ARCHITECTURE OF BRAID GRAPHS IN COXETER SYSTEMS

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ABSTRACT

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Any two reduced expressions for an element of a Coxeter group are related by a sequence of commutation and braid moves. Two reduced expressions are called braid equivalent if they are related by a sequence of only braid moves. Braid equivalence is an equivalence relation, and the corresponding equivalence classes are called braid classes. The braid class for a reduced expression can be encoded in a graph, called a braid graph, in a natural way. In a paper by Barnes, Breland, Ernst, and Perry, the authors proved that in a Coxeter system that is simply laced and triangle free (i.e., the corresponding Coxeter graph contains no three-cycles), every braid graph is median. In this thesis, we extend this result and prove that every braid graph in a Coxeter system whose corresponding Coxeter graph contains no three-cycles with the labels $3, 3, m \geq 3$ is median. To that end, we also generalize the theory presented in the aforementioned paper and a paper by Awik, Breland, Cadman, and Ernst. Many of the proofs and theorem statements in this thesis take inspiration from research done by three undergraduate students Atillio, Patrick, and Wilmer, under the guidance of Ernst during the 2024–2025 academic year.

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Chapter 1

Required graph theory

In this chapter, we discuss the graph theory concepts and terminology necessary to understand the results of this thesis. All of the graphs discussed throughout this thesis are assumed to be undirected, finite, connected, and simple. We will denote the vertex set of a graph G as $V(G)$ and the edge set as $E(G)$.

Let G be a graph and let $S \subseteq V(G)$. The graph whose vertex set is S and whose edges are all the edges of G incident to vertices in S is called the *subgraph induced by* S, denoted $G[S]$.

Example 1.1. Consider the subgraphs depicted in blue in Figure 1.1. The subgraph in Figure 1.1(a) is induced by $S = \{a, b, c, d, f\}$ while the subgraph in Figure 1.1(b) is not induced by $T = \{a, b, c, d, e\}$ since the edge $\{b, e\}$ is absent from the subgraph.

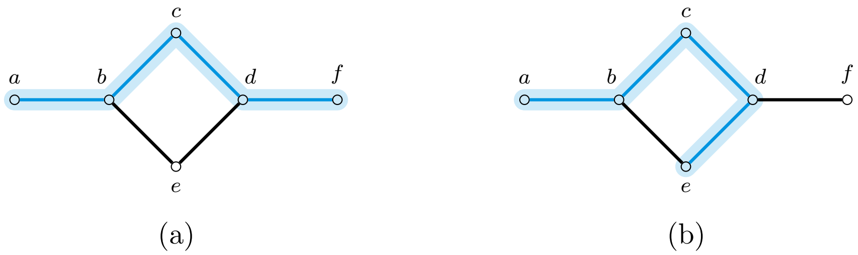
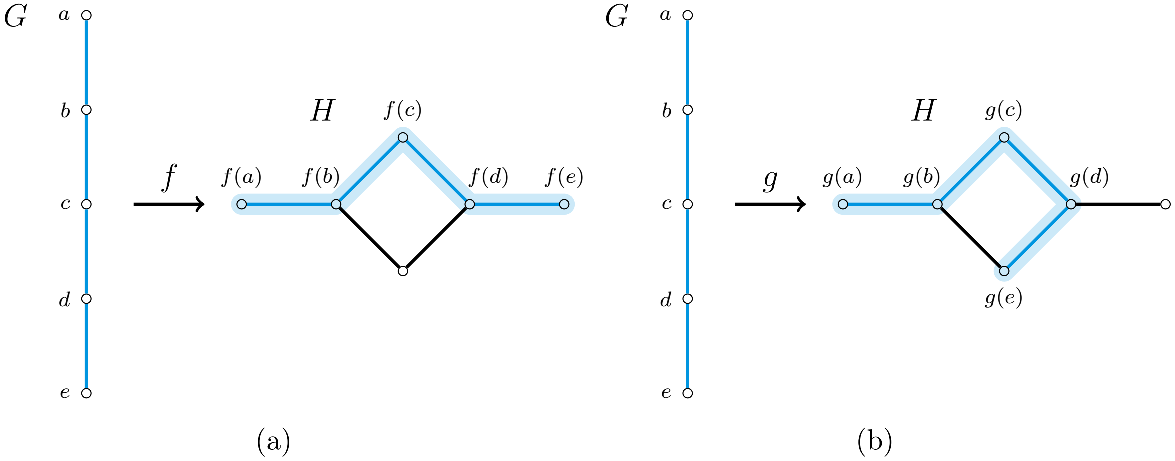


Figure 1.1: An example and non-example of an induced subgraph as described in Example 1.1.

For graphs G and H, a *graph homomorphism* $f\colon G \to H$ is a function $f\colon V(G) \to V(H)$ that satisfies $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(H)$. An injective graph homomorphism $f\colon G \to H$ is called an *embedding* of G into H. Additionally, if f is such that $\{u, v\} \in E(G)$ whenever $\{f(u), f(v)\} \in E(H)$, then we say that f is an *induced embedding*. If f is an induced embedding, then G is isomorphic to the subgraph of H induced by the image of f.

Example 1.2. The embedding depicted in Figure 1.2(a) is an induced embedding while the map shown in Figure 1.2(b) is not an induced embedding since $\{g(b), g(e)\} \in E(H)$ while $\{b, e\} \notin E(G)$.



(a) (b)

Figure 1.2: An induced embedding and an embedding that is not induced as described in Example 1.2.

In this thesis, we use the notion of distance between two vertices of a graph to establish several results. A *geodesic* in a graph G between two vertices u and v is a shortest path between u and v. We define the distance between u and v via

$$d_G(u, v) := \text{the length of any geodesic between } u \text{ and } v.$$

Note that we will simply write $d(u, v)$ in place of $d_G(u, v)$ when the context is clear. Using the given distance metric, we define the *diameter* of G to be

$$\operatorname{diam}(G) := \max\{d(u, v) : u, v \in V(G)\}.$$

In other words, $\operatorname{diam}(G)$ is the length of the longest geodesic between pairs of vertices in G. If $d(u, v) = \operatorname{diam}(G)$, then u and v are said to be *diametrical*.

Let G and H be graphs and let $f : G \to H$ be an embedding. We say that f is an *isometric embedding* if for all $u, v \in V(G)$, $d_G(u, v) = d_H(f(u), f(v))$; that is, f is distance preserving. It is not too hard to see that preserving distance also preserves adjacency, so every isometric embedding is also an induced embedding. We say that G is *isometric* to the subgraph induced by the image of f. The example below illustrates that not every induced embedding is an isometric embedding.

