mathrm{Cay}_n(12)$ is weakly decreasing. Using the reverse map, it is clear that $\mathrm{Cay}(12) = \mathrm{Cay}(21)$. Hence the results for $\mathrm{Cay}(21)$ also apply to $\mathrm{Cay}(12)$ as seen in Table 3.1.

Next, we will look at Cayley permutations that avoid pairs of patterns of length 2. We will show that for each n and all possible pairs of distinct patterns of length 2, there is only a single Cayley permutation that avoids both patterns. It follows that all three combinations of distinct pairs of patterns of length 2 yield the same counting formula, exponential generating function, ordinary generating function, and representation as species.

Proposition 3.14. For all $n \geq 1$, $\mathrm{Cay}_n(11, 12) = \{n(n-1)\cdots 321\}$.

Proof. Cayley permutations that avoid 11 contain no repeats and Cayley permutations that avoid 12 are weakly decreasing. This leaves the decreasing permutation $n(n-1)\cdots 321$ of length n. □

Proposition 3.15. For all $n \geq 1$, $\mathrm{Cay}_n(11, 21) = \{123\cdots(n-1)n\}$.

Proof. Cayley permutations that avoid 11 contain no repeats and Cayley permutations that avoid 21 are weakly increasing. This leaves the identity permutation of length n. □

Proposition 3.16. For all $n \geq 1$, $\mathrm{Cay}_n(12, 21) = \{11\cdots 1\}$.

Proof. Cayley permutations that avoid 12 cannot be increasing and Cayley permutations that avoid 21 cannot be decreasing. Cayley permutations must contain a 1 so that leaves the constant Cayley permutation $11\cdots 1$ of length n. □

As there is only one Cayley permutation in each of $\mathrm{Cay}_n(11, 12), \mathrm{Cay}_n(11, 21)$, and $\mathrm{Cay}_n(12, 21)$, they all clearly have the same count and therefore have the same exponential generating function, ordinary generating function, and species. It is worth mentioning that $\mathrm{Cay}_n(11, 12)$ and $\mathrm{Cay}_n(11, 21)$ are clearly Wilf equivalent using the reverse map but the previous result tells us that both are also Wilf equivalent to $\mathrm{Cay}_n(12, 21)$. This immediately produces the following results.

Proposition 3.17. For all $n \geq 0$, $\mathrm{cay}_n(11, 12) = \mathrm{cay}_n(11, 21) = \mathrm{cay}_n(12, 21) = 1$.

The exponential generating function and ordinary generating function for the constant sequence of all 1s are well known.

Proposition 3.18. $\mathrm{Cay}(11, 12)(x) = \mathrm{Cay}(11, 21)(x) = \mathrm{Cay}(12, 21)(x) = e^x$.

Propositions 1.18 and 3.18 yield the following.

Proposition 3.19. $\mathrm{Cay}(11, 12) = \mathrm{Cay}(11, 21) = \mathrm{Cay}(12, 21) = E$.

Proposition 3.20. $\hat{\mathrm{C}}\mathrm{ay}(11, 12)(x) = \hat{\mathrm{C}}\mathrm{ay}(11, 21)(x) = \hat{\mathrm{C}}\mathrm{ay}(12, 21)(x) = \dfrac{1}{1-x}$.

Note that in terms of ballots, since the only ordered set partition in $\mathrm{Bal}_n(12, 21)$ consists of a single block, we get $\mathrm{Cay}(12, 21) = E$ as $\mathbb{B}$-species.

The results from this section are summarized in Table 3.1. We have included references for the relevant entries in the Online Encyclopedia of Integer Sequences (OEIS).

Patterns	Species	EGF	OGF	Enumeration	OEIS
11	L	$\dfrac{1}{1-x}$	$\cfrac{1}{1-x-\cfrac{1^2x^2}{1-3x-\cfrac{2^2x^2}{1-5x-\cfrac{3^2x^2}{\ddots}}}}$	$n!$	A000142
12	$E_{\text{even}} \cdot E$	$\dfrac{e^{2x}+1}{2}$	$\dfrac{1-x}{1-2x}$	$\begin{cases} 1, & n=0 \\ 2^{n-1}, & n \geq 1 \end{cases}$	A011782
21					
11,12	E	e^x	$\dfrac{1}{1-x}$	1	A000012
11,21					
12,21					

Table 3.1: Results for patterns and pairs of distinct patterns of length 2.

3.3 Patterns of length 3

In this section, we will study Cayley permutations that avoid patterns of length three and pairs of distinct patterns of length three. In most cases, we will provide species, exponential generating functions, and counting formulas. When possible, we provide the species first.

We begin with Cayley permutations that avoid a single pattern of length three. We analyze 111-avoiding Cayley permutations first and provide the corresponding species and its exponential generating function, as well as a closed form for the counting formula. The next result is simply a special case of Proposition 3.1.

Proposition 3.21. $\text{Cay}(111) = L(E_1 + E_2)$.

Proposition 3.22. $\text{Cay}(111)(x) = \dfrac{2}{2-2x-x^2}$.

Proof. By Proposition 3.2, we see that

$$\text{Cay}(111)(x) = \frac{1}{1-(x+\frac{x^2}{2})} = \frac{2}{2-2x-x^2}.$$

□

Our next goal is to obtain a counting formula for Cay(111). We will use the exponential generating function for Cay(111) together with its partial fraction decomposition to do this. First, we need two lemmas.

Lemma 3.23. We have $\dfrac{1}{\sqrt{3}+1+x} = \displaystyle\sum_{n\geq 0} \frac{(-1)^n n!}{(\sqrt{3}+1)^{n+1}} \cdot \frac{x^n}{n!}$.

Proof. We see that

$$
\begin{aligned}
\frac{1}{\sqrt{3}+1+x} &= \frac{-1}{-\sqrt{3}-1-x} \\
&= \frac{\frac{-1}{-\sqrt{3}-1}}{\frac{-\sqrt{3}-1}{-\sqrt{3}-1} - \frac{x}{-\sqrt{3}-1}} \\
&= \frac{1}{\sqrt{3}+1} \cdot \frac{1}{1 - \frac{x}{-(\sqrt{3}+1)}} \\
&= \frac{1}{\sqrt{3}+1} \cdot \sum_{n\geq 0} \left(\frac{x}{-(\sqrt{3}+1)} \right)^n \\
&= \sum_{n\geq 0} \frac{(-1)^n}{(\sqrt{3}+1)^{n+1}} \cdot x^n \\
&= \sum_{n\geq 0} \frac{(-1)^n n!}{(\sqrt{3}+1)^{n+1}} \cdot \frac{x^n}{n!}.
\end{aligned}
$$

□

The next lemma follows from similar calculations and we omit the proof.