Example 1.3. The induced embedding g depicted in Figure 1.3 is not an isometric embedding because $d_G(a, e) = 4$ while $d_H(g(a), g(e)) = 2$.

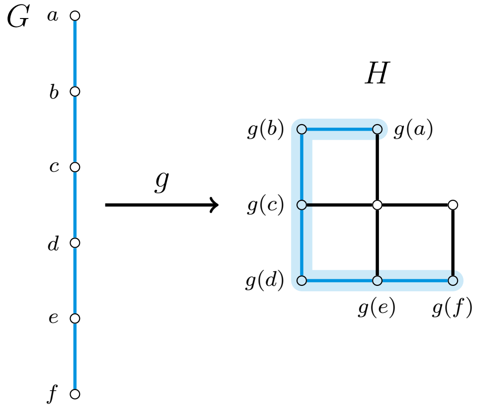


Figure 1.3: An embedding that is induced but not isometric.

Let G_1 and G_2 be graphs. The *box product*, denoted $G_1 \Box G_2$, is the graph whose vertex set is $V(G_1) \times V(G_2)$ and there is an edge from (x_1, y_1) to (x_2, y_2) provided either:

(a) $x_1 = x_2$ and there is an edge from y_1 to y_2 in G_2, or

(b) $y_1 = y_2$ and there is an edge from x_1 to x_2 in G_1.

In this thesis, we are particularly interested in how a graph can be decomposed with respect to the box product.

Example 1.4. Figure 1.4 depicts an example of the box product operator. We colored the graphs to help illustrate how the two graphs create the box product graph.

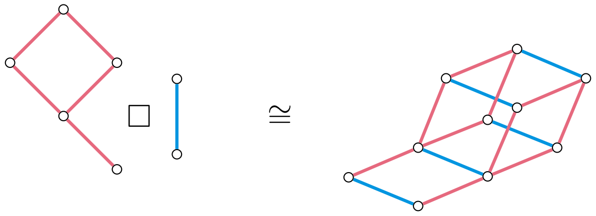


Figure 1.4: An example of the box product of graphs.

For $n \in \mathbb{N} \cup \{0\}$, the *hypercube* Q_n is the graph with vertex set $V(Q_n) = \{0,1\}^n$ and two vertices are adjacent when their corresponding binary strings differ by exactly one digit. Note that the empty string is the only string of length $n = 0$, so that Q_0 consists of a single vertex. A graph G is a *partial cube* if it can be isometrically embedded in some hypercube Q_n. Note that the *isometric dimension* of a partial cube G is defined as the minimum dimension of the hypercube into which the partial cube can be isometrically embedded, and is denoted $\dim_I(G)$.

The rest of this chapter mimics the development in [8] and [9]. We now define the notion of a semicube, which is an important feature used in the major result of this thesis. Let G be a graph and let u and v be distinct vertices. Define $W_{u,v} \subseteq V(G)$ via

$$W_{u,v} := \{w \in V(G) : d(w,u) < d(w,v)\}.$$

That is, $W_{u,v}$ is the set of vertices in G that are closer to u than v. Symmetrically, $W_{v,u}$ is the set of vertices that are closer to v than u. Note that it is possible for some vertices to not be in either semicube, namely the ones equidistant from u and v. Inspired by the role these play in partial cubes, both the subgraph $G[W_{u,v}]$ and the set $W_{u,v}$ are referred to as a *semicube* of G. The two semicubes $W_{u,v}$ and $W_{v,u}$ are called *opposite semicubes*. Note that the definition of semicubes does not require $\{u,v\} \in E(G)$, but in this thesis we only deal with the case in which $\{u,v\} \in E(G)$.

We also define an important subset of a semicube of G. If G is a graph and $\{u,v\} \in E(G)$, then we define the following set:

$$U_{u,v} := \{w \in W_{u,v} : w \text{ is adjacent to a vertex in } W_{v,u}\}.$$

That is, $U_{u,v}$ is the set of vertices that are closer to u than v and adjacent to a vertex in the opposite semicube $W_{v,u}$.

Example 1.5. Figure 1.5 depicts a partial cube. The opposite semicubes $W_{u,v}$ and $W_{v,u}$ are highlighted in pink and yellow, respectively. Also, note that $U_{u,v} = W_{u,v}$ in this case, but $U_{v,u}$ is properly contained in $W_{v,u}$, and is highlighted in grey.

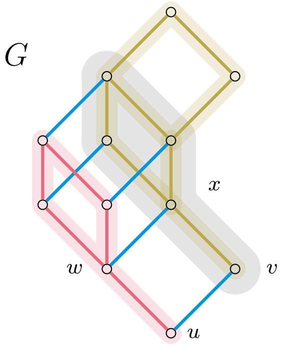


Figure 1.5: A partial cube with the semicubes $W_{u,v}$ and $W_{v,u}$ and the sets $U_{u,v}$ and $U_{v,u}$ highlighted.

The next proposition is from [9]. It states that if $\{u,v\} \in E(G)$, then all vertices in $W_{v,u}$ are exactly one step further from u than v in G.

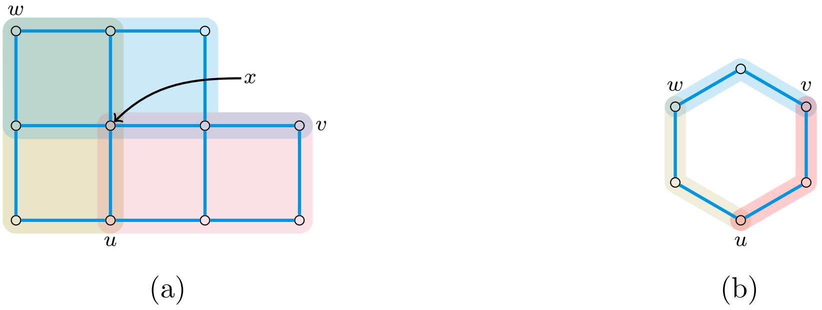
Proposition 1.6. Let G be a graph. If $w \in W_{u,v}$ for some edge $\{u,v\} \in E(G)$, then $d(w,v) = d(w,u) + 1$. Moreover, $W_{u,v} = \{w \in V(G) : d(w,v) = d(w,u) + 1\}$.

Now, we will focus on median graphs, a prominent idea in this thesis. Let G be a graph. The *interval* between vertices u and v, denoted $I(u,v)$, is the union of vertices on all geodesics between u and v. A graph G is *median* if

$$\operatorname{card}(I(u,v) \cap I(u,w) \cap I(v,w)) = 1$$

for all $u, v, w \in V(G)$. In other words, G is median if there is a unique vertex x that simultaneously lies on a geodesic between u and v, a geodesic between u and w, and a geodesic between v and w for all triples u, v, w. If G is a median graph, then we will let $\operatorname{med}(u,v,w)$ denote the unique vertex in $I(u,v) \cap I(u,w) \cap I(v,w)$.

Example 1.7. The shading in Figures 1.6(a) and 1.6(b) depicts $I(u,v)$ in pink, $I(v,w)$ in blue, and $I(u,w)$ in yellow. In Figure 1.6(a), we see that all three colors overlap at the vertex x, illustrating that $\operatorname{card}(I(u,v) \cap I(u,w) \cap I(v,w)) = 1$ implying that $\operatorname{med}(u,v,w) = x$. It turns out that for any three vertices in this graph, the intervals between any pair of the three will overlap at one vertex. Thus, the graph given in Figure 1.6(a) is median. On the other hand, in Figure 1.6(b), we see that there is no vertex common to all of these intervals. Hence, $I(u,v) \cap I(u,w) \cap I(v,w) = \emptyset$, and so the graph in Figure 1.6(b) is not median. Note that one can construct non-median graphs where $\operatorname{card}(I(u,v) \cap I(u,w) \cap I(v,w)) \geq 2$.



(a) (b)

Figure 1.6: Examples of a median graph and non-median graph from Example 1.7.

The following proposition is from [10], and it asserts that every median graph is also a partial cube.

Proposition 1.8. If a graph G is median, then G is a partial cube.

It turns out, however, that if a graph G is a partial cube, then G is not necessarily median. The next example illustrates this.

Example 1.9. A cycle graph with six vertices can be isometrically embedded into a hypercube of dimension three, and is therefore a partial cube. However, as shown in Example 1.7, this graph is not a median graph, so the converse of Proposition 1.8 does not hold.

The following result is well known, for example see [4], and states that the collection of median graphs is closed under the box product operation.

Proposition 1.10. If G_1 and G_2 are median graphs, then $G_1 \Box G_2$ is median.

A subset $S \subseteq V(G)$ is *convex* in G if every vertex on every geodesic connecting a pair of vertices in S is contained in S. That is, S is closed under geodesics. Note that since G is connected, if $C \subseteq V(G)$ is convex, then $G[C]$ is connected, as well.

Example 1.11. In Figure 1.5, notice that the vertex set $U_{v,u}$, highlighted in grey, is convex while the vertex set $\{u, v, w\}$ is not because it does not contain the vertex x, which lies on a geodesic between v and w.

To conclude this chapter on graphs, we discuss peripheral expansions and their relationship to median graphs. Given a graph G and a convex set $C \subseteq V(G)$, we define the *peripheral expansion of G along C* as follows:

- Start with the graph G;
- Make an isomorphic copy of $G[C]$, denoted G'_C, where each $u \in C$ corresponds to $u' \in C' := V(G'_C)$;
- For each $u \in C$, join u and u' with an edge.

The illustration given in Figure 1.7 shows a rough depiction of the process described above. The vertices in $G[C]$ mirror the vertices in the pink G'_C, and each pair of vertices u and u', are connected by a black edge. The rectangles $G[C]$ and G'_C are drawn the same size to indicate the isomorphism between the two graphs: $\{u, v\} \in E(G[C])$ if and only if $\{u', v'\} \in E(G'_C)$. When G is a partial cube, the blue G and pink G'_C are opposite semicubes and the black edges joining each u and u' pair essentially exhibit the isomorphism. Notice that, in this case, $U_{u,u'} = G[C]$ and $U_{u',u} = G'_C$.

Example 1.12. Notice that we can obtain the graph G in Figure 1.5 by doing a peripheral expansion of $G[W_{v,u}]$ along $U_{v,u}$.

Example 1.13. Figure 1.8 illustrates a sequence of peripheral expansions. The grey highlighted portion of each subfigure shows which subgraph is playing the role of $G[C]$. Each subsequent graph represents the graph obtained when the peripheral expansion is performed on the grey portion.

The following theorem from [8] is sometimes referred to as Mulder's Theorem, and it states that a median graph can always be obtained through a sequence of peripheral expansions that begin from a single vertex.

Proposition 1.14. A graph G is median if and only if it can be obtained from a single vertex by a sequence of peripheral expansions.

Example 1.15. Proposition 1.14 implies that each graph in Figure 1.8 is median.

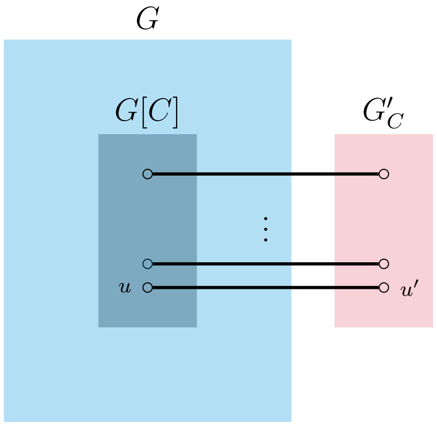


Figure 1.7: A rough illustration of the peripheral expansion process.