Lemma 3.24. We have $\dfrac{1}{\sqrt{3}-1-x} = \displaystyle\sum_{n\geq 0} \frac{n!}{(\sqrt{3}-1)^{n+1}} \cdot \frac{x^n}{n!}$.

We utilize the two previous lemmas in the next result.

Proposition 3.25. For all $n \geq 0$, $\mathrm{cay}_n(111)(x) = n! \cdot \dfrac{(1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}}{2^{n+1}\sqrt{3}}$.

Proof. Using Proposition 3.22 together with the partial fraction decomposition and Lemmas 3.23 and 3.24, we see that

$$
\begin{aligned}
\mathrm{Cay}(111)(x) &= \frac{2}{2-2x-x^2} \\
&= \frac{2}{(\sqrt{3}+1+x)(\sqrt{3}-1-x)} \\
&= \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{3}+1+x} + \frac{1}{\sqrt{3}-1-x} \right) \\
&= \frac{1}{\sqrt{3}} \left(\sum_{n\geq 0} \frac{n!}{(\sqrt{3}-1)^{n+1}} \cdot \frac{x^n}{n!} + \sum_{n=0}^{n} \frac{n!(-1)^n}{(\sqrt{3}+1)^{n+1}} \cdot \frac{x^n}{n!} \right).
\end{aligned}
$$

It follows that

$$\begin{aligned}\operatorname{cay}_n(111) &= \frac{1}{\sqrt{3}}\left(\frac{n!}{(\sqrt{3}-1)^{n+1}} + \frac{n!(-1)^n}{(\sqrt{3}+1)^{n+1}}\right)\\ &= n! \cdot \frac{(1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1})}{2^{n+1}\sqrt{3}}.\end{aligned}$$

□

Recall that Proposition 3.4 provides a recursive enumeration for all $\operatorname{Cay}(1^k)$, whereas Proposition 3.25 provides a closed form for the special case $\operatorname{Cay}(111)$. Our approach for 111-avoiding Cayley permutations works because we were able to factor the denominator of the exponential generating function for $\operatorname{Cay}(111)$. It is unlikely that this technique generalizes because we do not know the appropriate splitting fields.

Now, although $\operatorname{Cay}_n(212)$ and $\operatorname{Cay}_n(112)$ are Wilf equivalent by Proposition 2.13, we will present two different methods for obtaining the same exponential generating function to show $\operatorname{Cay}(212) = \operatorname{Cay}(112)$. We will first provide a functional species equation for $\operatorname{Cay}(212)$. For a Cayley permutation $p = p_1 \cdots p_n$, define $\max(p) = \max\{p_i \mid i \in [n]\}$.

Proposition 3.26. $\operatorname{Cay}(212) = 1 + E * \operatorname{Cay}(212) + E * \operatorname{Cay}(212)^\bullet$.

Proof. We provide a combinatorial interpretation. The empty Cayley permutation is certainly 212-avoiding, which is accounted for by the species 1 on the right hand side. Now consider a nonempty 212-avoiding Cayley permutation. Observe that the occurrences of the maximum value of any 212-avoiding Cayley permutation must occur as a single contiguous string. That is, every 212-avoiding Cayley permutation p is of the form

$$p_1 \cdots p_i m \cdots m p_j \cdots p_n,$$

where the maximum value is $m = \max(p)$ and only occurs in positions $i+1, \ldots, j-1$ and $p_1 \cdots p_i, p_j \cdots p_n$ are both 212-avoiding in their own right. We consider two cases. Suppose the contiguous string of the maximum value occurs on the far right. Then we have

$$\underbrace{p_1 \cdots p_i}_{\operatorname{Cay}(212)} \underbrace{m}_{X} \underbrace{m \cdots m}_{E},$$

which yields the species $\operatorname{Cay}(212) \odot X \odot E = \operatorname{Cay}(212) * E$. Now, assume that the contiguous string of the maximum value does not occur on the far right. In this case, we can identify a distinguished position in a 212-avoiding permutation, corresponding to the species $\operatorname{Cay}(212)^\bullet$. Then, immediately to the left of the distinguished position, we insert a contiguous string of the appropriate maximum value. This means

$$\underbrace{(p_1 \cdots p_i \overset{\bullet}{p}_{i+1} \cdots p_k,}_{\operatorname{Cay}(212)} \underbrace{m,}_{X} \underbrace{m \cdots m)}_{E} \mapsto \underbrace{p_1 \cdots p_i m \cdots m p_j \cdots p_n}_{\operatorname{Cay}(212)^\bullet \odot X \odot E}.$$

This generates the species $\mathrm{Cay}(212)^\bullet \odot X \odot E = \mathrm{Cay}(212)^\bullet * E$. Note that because the convolution product is commutative, $\mathrm{Cay}(212)^\bullet * E = E * \mathrm{Cay}(212)^\bullet$. Combining the cases, we get $\mathrm{Cay}(212) = 1 + E * \mathrm{Cay}(212) + E * \mathrm{Cay}(212)^\bullet$. $\square$

We use the previous functional species equation to determine the following exponential generating function.

Proposition 3.27. $\mathrm{Cay}(212)(x) = \dfrac{x^2 - 2x + 2}{2(x-1)^2}$.

Proof. Let $F = \mathrm{Cay}(212)$. By Proposition 3.26, we have

$$F(x) = 1 + e^x * F(x) + e^x * (xF'(x)).$$

On the other hand, we see that

$$\begin{aligned}
&1 + e^x * \frac{x^2 - 2x + 2}{2(x-1)^2} + e^x * \frac{-x}{(x-1)^3} \\
&= 1 + \int_0^x e^{x-t} \frac{t^2 - 2t + 2}{2(t-1)^2} dt + \int_0^x e^{x-t} \frac{-t}{(t-1)^3} dt \\
&= \frac{x^2 - 2x + 2}{2(x-1)^2},
\end{aligned}$$

where the last equality was verified using Mathematica [7]. Since both functions satisfy the same functional equation, the result follows.[1] $\square$

Our next goal is to obtain a counting formula for $\mathrm{Cay}(212)$. We first provide a recursive formula.

Proposition 3.28. For $n \geq 1$, $\mathrm{cay}_n(212) = \sum_{k=0}^{n-1} (k+1)\,\mathrm{cay}_k(212)$ and $\mathrm{cay}_0(212) = 1$.

Proof. As in the case analysis of the proof of Proposition 3.26, to construct a 212-avoiding Cayley permutation of length n, we take $p \in \mathrm{Cay}_k(212)$ and insert a string of $\max(p)+1$ of length $n-k$ into the prefix, suffix, or between one of the $k-1$ gaps. This produces $k+1$ places to insert the string. We do this for all $0 \leq k \leq n-1$. It follows that

$$\mathrm{cay}_n(212) = \sum_{k=0}^{n-1} (k+1)\,\mathrm{cay}_k(212).$$

$\square$

We prove the following lemma to be used in the proof of Proposition 3.30.

[1] The proof that appears in print contained an error.

Lemma 3.29. For $n \geq 2$, $\displaystyle\sum_{k=1}^{n-1}(k+1)(k+1)! = (n+1)! - 2$.

Proof. We will proceed by induction. If $n = 2$, then the left-hand side is $\sum_{k=1}^{1}(k+1)(k+1)! = (2)(2)! = 4$ while the right-hand side is $(2+1)! - 2 = 3! - 2 = 6 - 2 = 4$. Therefore, the equality holds when $n = 2$. Now, suppose the equation holds for a fixed $n \geq 2$. Then

$$\begin{aligned}
\sum_{k=1}^{n}(k+1)(k+1)! &= \sum_{k=1}^{n-1}(k+1)(k+1)! + (n+1)(n+1)! \\
&= (n+1)! - 2 + (n+1)(n+1)! && \text{(by inductive hypothesis)} \\
&= (n+1)!(1 + (n+1)) - 2 \\
&= (n+1)!(n+2) - 2 \\
&= (n+2)! - 2,
\end{aligned}$$

as desired. ∎

Proposition 3.30. For all $n \geq 0$,

$$\operatorname{cay}_n(212) = \begin{cases} 1, & n = 0 \\ \frac{(n+1)!}{2}, & n \geq 1. \end{cases}$$

Proof. We will proceed by induction. It is easily seen that $\operatorname{Cay}_0(212) = 1 = \operatorname{Cay}_1(212)$, which verifies the base cases. Now, let $n \geq 2$ and assume $\operatorname{Cay}_k(212) = \frac{(k+1)!}{2}$ for all $1 \leq k \leq n-1$. Then

$$\begin{aligned}
\operatorname{cay}_n(212) &= \sum_{k=0}^{n-1}(k+1)\operatorname{cay}_k(212) && \text{(by Proposition 3.28)} \\
&= 1 + \sum_{k=1}^{n-1}(k+1)\operatorname{cay}_k(212) \\
&= 1 + \sum_{k=1}^{n-1}\frac{(k+1)(k+1)!}{2} && \text{(by inductive hypothesis)} \\
&= 1 + \frac{1}{2}\sum_{k=1}^{n-1}(k+1)(k+1)! \\
&= 1 + \frac{1}{2}((n+1)! - 2) && \text{(by Lemma 3.29)} \\
&= 1 + \frac{(n+1)!}{2} - 1 \\
&= \frac{(n+1)!}{2}.
\end{aligned}$$

∎

Recall that $\mathrm{Cay}_n(212)$ and $\mathrm{Cay}_n(121)$ are Wilf equivalent by Proposition 2.11. So, the results provided for Cay(212) also apply to Cay(121).

We now transition to 112-avoiding Cayley permutations. The next result provides a functional species equation for Cay(112). Recall that Cay(112) = Bal(112) by Proposition 2.9.

Proposition 3.31. $\mathrm{Cay}(112)' = L \cdot \mathrm{Cay}_+(112) + L \cdot \mathrm{Cay}(112)$.

Proof. We will verify the result in terms of ballots. In a 112-avoiding ballot, all the blocks preceding the block containing the maximum value must be singletons. First, identify the maximum value with $\star$. If $\star$ is not in a block by itself, then a 112-avoiding ballot is of the form

$$(\underbrace{\{a_1\}, \{a_2\}, \dots \{a_k\}}_{L}, \underbrace{\{\star, b_1, \dots, b_l\}, \{\cdots\}, \dots, \{\cdots\}}_{\mathrm{Bal}_+(112) \text{ ignoring } \star}),$$

where $l \geq 1$ and the blocks to the right of the block containing $\star$ may or may not exist. Now, if $\star$ is in a block by itself, then a 112-avoiding Cayley permutation is of the form

$$(\underbrace{\{a_1\}, \{a_2\}, \dots \{a_k\}}_{L}, \underbrace{\{\star\}, \{\cdots\}, \dots, \{\cdots\}}_{\mathrm{Bal}(112) \text{ ignoring } \star}),$$

where again, blocks to the right of the block containing $\{\star\}$ may or may not exist. It follows that

$$\mathrm{Bal}(112)' = L \cdot \mathrm{Bal}_+(112) + L \cdot \mathrm{Bal}(112),$$

which implies

$$\mathrm{Cay}(112)' = L \cdot \mathrm{Cay}_+(112) + L \cdot \mathrm{Cay}(112).$$

$\square$

Proposition 3.32. $\mathrm{Cay}(112)(x) = \dfrac{x^2 - 2x + 1}{2(x-1)^2}$.