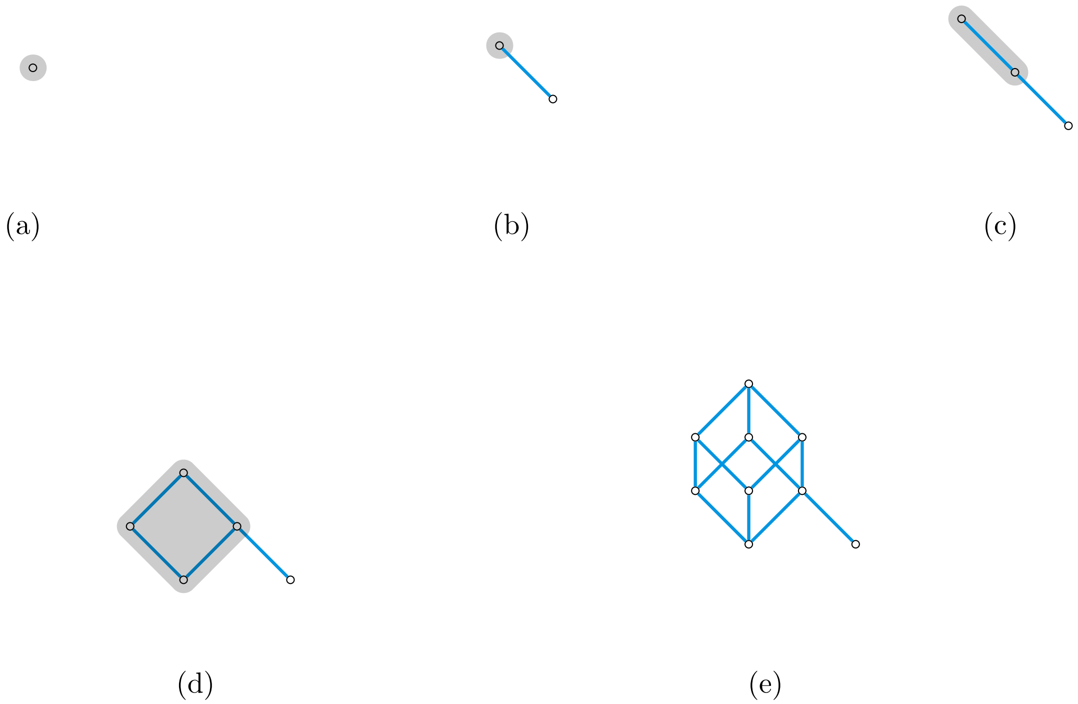


Figure 1.8: A sequence of peripheral expansions starting from a single vertex.

Chapter 2

Braid classes and braid graphs in Coxeter systems

This chapter introduces Coxeter systems, braid classes, and the braid graphs corresponding to braid classes.

For $n \in \mathbb{N}$, a *Coxeter matrix* is an $n \times n$ symmetric matrix $M = (m_{ij})$ with entries $m_{ij} \in \mathbb{N} \cup \{\infty\}$ such that $m_{ii} = 1$ for all $1 \leq i \leq n$ and $m_{ij} \geq 2$ for $i \neq j$. A *Coxeter system* is a pair (W, S) where $S = \{s_1, s_2, \ldots, s_n\}$ is a set of generators and W is a group, called a *Coxeter group*, with presentation

$$W = \langle s_1, s_2, \ldots, s_n : (s_i s_j)^{m(s_i,s_j)} = e \rangle,$$

where $m(s_i, s_j) := m_{ij}$ for some $n \times n$ Coxeter matrix $M = (m_{ij})$. For $s, t \in S$, the condition $m(s,t) = \infty$ means that there is no relation imposed between s and t. In [7], it is shown that the elements of S are distinct as group elements and for $s \neq t$, $m(s,t)$ is the order of st. Since elements of S have order two, the relation $(st)^{m(s,t)} = e$ can be written as

$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)}$$

with $m(s,t) \geq 2$ letters. When $m(s,t) = 2$, $st = ts$ is called a *commutation relation* and when $m(s,t) \geq 3$, the corresponding relation is called a *braid relation*. For $m(s,t) < \infty$, the replacement

$$\underbrace{sts\cdots}_{m(s,t)} \longmapsto \underbrace{tst\cdots}_{m(s,t)}$$

is called a *commutation move* if $m(s,t) = 2$ and a *braid move* if $m(s,t) \geq 3$.

We can visually encode the information given in a Coxeter system into a *Coxeter graph*, Γ, having vertex set S and edges $\{s,t\}$ for each $m(s,t) \geq 3$. Moreover, each edge is labeled with the corresponding $m(s,t)$. Note that the labels of 3 are often omitted because they are the most common. We say that (W, S), or just W, is of type Γ, and we may denote the Coxeter group as $W(\Gamma)$ and the generating set as $S(\Gamma)$ for emphasis.

A Coxeter system is *simply laced* if for all $s, t \in S$, $m(s,t) \leq 3$. That is, a Coxeter system is said to be simply laced if the generators have either a commutation relation or a braid relation of length 3 imposed upon them.

Example 2.1. The Coxeter graphs given in Figure 2.1 correspond to four common Coxeter systems. Using the Coxeter graphs, we can determine the defining relations between the generators of these Coxeter systems. Below we elaborate on these types.

(a) The Coxeter system of type A_n is given by the Coxeter graph in Figure 2.1(a). The Coxeter group $W(A_n)$ has generating set $S(A_n) = \{s_1, s_2, \ldots, s_n\}$ with defining relations

  - $s_i^2 = e$ for all i;
  - $s_i s_j = s_j s_i$ when $|i-j| > 1$;
  - $s_i s_j s_i = s_j s_i s_j$ when $|i-j| = 1$.

  The Coxeter group $W(A_n)$ is isomorphic to the symmetric group S_{n+1} under the mapping that sends s_i to the adjacent transposition $(i, i+1)$.

(b) The Coxeter system of type D_n is given by the Coxeter graph in Figure 2.1(b). The Coxeter group $W(D_n)$ has generating set $S(D_n) = \{s_1, s_2, \ldots, s_n\}$ and has defining relations

  - $s_i^2 = e$ for all i;
  - $s_i s_j = s_j s_i$ if $|i-j| > 1$ and $i, j \neq 1$, or if $i = 1$ and $j \neq 3$;
  - $s_1 s_3 s_1 = s_3 s_1 s_3$ and $s_i s_j s_i = s_j s_i s_j$ if $|i-j| = 1$.

  The Coxeter group $W(D_n)$ is isomorphic to the index two subgroup of the group of signed permutations on n letters having an even number of sign changes.

(c) The Coxeter system of type $\widetilde{A}_n$ depicted in Figure 2.1(c) turns out to yield an infinite Coxeter group.

(d) All of the Coxeter systems determined by the graphs in the aforementioned figure are simply laced, except for type F_4 in Figure 2.1(d) since $m(s_2, s_3) = 4$.

If a Coxeter graph Γ contains no three-cycles, we say that the corresponding Coxeter system (W, S) is *triangle free*. A Coxeter system that is both simply laced and triangle free is said to be of type Λ. In [1] and [2], the focus was on Coxeter systems of type Λ, but in this thesis we focus on Coxeter systems of a more general type. We no longer require that the Coxeter graph Γ be triangle free or simply laced, but we do require that Γ not contain any three cycles with the edges labeled 3, 3, and m, for any $m \geq 3$. We call this kind of Coxeter system $(3,3,m)$-*avoiding*, or simply $\underset{m}{\triangle}$-avoiding.

(a) A_n $(n \geq 1)$ (b) D_n $(n \geq 4)$

(c) $\widetilde{A}_n$ $(n \geq 2)$ (d) F_4

Figure 2.1: Examples of common Coxeter graphs.

Example 2.2. Notice that Coxeter graphs depicted in Figures 2.1(a) and 2.1(b) are both type Λ and $\underset{m}{\triangle}$-avoiding. Except when $n = 2$, the Coxeter graph in Figure 2.1(c) is of type Λ and $\underset{m}{\triangle}$-avoiding. The Coxeter graph in Figure 2.1(d) is triangle free and $\underset{m}{\triangle}$-avoiding. The Coxeter graph depicted in Figure 2.2 is another example of a graph that is not $\underset{m}{\triangle}$-avoiding, and it will be referenced later on in this thesis.

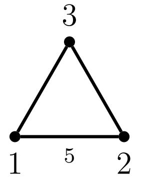


Figure 2.2: The Coxeter graph of type Γ_1.

Consider a Coxeter system (W, S). Define S^* to be the free monoid on S. We call $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ a *word* while a *factor* of $\boldsymbol{\alpha}$ is a word of the form $s_{x_i} s_{x_{i+1}} \cdots s_{x_{j-1}} s_{x_j}$ for $1 \leq i \leq j \leq m$. Now, let $w \in W$. If $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is equal to w when considered as an element of the group W, we say that $\boldsymbol{\alpha}$ is an *expression* for w. If m is minimal among all possible expressions for w, we say that $\boldsymbol{\alpha}$ is a *reduced expression* for w. We define the *length* of w, denoted $\ell(w)$, to be the number of letters in a reduced expression. We will also say that any reduced expression $\boldsymbol{\alpha}$ for w has length $\ell(\boldsymbol{\alpha}) := \ell(w)$. Note that any factor of a reduced expression is also reduced. We denote the set of all reduced expressions for a group element $w \in W$ by $\mathcal{R}(w)$. For brevity, if we are considering a particular labeling of a Coxeter graph, we will replace s_i with i.

The following proposition from [5] is commonly referred to as Matsumoto's Theorem, and it describes the relationships between any two reduced expressions for a single group element.

Proposition 2.3. In a Coxeter system (W, S), any two reduced expressions for the same group element differ by a sequence of commutation and braid moves.

Matsumoto's Theorem naturally leads to a graphical representation of the collection of reduced expressions for an element of a Coxeter group. Let (W, S) be a Coxeter system and let $w \in W$. The *Matsumoto graph* $\mathcal{G}(w)$ is defined to be the graph whose vertex set is $\mathcal{R}(w)$, where two vertices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are connected by an edge if and only if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related via a single commutation or braid move. Temporarily, we will color an edge pink if it corresponds to a commutation move and we will color an edge blue if it corresponds to a braid move. Matsumoto's Theorem implies that $\mathcal{G}(w)$ is connected.

Example 2.4. Consider the reduced expression $\boldsymbol{\gamma}_1 = 12132434$ for some w in the Coxeter system of type F_4. There are 14 reduced expressions in $\mathcal{R}(w)$ and the corresponding Matsumoto graph $\mathcal{G}(w)$ is given in Figure 2.3. The edges of $\mathcal{G}(w)$ show how pairs of reduced expressions are related via commutation or braid moves. We have used overlines and underlines to clarify the locations where braid moves may occur.

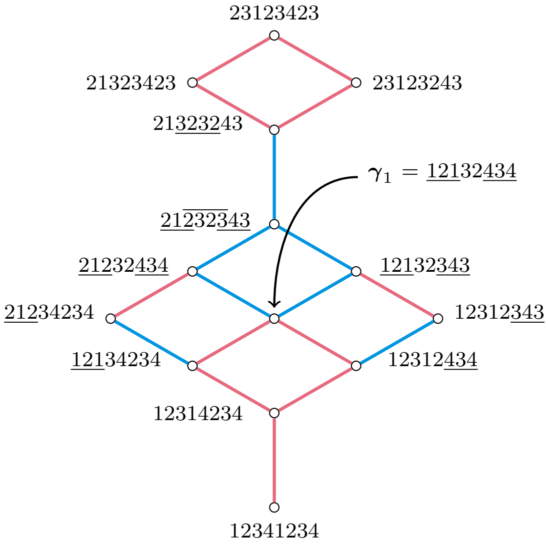


Figure 2.3: The Matsumoto graph for the group element from Example 2.4 in the Coxeter system of type F_4.

In [3], Bergeron, Ceballos, and Labbé proved that every cycle in a Matsumoto graph for finite Coxeter groups is of even length. This result was extended to arbitrary Coxeter systems in [6]. As a result of this fact, we get the following proposition.

Proposition 2.5. If (W, S) is a Coxeter system and $w \in W$, then $\mathcal{G}(w)$ is bipartite.

Take (W,S) to be a Coxeter system and let $w \in W$. In light of Matsumoto's Theorem, we can restrict to braid moves and define $\sim_b$ via $\boldsymbol{\alpha} \sim_b \boldsymbol{\beta}$ if $\boldsymbol{\alpha}$ may be obtained from $\boldsymbol{\beta}$ by applying a single braid move of the form

$$\underbrace{sts\cdots}_{m(s,t)} \longmapsto \underbrace{tst\cdots}_{m(s,t)}$$

with $m(s,t) \geq 3$. We define the equivalence relation $\approx_b$ by taking the reflexive and transitive closure of $\sim_b$, and call each equivalence class under $\approx_b$ a *braid class*, denoted $[\boldsymbol{\alpha}]$. If two reduced expressions are in the same braid class, we say that these expressions are *braid equivalent*. In addition to [1] and [2], braid classes have appeared in the work of Bergeron, Ceballos, and Labbé [3] while Zollinger [11] provided formulas for the cardinality of braid classes in the case of Coxeter systems of type A_n. Unlike the analogous commutation classes, braid classes have received little attention until recently.

Example 2.6. We describe five different braid classes below. Note that we have used overlines and underlines to clarify where braid moves may occur.

(a) Considering the Coxeter system of type Γ_2 given in Figure 2.4, one can show that the expression $\boldsymbol{\alpha}_1 = 21213232$ is reduced and its braid class consists of the following reduced expressions:

$$\boldsymbol{\alpha}_1 = \underline{2121}3232,\ \boldsymbol{\alpha}_2 = \underline{121\overline{2}}\overline{3232},\ \boldsymbol{\alpha}_3 = 121\overline{32323}.$$

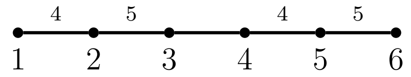


Figure 2.4: The Coxeter graph of type Γ_2.