Proof. Using the functional species equation in Proposition 3.31 together with Proposition 1.6(h), we obtain

$$\mathrm{Cay}(112)'(x) = \frac{1}{1-x} \cdot (\mathrm{Cay}(112)(x) - 1) + \frac{1}{1-x} \cdot (\mathrm{Cay}(112)(x)).$$

To ease notation, we set $F = \mathrm{Cay}(112)$. Simplifying, we get

$$\begin{aligned} F'(x) &= \frac{F(x)-1}{1-x} + \frac{F(x)}{1-x} \\ &= \frac{2F(x)-1}{1-x}. \end{aligned}$$

Solving this differential equation yields $F(x) = \dfrac{x^2 - 2x + 2}{2(x-1)^2}$. $\square$

By Proposition 2.12, $\mathrm{Cay}_n(112), \mathrm{Cay}_n(221), \mathrm{Cay}_n(211)$, and $\mathrm{Cay}_n(122)$ are all Wilf equivalent. This implies that the results for Cay(112) also apply to Cay(221), Cay(211), and Cay(122). However, the exponential generating functions for Cay(212) and Cay(112) are the same according to Propositions 3.27 and 3.32. This implies that all six of $\mathrm{Cay}_n(112)$, $\mathrm{Cay}_n(221)$, $\mathrm{Cay}_n(211)$, $\mathrm{Cay}_n(122)$, $\mathrm{Cay}_n(121)$, and $\mathrm{Cay}_n(212)$ are Wilf equivalent independent of Proposition 2.13. Moreover, all six species are isomorphic by Proposition 1.18.

Next, we consider Cayley permutations avoiding the ordinary permutations of length 3. After translating between contexts, the next result follows by combining results from [3] and [9].

Proposition 3.33. $\hat{\mathrm{C}}\mathrm{ay}(123)(x) = \dfrac{1}{2} + \dfrac{1}{1+\sqrt{1-8x+8x^2}}$.

By Proposition 2.13, all ordinary permutations $p \in \{123, 321, 312, 213, 132, 231\}$ are Wilf equivalent, so the ordinary generating function for Cay(123) is also the ordinary generating function for all other ordinary permutations of length 3.

We will now look at pairs of patterns of length 3. First, we will provide a functional species equation for Cay(111, 112) by altering the thinking we used to obtain the species Cay(112).

Proposition 3.34. $\mathrm{Cay}(111, 112)' = L \cdot X \cdot \mathrm{Cay}(111, 112) + L \cdot \mathrm{Cay}(111, 112)$.

Proof. In terms of ballots, in a 112-avoiding and 111-avoiding ballot, all the blocks preceding the block containing the maximum value must be singletons and all blocks are at most size 2. Identify the maximum value with $\star$. If $\star$ is not in a block by itself, then a 111-avoiding and 112-avoiding ballot is of the form

$$(\underbrace{\{a_1\}, \{a_2\}, \ldots \{a_k\}}_{L}, \underbrace{\{\star, b_1\}}_{X \text{ ignoring } \star}, \underbrace{\{\cdots\}, \ldots, \{\cdots\}}_{\mathrm{Bal}(111,112)}),$$

where the blocks to the right of the block containing $\star$ may or may not exist. Now, if $\star$ is in a block by itself, then a 111-avoiding and 112-avoiding Cayley permutation is of the form

$$(\underbrace{\{a_1\}, \{a_2\}, \ldots \{a_k\}}_{L}, \underbrace{\{\star\}, \{\cdots\}, \ldots, \{\cdots\}}_{\mathrm{Bal}(111,112) \text{ ignoring } \star}),$$

where again, blocks to the right of a block containing $\{\star\}$ may or may not exist. It follows that

$$\mathrm{Bal}(111, 112)' = L \cdot X \cdot \mathrm{Bal}(111, 112) + L \cdot \mathrm{Bal}(111, 112),$$

which implies

$$\mathrm{Cay}(111, 112)' = L \cdot X \cdot \mathrm{Cay}(111, 112) + L \cdot \mathrm{Cay}(111, 112).$$

□

Using integration, we acquire the corresponding exponential generating function.

Proposition 3.35. $\operatorname{Cay}(111, 112)(x) = \dfrac{e^{-x}}{(1-x)^2}$.

Proof. Using the functional species equation in Proposition 3.34 together with Proposition 1.6, we obtain

$$\operatorname{Cay}(111, 112)'(x) = \frac{x}{1-x}\operatorname{Cay}(111, 112)(x) + \frac{1}{1-x}\operatorname{Cay}(111, 112)(x).$$

To ease notation, we set $F = \operatorname{Cay}(111, 112)$. Simplifying, we get

$$\begin{aligned} F'(x) &= \frac{xF(x)}{1-x} + \frac{F(x)}{1-x} \\ &= F(x)\left(\frac{x+1}{1-x}\right). \end{aligned}$$

Solving the differential equation, we get

$$\begin{aligned} F(x) &= e^{-2\ln|1-x|}e^{-x} \\ &= e^{\ln(1-x)^{-2}}e^{-x} \\ &= \frac{e^{-x}}{(1-x)^2}, \end{aligned}$$

and so $\operatorname{Cay}(111, 112)(x) = \dfrac{e^{-x}}{(1-x)^2}$. □

Proposition 3.36. $\operatorname{Cay}(111, 112) = L \cdot \operatorname{Der}$.

Proof. Recall that $L(x) = \frac{1}{1-x}$ and $\operatorname{Der}(x) = \frac{e^{-x}}{1-x}$ by Proposition 1.6. By Propositions 1.18 and 3.35, it follows that $\operatorname{Cay}(111, 112) = L \cdot \operatorname{Der}$. □

We currently do not have a combinatorial interpretation of the previous result.

By Proposition 2.11, $\operatorname{Cay}_n(111, 112)$, $\operatorname{Cay}_n(111, 221)$, $\operatorname{Cay}_n(111, 122)$, and $\operatorname{Cay}_n(111, 211)$ are all Wilf equivalent. So, the results for $\operatorname{Cay}(111, 112)$ also apply to $\operatorname{Cay}(111, 221)$, $\operatorname{Cay}(111, 122)$, and $\operatorname{Cay}(111, 211)$. The pair $\operatorname{Cay}(111, 121)$ and $\operatorname{Cay}(111, 212)$ are Wilf equivalent by Proposition 2.11 using the complement map, and data produced using Python suggests this pair is Wilf equivalent to the previously mentioned four sets. This yields the following conjecture.

Conjecture 3.37. We claim that

$$[111, 112]_w = \{\{111, 112\}, \{111, 211\}, \{111, 221\}, \{111, 122\}, \{111, 121\}, \{111, 212\}\},$$

which would imply that the functional species equation, the derivative of the functional species equation, and the exponential generating function for $\operatorname{Cay}(111, 112)$ also apply to $\operatorname{Cay}(111, 121)$ and $\operatorname{Cay}(111, 212)$.

Next, we look at Cay(221, 212).

Proposition 3.38. $\text{Cay}(221, 212) = 1 + \text{Cay}(221, 212) * E + \text{Cay}(221, 212)^{\bullet}$.

Proof. We provide a combinatorial interpretation. The empty Cayley permutation is certainly both 221-avoiding and 212-avoiding, which is accounted for by the species 1 on the right hand side. Now consider a nonempty 221-avoiding and 212-avoiding Cayley permutation. As in Proposition 3.26, a Cayley permutation avoiding 212 has its maximum value as a contiguous string. We consider two cases. Assume that the contiguous string consisting of maximum value m occurs on the far right. Then we have

$$\underbrace{p_1 \cdots p_i}_{\text{Cay}(221,212)} \underbrace{m}_{X} \underbrace{m \cdots m}_{E},$$

where the maximum value is m. Now, assume that the contiguous string of the maximum value does not occur on the far right. But to be 221-avoiding, there can only be one occurrence of the maximum value. We can then identify a distinguished position in a 221-avoiding and 212-avoiding permutation, and insert a singular new maximum value to the left of the distinguished position. That is, we have the form

$$\underbrace{p_1 \cdots p_i m p_{i+2} \cdots p_n}_{\text{Cay}(221,212)^{\bullet}},$$

producing the species $\text{Cay}(221, 212)^{\bullet}$. Combining the cases, we get $\text{Cay}(221, 212) = 1 + \text{Cay}(221, 212) * E + \text{Cay}(221, 212)^{\bullet}$. □

We need the following proposition before we use the functional species equation to derive the corresponding exponential generating function.

Proposition 3.39. $\text{Cay}(221, 212)'(x) = e^x + \text{Cay}(221, 212)'(x) * e^x + \text{Cay}(221, 212)^{\bullet}(x)$.

Proof. Let $F = \text{Cay}(221, 212)$. From Propositions 1.18 and 3.38, we have

$$F(x) = 1 + e^x * F(x) + F^{\bullet}(x) = 1 + F(x) * e^x + 1 * F^{\bullet}(x)$$

since the convolution operation is commutative and $F^{\bullet}(x) = 1 * F^{\bullet}(x)$. Using the Product Rule and Proposition 1.21 (Leibniz Rule), we have

$$\begin{aligned} F'(x) &= 0 + F(0) \cdot e^x + F'(x) * e^x + 1 \cdot F^{\bullet}(x) + 0 * F^{\bullet}(x) \\ &= e^x + F'(x) * e^x + F^{\bullet}(x) \end{aligned}$$

since $F(0) = 1$ and $0 * F^{\bullet}(x) = 0$. □

The previous proposition gives us the functional species equation by Proposition 1.18.

Proposition 3.40. $\mathrm{Cay}(221,212)' = E + \mathrm{Cay}(221,212)' * E + \mathrm{Cay}(221,212)^{\bullet}$.

We now use the functional species equation to determine the following exponential generating function.

Proposition 3.41. $\mathrm{Cay}(221,212)(x) = 1 + \displaystyle\int_0^x \frac{e^t}{(1-t)^2}dt$.

Proof. Define $G(x) := 1 + \displaystyle\int_0^x \frac{e^t}{(1-t)^2}dt$ such that $G(0) = 1$. Recall that $G^{\bullet}(x) = x \cdot G'(x)$ and notice that $G'(x) = \dfrac{e^x}{(1-x)^2} = E(x) \cdot L^2(x)$.

We will show that $G(x)$ satisfies the functional species equation in Proposition 3.40. We see that

$$\begin{aligned}
e^x + G'(x) * e^x + G^{\bullet}(x) &= e^x + e^x * \frac{e^x}{(1-x)^2} + \frac{xe^x}{(1-x)^2} \\
&= e^x + \int_0^x \frac{e^t}{(1-t)^2} \cdot e^{(x-t)}dt + \frac{xe^x}{(1-x)^2} \\
&= e^x + e^x \int_0^x \frac{1}{(1-t)^2}dt + \frac{xe^x}{(1-x)^2} \\
&= e^x(1 + x \cdot L(x) + x \cdot L^2(x)) \\
&= e^x(1 + L(x) - 1 + x \cdot L^2(x)) \qquad \text{(by Example 1.14)} \\
&= e^x L(x)(1 + x \cdot L(x)) \\
&= e^x L(x)L(x) \\
&= E(x)L^2(x) \\
&= G'(x).
\end{aligned}$$

Therefore, we have $\mathrm{Cay}(221,212)(x) = 1 + \displaystyle\int_0^x \frac{e^t}{(1-t)^2}dt$. □

It appears that $\int_0^x \frac{e^t}{(1-t)^2}$ is not equal to an elementary function, which is why we have left the exponential generating function for $\mathrm{Cay}(221,212)$ in terms of this integral.

By Proposition 2.11, $\mathrm{Cay}_n(221,212)$, $\mathrm{Cay}_n(112,121)$, $\mathrm{Cay}_n(211,121)$, and $\mathrm{Cay}_n(112,212)$ are all Wilf equivalent. Thus, the results for $\mathrm{Cay}(221,212)$ also apply to $\mathrm{Cay}(112,121)$, $\mathrm{Cay}(211,121)$, and $\mathrm{Cay}(122,212)$. Data, again produced using Python, suggests that the previous four sets are also Wilf equivalent to $\mathrm{Cay}_n(121,212)$, which is in its own symmetry class. This yields the following conjecture.

Conjecture 3.42. We claim that

$$\begin{aligned}
[221,212]_w &= [221,212]_s \cup [121,212]_s \\
&= \{\{221,212\},\{112,121\},\{211,121\},\{112,212\},\{121,212\}\},
\end{aligned}$$

which would imply that the functional species equation, the derivative of the functional species equation, and the exponential generating function for $\mathrm{Cay}(221, 212)$ also apply to $\mathrm{Cay}(121, 212)$.

We were unable to determine a species equation, exponential generating function, ordinary generating function, or counting formula for the remaining pairs of patterns of length three. However, based on numerical data produced using Python, we have two more conjectures.