(b) Again, considering the Coxeter system of type Γ_2, the expression $\boldsymbol{\beta}_1 = 212132324356565$ is reduced and its braid class consists of the following reduced expressions:

$$\boldsymbol{\beta}_1 = \underline{2121}323243\underline{56565},\ \boldsymbol{\beta}_2 = \underline{121\overline{2}}\overline{3232}43\underline{56565},\ \boldsymbol{\beta}_3 = 121\overline{3232\underline{3}}\underline{43}\overline{56565},$$

$$\boldsymbol{\beta}_4 = 1213232\underline{434}\overline{56565},\ \boldsymbol{\beta}_5 = \underline{2121}323243\underline{65656},\ \boldsymbol{\beta}_6 = \underline{121\overline{2}}\overline{3232}43\underline{65656},$$

$$\boldsymbol{\beta}_7 = 121\underline{3232\overline{3}}\overline{43}\underline{65656},\ \boldsymbol{\beta}_8 = 1213232\underline{434}\overline{65656}.$$

(c) Now, consider the Coxeter system of type F_4 from Figure 2.1(d). Recall the reduced expression $\boldsymbol{\gamma}_1 = 12132343$ from Example 2.4. Its braid class consists of the following reduced expressions:

$$\boldsymbol{\gamma}_1 = \underline{121}32\underline{434},\ \boldsymbol{\gamma}_2 = \underline{212}32\underline{434},\ \boldsymbol{\gamma}_3 = \underline{121}32\underline{343},\ \boldsymbol{\gamma}_4 = \underline{21\overline{2}}\overline{32\underline{3}}\underline{43},\ \boldsymbol{\gamma}_5 = 21\overline{3232}43.$$

(d) Next, we consider the Coxeter system of type Γ_3 given in Figure 2.5 and the reduced expression $\boldsymbol{\delta}_1 = 3232143454$. Its braid class consists of the following reduced expressions:

$$\boldsymbol{\delta}_1 = \underline{3232}1\underline{43\overline{4}}\overline{54},\ \boldsymbol{\delta}_2 = \underline{2323}1\underline{43\overline{4}}\overline{54},\ \boldsymbol{\delta}_3 = \underline{3232}1\underline{343}54,\ \boldsymbol{\delta}_4 = \underline{232\overline{3}}\overline{1}\underline{\overline{3}43}54,$$

$$\boldsymbol{\delta}_5 = 232\underline{131}4354,\ \boldsymbol{\delta}_6 = \underline{3232}143\underline{545},\ \boldsymbol{\delta}_7 = \underline{2323}143\underline{545}.$$

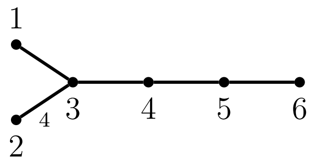


Figure 2.5: The Coxeter graph of type Γ_3.

(e) Lastly, consider the Coxeter system of type Γ_1 given in Figure 2.2 and the reduced expression $\boldsymbol{\nu}_1 = 12121312121$. The braid class consists of the following reduced expressions:

$$\boldsymbol{\nu}_1 = \underline{1212\overline{1}}\overline{3}\underline{\overline{1}2121},\ \boldsymbol{\nu}_2 = 1212\overline{313}2121,\ \boldsymbol{\nu}_3 = \underline{21212}3\underline{12121},$$

$$\boldsymbol{\nu}_4 = \underline{12121}3\underline{21212},\ \boldsymbol{\nu}_5 = \underline{2121\overline{2}}\overline{3}\underline{\overline{2}1212},\ \boldsymbol{\nu}_6 = 2121\overline{323}1212.$$

This thesis focuses heavily on how braid classes can be represented graphically. Recall that we color the edges representing braid moves in a Matsumoto graph blue. Focusing on the maximal blue connected components of a Matsumoto graph yields graphical representations of the corresponding braid class. Each maximal blue connected component defines a braid graph for a braid class. More formally, for a reduced expression $\boldsymbol{\alpha}$, the *braid graph* of $\boldsymbol{\alpha}$, denoted $\mathcal{B}(\boldsymbol{\alpha})$, is the graph whose vertex set is $[\boldsymbol{\alpha}]$ and $\boldsymbol{\beta}, \boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$ are connected by an edge if and only if $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are related by a single braid move. Braid graphs are defined with respect to a fixed reduced expression (or braid class) as opposed to the corresponding group element. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are braid equivalent, then $\mathcal{B}(\boldsymbol{\alpha}) = \mathcal{B}(\boldsymbol{\beta})$. On the other hand, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related via a commutation move, then $\mathcal{B}(\boldsymbol{\alpha}) \neq \mathcal{B}(\boldsymbol{\beta})$ but they might be isomorphic.

Example 2.7. The braid graphs $\mathcal{B}(\boldsymbol{\alpha}_1)$, $\mathcal{B}(\boldsymbol{\beta}_1)$, $\mathcal{B}(\boldsymbol{\gamma}_1)$, $\mathcal{B}(\boldsymbol{\delta}_1)$, and $\mathcal{B}(\boldsymbol{\nu}_1)$ corresponding to the reduced expressions in Example 2.6 are depicted in Figure 2.6.

The next proposition is a direct result of Proposition 2.5.

Corollary 2.8. If (W, S) is a Coxeter system and $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$, then $\mathcal{B}(\boldsymbol{\alpha})$ is bipartite.

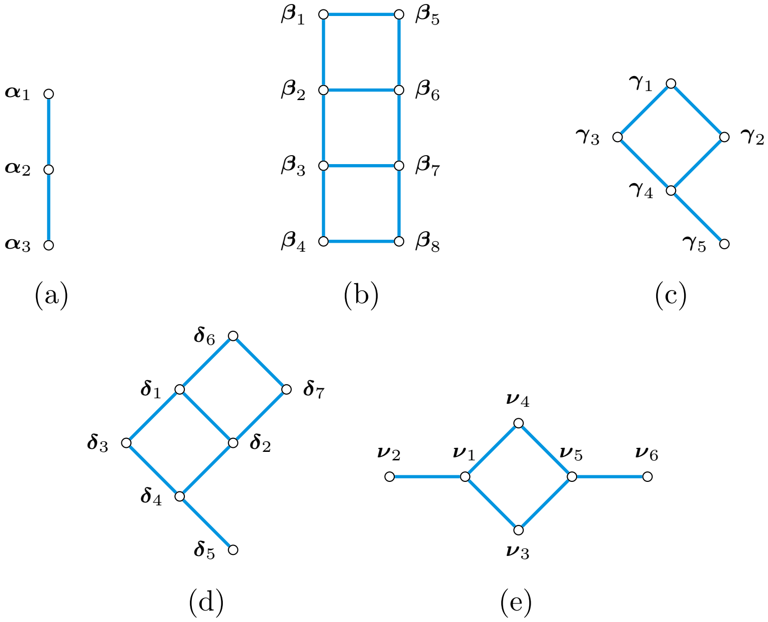


Figure 2.6: Braid graphs generated by the various reduced expressions in Example 2.6.