Recall parts (h)–(m) from Proposition 2.14. The data suggests that the corresponding six symmetry classes can be merged into two. The counting sequences for $\mathrm{Cay}(112, 123)$, $\mathrm{Cay}(112, 213)$, $\mathrm{Cay}(121, 132)$, and $\mathrm{Cay}(121, 231)$ appear to match OEIS entry A001003 up to at least the first eight terms. The first seven terms of the counting sequences for $\mathrm{Cay}(123, 132)$ and $\mathrm{Cay}(312, 132)$ appear to match OEIS entry A007583 after inserting a leading 1 into the OEIS sequence. This produces the following conjecture.

Conjecture 3.43. We claim that

- $$\begin{aligned}[112, 123]_w &= [112, 123]_s \cup [112, 213]_s \cup [121, 132]_s \cup [121, 231]_s \\ &= \{\{112, 123\}, \{211, 321\}, \{221, 321\}, \{122, 123\}, \{112, 213\}, \{211, 312\}, \\ &\quad \{221, 231\}, \{122, 132\}, \{121, 132\}, \{212, 312\}, \{121, 231\}, \{212, 213\}\}.\end{aligned}$$

- $$\begin{aligned}[123, 132]_w &= [123, 132]_s \cup [312, 132]_s \\ &= \{\{123, 132\}, \{321, 231\}, \{321, 312\}, \{123, 213\}, \{312, 132\}, \{213, 231\}\}.\end{aligned}$$

The findings from this section are summarized in Table 3.2 with the exception of the ordinary generating function in Proposition 3.33. Note that we utilize F for ease of notation when the species is written in terms of a functional species equation.

Patterns	Species	EGF	Enumeration	OEIS
111	$L(E_1 + E_2)$	$\dfrac{2}{2-2x-x^2}$	$n! \cdot \dfrac{(1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}}{2^{n+1}\sqrt{3}}$	A080599
212	$F = 1 + E * F + E * F^{\bullet}$	$\dfrac{x^2-2x+2}{2(x-1)^2}$	$\begin{cases} 1, & n = 0 \\ \frac{(n+1)!}{2}, & n \geq 1 \end{cases}$	A001710
121				
112	$F' = L \cdot F_{+} + L \cdot F$			
211				
221				
122				
221,212	$F = 1 + E * F + 1 * F^{\bullet}$	$1 + \displaystyle\int_0^x \frac{e^t}{(1-t)^2} dt$		A001339
211,121				
112,121	$F' = E + F' * E + F^{\bullet}$			
122,212				
111,112	$F' = L \cdot F + L \cdot x \cdot F$	$\dfrac{e^{-x}}{(1-x)^2}$		A000255
111,211				
111,122				
111,221				

Table 3.2: Results for patterns and pairs of distinct patterns of length 3.

Chapter 4

Primitive Cayley permutations

This chapter focuses on primitive Cayley permutations. A *primitive Cayley permutation* of length n is a Cayley permutation $p = p_1 \cdots p_n$ such that $p_i \neq p_{i+1}$ for $1 \leq i \leq n-1$. That is, a primitive Cayley permutation has no "flat steps". We will analyze primitive Cayley permutations that avoid patterns and distinct pairs of patterns up to length 3. For a set of patterns P, let $\mathrm{Prim}(P)$ be the species of primitive Cayley permutations and let $\mathrm{Prim}_n(P), \mathrm{prim}_n(P), \mathrm{Prim}(P)(x)$ and $\hat{\mathrm{Prim}}(P)(x)$ have the naturally intended meaning.

We first provide species, exponential generating functions, counting formulas, and ordinary generating functions for primitive Cayley permutations avoiding patterns of length 2.

Proposition 4.1. For $n \geq 1$, $\mathrm{Prim}_n(11)$ is the set of all ordinary permutations of length n. In particular, $\mathrm{Prim}_n(11) = \mathrm{Cay}_n(11)$.

Proof. Primitive Cayley permutations that are 11-avoiding do not contain any repeats. This leaves the ordinary permutations, which is the same as $\mathrm{Cay}_n(11)$. $\square$

The next four results follow immediately from the fact that $\mathrm{Prim}_n(11) = \mathrm{Cay}_n(11)$. See Propositions 3.6–3.9.

Proposition 4.2. $\mathrm{Prim}(11) = L$.

Proposition 4.3. $\mathrm{Prim}(11)(x) = \dfrac{1}{1-x}$.

Proposition 4.4. For $n \geq 0$, $\mathrm{prim}_n(11) = n!$.

Proposition 4.5. We have

$$\hat{\mathrm{Prim}}(11)(x) = \cfrac{1}{1 - x - \cfrac{1^2x^2}{1-3x-\cfrac{2^2x^2}{1-5x-\cfrac{3^2x^2}{\ddots}}}},$$

where odd numbers are the coefficients of x and squares are the coefficients of x^2.

We now transition to other patterns of length 2. While looking at patterns of length 2, we also obtained results on pairs of distinct patterns of length 2 simultaneously.

Proposition 4.6. For all $n \geq 1$, $\mathrm{Prim}_n(12) = \mathrm{Prim}_n(11, 12) = \mathrm{Cay}_n(11, 12) = \{n(n-1)\cdots 321\}$.

Proof. Primitive Cayley permutations that avoid 12 are weakly decreasing. Additionally, primitive Cayley permutations do not contain any consecutive strings of the same value. This leaves the decreasing permutation $n(n-1)\cdots 321$ of length n. It is easily seen that $\mathrm{Prim}_n(11, 12) = \{n(n-1)\cdots 321\} = \mathrm{Cay}_n(11, 12)$ (see Proposition 3.14). $\square$

We immediately get the following from Propositions 3.17–3.19.

Proposition 4.7. For all $n \geq 0$, $\mathrm{prim}_n(12) = \mathrm{prim}_n(11, 12) = 1$.

Proposition 4.8. $\mathrm{Prim}(12)(x) = \mathrm{Prim}(11, 12)(x) = e^x$.

Proposition 4.9. $\mathrm{Prim}(12) = \mathrm{Prim}(11, 12) = E$.

Proposition 4.10. $\hat{\mathrm{P}}\mathrm{rim}(12)(x) = \hat{\mathrm{P}}\mathrm{rim}(11, 12)(x) = \dfrac{1}{1-x}$.