Chapter 3

Structure of reduced expressions in braid classes

Throughout this chapter, assume (W, S) is a Coxeter system.

For natural numbers $i \leq j$, define the interval $\llbracket i, j \rrbracket := \{i, i+1, \ldots, j-1, j\}$. Note that $\llbracket i, i \rrbracket = \{i\}$. Let $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m}$ be a reduced expression for $w \in W$. We define the *support of* $\boldsymbol{\alpha}$ *over* $\llbracket i, j \rrbracket$ via

$$\operatorname{supp}_{\llbracket i,j \rrbracket}(\boldsymbol{\alpha}) := \{s_{x_k} : k \in \llbracket i, j \rrbracket\}$$

and the *support of the braid class* $[\boldsymbol{\alpha}]$ *over* $\llbracket i, j \rrbracket$ via

$$\operatorname{supp}_{\llbracket i,j \rrbracket}([\boldsymbol{\alpha}]) := \bigcup_{\boldsymbol{\beta} \in [\boldsymbol{\alpha}]} \operatorname{supp}_{\llbracket i,j \rrbracket}(\boldsymbol{\beta}).$$

That is, $\operatorname{supp}_{\llbracket i,j \rrbracket}(\boldsymbol{\alpha})$ is the set of the generators that appear in consecutive positions i through j of $\boldsymbol{\alpha}$ while $\operatorname{supp}_{\llbracket i,j \rrbracket}([\boldsymbol{\alpha}])$ is the set of the generators that appear in these positions for any reduced expression braid equivalent to $\boldsymbol{\alpha}$.

Example 3.1. Recall the reduced expression $\boldsymbol{\alpha}_1$ from Example 2.6(a). One can calculate the following supports of the reduced expression $\boldsymbol{\alpha}_1$ and the braid class $[\boldsymbol{\alpha}_1]$:

$$\operatorname{supp}_{\llbracket 1,4 \rrbracket}(\boldsymbol{\alpha}_1) = \{1, 2\}, \ \operatorname{supp}_{\llbracket 1,4 \rrbracket}([\boldsymbol{\alpha}_1]) = \{1, 2, 3\}.$$

Observe that the support over an interval across the braid class of a reduced expression need not always match the support for a given reduced expression in the class.

Generalizing from the simply-laced case given in [1], we call an interval $I = \llbracket i, j \rrbracket$ a *braid shadow* for a reduced expression $\boldsymbol{\alpha}$ if there exists $s, t \in S$ such that $\operatorname{supp}_I(\boldsymbol{\alpha}) = \{s, t\}$ and $j + 1 - i = m(s, t) \geq 3$. That is, I is a braid shadow for $\boldsymbol{\alpha}$ if we have the opportunity to perform a braid move over the interval I. We denote the collection of braid shadows for $\boldsymbol{\alpha}$

by $\mathcal{S}(\boldsymbol{\alpha})$. In other words, $\mathcal{S}(\boldsymbol{\alpha})$ is the set of locations of available braid moves in $\boldsymbol{\alpha}$. The set of braid shadows for the braid class $[\boldsymbol{\alpha}]$ is given by

$$\mathcal{S}([\boldsymbol{\alpha}]) := \bigcup_{\boldsymbol{\beta} \in [\boldsymbol{\alpha}]} \mathcal{S}(\boldsymbol{\beta}).$$

That is, $\mathcal{S}([\boldsymbol{\alpha}])$ is the set of locations of possible braid moves in any reduced expression braid equivalent to $\boldsymbol{\alpha}$. The cardinality of $\mathcal{S}([\boldsymbol{\alpha}])$ is called the *rank* of $\boldsymbol{\alpha}$, which we denote by $\operatorname{rank}(\boldsymbol{\alpha})$.

Example 3.2. Recall the reduced expressions given in Example 2.6. Below are the braid shadows for those reduced expressions and their respective braid classes:

(a) $\mathcal{S}(\boldsymbol{\alpha}_1) = \{[\![1,4]\!]\}$, $\mathcal{S}([\boldsymbol{\alpha}_1]) = \{[\![1,4]\!], [\![4,8]\!]\}$;

(b) $\mathcal{S}(\boldsymbol{\beta}_1) = \{[\![1,4]\!], [\![11,15]\!]\}$, $\mathcal{S}([\boldsymbol{\beta}_1]) = \{[\![1,4]\!], [\![4,8]\!], [\![8,10]\!], [\![11,15]\!]\}$;

(c) $\mathcal{S}(\boldsymbol{\gamma}_1) = \{[\![1,3]\!], [\![6,8]\!]\}$, $\mathcal{S}([\boldsymbol{\gamma}_1]) = \{[\![1,3]\!], [\![3,6]\!], [\![6,8]\!]\}$;

(d) $\mathcal{S}(\boldsymbol{\delta}_1) = \{[\![1,4]\!], [\![6,8]\!], [\![8,10]\!]\}$, $\mathcal{S}([\boldsymbol{\delta}_1]) = \{[\![1,4]\!], [\![4,6]\!], [\![6,8]\!], [\![8,10]\!]\}$;

(e) $\mathcal{S}(\boldsymbol{\nu}_2) = \{[\![5,7]\!]\}$, $\mathcal{S}([\boldsymbol{\nu}_2]) = \{[\![1,5]\!], [\![5,7]\!], [\![7,11]\!]\}$.

Let $\boldsymbol{\alpha}$ be a reduced expression and consider the braid graph $\mathcal{B}(\boldsymbol{\alpha})$. For $\boldsymbol{\beta} \in [\boldsymbol{\alpha}]$, recall that $d(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the length of any geodesic between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Each geodesic corresponds to a minimal sequence of braid moves of length $d(\boldsymbol{\alpha}, \boldsymbol{\beta})$ that transforms $\boldsymbol{\alpha}$ into $\boldsymbol{\beta}$. We will utilize this concept in the proof of the following proposition, which tells us that braid shadows across a braid class for a reduced expression must either be disjoint or overlap by exactly one position. This generalizes Proposition 3.5 in [1], which established the result in the simply-laced case.