By Proposition 2.11 restricted to Prim_n, $\mathrm{Prim}_n(12)$ and $\mathrm{Prim}_n(21)$ are Wilf equivalent (using the reverse map). Also, $\mathrm{Prim}_n(11, 12)$ and $\mathrm{Prim}_n(11, 21)$ are Wilf equivalent (using the reverse map). Hence, the previous four results also apply to $\mathrm{Prim}_n(21)$ and $\mathrm{Prim}_n(11, 21)$.

We only have one pair of patterns of length 2 left to consider.

Proposition 4.11. For $n \geq 0$,

$$\mathrm{Prim}_n(12, 21) = \begin{cases} \{\text{empty word}\}, & n = 0 \\ \{1\}, & n = 1 \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proof. The claim is clear when $n = 0$ and $n = 1$. Otherwise, primitive Cayley permutations avoiding 12 and 21 cannot have an increase nor a decrease in value but also cannot have any consecutive repeats. Therefore, for $n \geq 2$, there does not exist a primitive Cayley permutation that avoids both 12 and 21, and hence $\mathrm{Prim}_n(12, 21) = \emptyset$ for $n \geq 2$. $\square$

The next result is immediate.

Proposition 4.12. For $n \geq 0$,

$$\mathrm{prim}_n(12, 21) = \begin{cases} 1, & n \in \{0, 1\} \\ 0, & \text{otherwise.} \end{cases}$$

Using the previous proposition, we obtain the species and corresponding exponential generating function for $\mathrm{Prim}(12, 21)$.

Proposition 4.13. $\mathrm{Prim}(12, 21)(x) = 1 + x$.

By Proposition 1.18, we get the next result.

Proposition 4.14. $\mathrm{Prim}(12, 21) = 1 + E_1$.

It is not hard to see that the ordinary generating function is the same as the exponential generating function in this case.

Proposition 4.15. $\hat{\mathrm{P}}\mathrm{rim}(12, 21)(x) = 1 + x$.

For pairs of patterns of length 3, we were only able to obtain results for $\mathrm{Prim}(123, 321)$. First, we cite a necessary result known as the Erdős–Szekeres Theorem [5].

Proposition 4.16 (Erdős–Szekeres Theorem)**.** For $r, s \in \mathbb{N}$, any (ordinary) permutation of length at least $rs + 1$ contains either the pattern $123 \cdots (r + 1)$ or $(s + 1) \cdots 321$.

Example 4.17. We look at the set $\mathrm{Prim}_4(123, 321)$ to build some intuition. We see that

$$\mathrm{Prim}_4(123, 321) = \{\underline{1212, 1312}, \underline{2121, 2131, 2132, 2143, 2312, 2313, 2413}, \underline{3132, 3142, 3412}\}.$$

We have underlined primitive Cayley permutations based on their initial value. Notice that two of the permutations start with 1, seven start with 2, and three start with 3. We will argue that this phenomenon holds for all $n \geq 4$.

Proposition 4.18. For $n \geq 0$,

$$\mathrm{prim}_n(123, 321) = \begin{cases} 1, & n \leq 1 \\ 2, & n = 2 \\ 6, & n = 3 \\ 12, & n \geq 4. \end{cases}$$

Proof. By brute force, we get the results for $n \in \{0, 1, 2, 3\}$. We have verified the result for $n = 4$ in Example 4.17. Now suppose that $n \geq 5$. By the Erdős–Szekeres Theorem, every ordinary permutation of length at least 5 contains the pattern 123 or 321. It follows that every primitive Cayley permutation in $\mathrm{Prim}_n(123, 321)$ has maximum value of at most 4. We have three cases to consider based on the initial value of the corresponding primitive Cayley permutations in $\mathrm{Prim}_n(123, 321)$.

If $p \in \mathrm{Prim}_n(123, 321)$ starts with a 1, then the first two values are either 12 or 13. If p starts with a 12, we have

$$\underbrace{12 \cdots 1}_{n \text{ odd}} \text{ or } \underbrace{12 \cdots 12}_{n \text{ even}}.$$

If p starts with a 13, we have

$$\underbrace{13\cdots 132}_{n\text{ odd}} \text{ or } \underbrace{13\cdots 1312}_{n\text{ even}}.$$

Notice that if p started with 14, we would need a 2 and 3 later in our permutation, resulting in a 123 or 321 pattern. Therefore, regardless of whether n is odd or even, we have two primitive Cayley permutations that begin with 1.

If $p \in \mathrm{Prim}_n(123, 321)$ starts with a 3, then the first two values are either 31 or 34. If p starts with a 31, then we can have the first three positions be 313 or 314. We then have

$$\underbrace{313\cdots 12}_{n\text{ odd}} \text{ or } \underbrace{313\cdots 132}_{n\text{ even}}$$

or

$$\underbrace{314\cdots 12}_{n\text{ odd}} \text{ or } \underbrace{314\cdots 142}_{n\text{ even}}.$$

We could also have p start with 34, in which case we have

$$\underbrace{3414\cdots 142}_{n\text{ odd}} \text{ or } \underbrace{341\cdots 412}_{n\text{ even}}.$$

If p started with 32, we would need a 1 later in the permutation, resulting in a 321 pattern. Hence, regardless of whether n is odd or even, we have three primitive Cayley permutations that begin with 3.

If $p \in \mathrm{Prim}_n(123, 321)$ starts with a 2, then the first three values could be 212, 213, 214, 231, or 241. If p starts with 212, then we have

$$\underbrace{212\cdots 12}_{n\text{ odd}} \text{ or } \underbrace{212\cdots 1}_{n\text{ even}}.$$

If p starts with a 213, then we could have

$$\underbrace{213\cdots 13}_{n\text{ odd}} \text{ or } \underbrace{213\cdots 131}_{n\text{ even}}$$

or

$$\underbrace{213\cdots 1312}_{n\text{ odd}} \text{ or } \underbrace{213\cdots 132}_{n\text{ even}}.$$

We could also have p start with 214 in which case we have

$$\underbrace{214\cdots 413}_{n\text{ odd}} \text{ or } \underbrace{214\cdots 143}_{n\text{ even}}.$$

If p starts with 231, then we could have

$$\underbrace{231\cdots 132}_{n\text{ odd}} \text{ or } \underbrace{231\cdots 312}_{n\text{ even}}$$

or

$$\underbrace{231\cdots 31}_{n \text{ odd}} \text{ or } \underbrace{231\cdots 313}_{n \text{ even}}.$$

Lastly, p could start with 241 and we could have

$$\underbrace{241\cdots 143}_{n \text{ odd}} \text{ or } \underbrace{241\cdots 413}_{n \text{ even}}.$$

In the final case, seven primitive Cayley permutations begin with 2.

Combining all possible cases, we see that for $n \geq 4$, $\text{prim}_n(123, 321) = 12$. $\square$

Proposition 4.19. $\text{Prim}(123, 321)(x) = 12e^x - (11 + 11x + 5x^2 + x^3)$.

Proof. The function $12e^x$ is the exponential generating function for the constant sequence of 12s. To produce the desired exponential generating function we need to subtract the appropriate terms, and hence we obtain the desired result. $\square$

The results of Proposition 2.11 certainly apply to primitive Cayley permutations avoiding a set of patterns. Interestingly, numerical investigations have suggested that several collections of primitive Cayley permutations avoiding distinct pairs of patterns of length 3 are enumerated by a sequence appearing in OEIS. In the list below, we have indicated which sets are symmetric and have a corresponding hit in OEIS. After potentially inserting or deleting one or two leading terms of the corresponding OEIS entry, each counting sequence matches at least the first seven terms. In most cases, we either have to insert or delete one or two leading terms from the OEIS sequence.

- $\text{Prim}(121) \sim \text{Prim}(212)$ (A000153);
- $\text{Prim}(112, 221) \sim \text{Prim}(211, 122)$ (A052582);
- $\text{Prim}(121, 123) \sim \text{Prim}(121, 321) \sim \text{Prim}(212, 321) \sim \text{Prim}(212, 123)$ (A119370);
- $\text{Prim}(123, 132) \sim \text{Prim}(321, 231) \sim \text{Prim}(321, 312) \sim \text{Prim}(123, 213)$ (A025192);
- $\text{Prim}(231, 132) \sim \text{Prim}(132, 231) \sim \text{Prim}(213, 312) \sim \text{Prim}(312, 213)$ (A265278);
- $\text{Prim}(123, 231) \sim \text{Prim}(321, 132) \sim \text{Prim}(321, 213) \sim \text{Prim}(123, 312)$ (A038503).

The findings from this chapter are summarized in Table 4.1.

Patterns	Species	EGF	OGF	Enumeration	OEIS
11	L	$\dfrac{1}{1-x}$	$\dfrac{1}{1-x-\dfrac{1^2x^2}{1-3x-\dfrac{2^2x^2}{1-5x-\dfrac{3^2x^2}{\ddots}}}}$	$n!$	A000142
12 21 11,12 11,21	E	e^x	$\dfrac{1}{1-x}$	1	A000012
12,21	$1+E_1$	$1+x$	$1+x$	$\begin{cases} 1, & n \in \{0,1\} \\ 0, & \text{otherwise} \end{cases}$	A019590
123,321		$12e^x-(11+11x+5x^2+x^3)$		$\begin{cases} 1, & n \in \{0,1\} \\ 2, & n=2 \\ 6, & n=3 \\ 12, & n \geq 4 \end{cases}$	

Table 4.1: Results for patterns and pairs of distinct patterns of primitive Cayley permutations.

Chapter 5

Conclusion

In this thesis, we analyzed Cayley permutations avoiding patterns and pairs of patterns of lengths 2 and 3 using a species-first approach when possible.

In Chapter 1, we established the basics of species through motivating examples and definitions. Of particular importance, we established the difference between $\mathbb{B}$-species and $\mathbb{L}$-species. The key difference between the two is that the underlying sets are ordered for $\mathbb{L}$-species.

In Chapter 2, we introduced Cayley permutations and pattern avoidance along with an isomorphism between the species of Cayley permutations and ballots. We also summarized symmetry and Wilf classes in the context of Cayley permutations avoiding a set of patterns.

In the next chapter, we explored Cayley permutations avoiding certain sets of patterns. In the first section, we obtained the species, exponential generating function for $\mathrm{Cay}(1^k)$, and a counting formula for any $k \geq 2$. In the second section, we obtained the species, exponential generating function, counting formula, and ordinary generating function for all patterns and pairs of patterns of length 2. Our results are summarized in Table 3.1. In the last section, we investigated Cayley permutations avoiding patterns of length 3 and pairs of patterns of length 3. We were able to characterize all sets of Cayley permutations avoiding patterns of length 3 that do not correspond to ordinary permutations. In each of these cases, we obtained the species, exponential generating function, and counting formula. We also obtained a few species and exponential generating functions for Cayley permutations avoiding pairs of patterns of length 3. Our results are summarized in Table 3.2.

In Chapter 4, we touched on primitive Cayley permutations avoiding patterns of length 2, pairs of patterns of length 2, and a pair of patterns of length 3. The results for this chapter are summarized in Table 4.1.

We have the following conjectures concerning Wilf classes of pairs of patterns of length 3 inspired by numerical investigations.

- $[111, 112]_w = \{\{111, 112\}, \{111, 211\}, \{111, 221\}, \{111, 122\}, \{111, 121\}, \{111, 212\}\}$;
- $[221, 212]_w = \{\{221, 212\}, \{112, 121\}, \{211, 121\}, \{112, 212\}, \{121, 212\}\}$;

- $[112, 123]_w = \{\{112, 123\}, \{211, 321\}, \{221, 321\}, \{122, 123\}, \{112, 213\}, \{211, 312\},$
$\{221, 231\}, \{122, 132\}, \{121, 132\}, \{212, 312\}, \{121, 231\}, \{212, 213\}\};$
- $[123, 132]_w = \{\{123, 132\}, \{321, 231\}, \{321, 312\}, \{123, 213\}, \{312, 132\}, \{213, 231\}\}.$

We conclude with a list of open problems. We wish to characterize and enumerate:

- Cayley permutations that avoid pairs of patterns of length 3 that we did not address;
- Cayley permutations that avoid patterns and pairs of patterns of lengths greater than 3;
- Cayley permutations that avoid sets of patterns of different lengths;
- Primitive Cayley permutations that avoid sets of patterns of length 3 that we did not address, as well as sets of patterns greater than 3.

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