Proposition 3.3. Suppose $\boldsymbol{\alpha}$ is a reduced expression in a Coxeter system. If $I, J \in \mathcal{S}([\boldsymbol{\alpha}])$ with $I \neq J$, then $\operatorname{card}(I \cap J) \leq 1$.

Proof. For sake of contradiction, assume there exists $I, J \in \mathcal{S}([\boldsymbol{\alpha}])$ with $I \neq J$ such that $\operatorname{card}(I \cap J) \geq 2$. Choose $I \neq J$ and $\boldsymbol{\beta}, \boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$ with $I \in \mathcal{S}(\boldsymbol{\beta})$ and $J \in \mathcal{S}(\boldsymbol{\gamma})$ such that $d(\boldsymbol{\beta}, \boldsymbol{\gamma}) = k$ is minimal. Note that $k \geq 2$ since $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are reduced and we have chosen the distance to be minimal. Let $I = [\![i, i+\delta]\!]$ and $J = [\![j, j+\varepsilon]\!]$ for $\delta, \varepsilon \geq 2$.

Suppose $\boldsymbol{\alpha}_0 := \boldsymbol{\beta}, \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{k-1}, \boldsymbol{\alpha}_k := \boldsymbol{\gamma}$ is a minimal sequence of braid equivalent reduced expressions, each exactly one braid move apart, that transforms $\boldsymbol{\beta}$ into $\boldsymbol{\gamma}$ in k moves. Let b_l denote the braid move that transforms $\boldsymbol{\alpha}_{l-1}$ into $\boldsymbol{\alpha}_l$, and let X_l be the braid shadow that b_l acts upon. Consider the collection $X_2, X_3, \ldots, X_{k-1}$, which is made up of all the braid shadows that the sequence $b_2, b_3 \ldots, b_{k-1}$ acts upon. Notice that

$$\left(\bigcup_{l=2}^{k-1} X_l\right) \cap [\![i+1, i+\delta-1]\!] = \emptyset,$$

otherwise there would exist some X_l that overlaps I by two or more positions, contradicting minimality.

We will explore all cases for X_1, and ultimately show that no such first braid move exists. We will implicitly leverage symmetry to minimize the number of cases. Without loss of generality, assume $i < j$.

Case 1. Assume $X_1 \cap I = \emptyset$. Since $I \in \mathcal{S}(\boldsymbol{\beta})$, we have $I \in \mathcal{S}(\boldsymbol{\alpha}_1)$. But then $I \in \mathcal{S}(\boldsymbol{\alpha}_1)$ and $J \in \mathcal{S}(\boldsymbol{\gamma})$ with $d(\boldsymbol{\alpha}_1, \boldsymbol{\gamma}) = k-1$, which contradicts minimality. So, $\operatorname{card}(X_1 \cap I) \geq 1$.

Case 2. Now, assume $X_1 = I$. Then $I \in \mathcal{S}(\boldsymbol{\alpha}_1)$ so that $d(\boldsymbol{\alpha}_1, \boldsymbol{\gamma}) = k-1$, which again contradicts minimality.

Case 3. Suppose $X_1 \neq I$ yet $\operatorname{card}(X_1 \cap I) \geq 2$. Then $\boldsymbol{\alpha}$ would not be reduced.

Case 4. The only remaining case is $\operatorname{card}(X_1 \cap I) = 1$. The preceding arguments imply that J cannot be nested inside I, so it must be that $i + \delta < j + \varepsilon$. That is, the right endpoint of I must occur to the left of the right endpoint of J. There are now two subcases: X_1 on the left or the right end of I. We start with X_1 overlapping I by one position on the right, so that $X_1 \cap I = \{i + \delta\}$. Then $X_1 \in \mathcal{S}(\boldsymbol{\alpha}_1)$ and $\operatorname{card}(X_1 \cap J) \geq 2$ with $d(\boldsymbol{\alpha}_1, \boldsymbol{\gamma}) = k-1$, which contradicts minimality.

Finally, we consider the subcase in which $X_1 \cap I = \{i\}$. By symmetry, we know that X_k overlaps J by one position on the right of J. More specifically, $X_k \cap J = \{j + \varepsilon\}$. Recall that none of the braid shadows $X_2, \ldots, X_{k-1}$ overlap I by more than one, which implies that there exists two disjoint sets of braid moves:

$$L = \{b_l : X_l \subseteq [\![1, i]\!]\} \text{ and } R = \{b_r : X_r \subseteq [\![i+1, \ell(\boldsymbol{\alpha})]\!]\}.$$

Clearly, L and R are disjoint, each is nonempty since $b_1 \in L$ and $b_k \in R$, and we have $L \cup R = \{b_1, \ldots, b_k\}$. Moreover, all braid moves in L collectively commute with all braid moves in R since the respective collections of braid shadows are disjoint. In particular, we can perform some element of R as the opening move in a minimal sequence of braid moves that transforms $\boldsymbol{\beta}$ into $\boldsymbol{\gamma}$, but this is a contradiction because we determined that the opening braid move must occur on the left side of I.

Thus, b_1 cannot exist for our proposed $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. It follows that no such pair exists, so $\operatorname{card}(I \cap J) \leq 1$. □

Proposition 3.3 motivates the following definition, which extends the definition in [1] and [2] for the simply-laced case. We call a reduced expression $\boldsymbol{\alpha}$ a *link* if $\ell(\boldsymbol{\alpha}) = 1$, or $\ell(\boldsymbol{\alpha}) \geq 3$ and

$$\mathcal{S}([\boldsymbol{\alpha}]) = \{[\![i_1, i_2]\!], [\![i_2, i_3]\!], \ldots, [\![i_r, i_{r+1}]\!]\},$$

where $1 = i_1 < i_2 < \cdots < i_r < i_{r+1} = \ell(\boldsymbol{\alpha})$ and $i_{k+1} + 1 - i_k \geq 3$ for all $1 \leq k \leq r$. Assuming $\boldsymbol{\alpha}$ is a link, note that if $\ell(\boldsymbol{\alpha}) = 1$, then $\operatorname{rank}(\boldsymbol{\alpha}) = 0$, otherwise $\operatorname{rank}(\boldsymbol{\alpha}) = r \geq 1$. If $\boldsymbol{\alpha}$ is a link of rank $r \geq 1$, then we will often denote the k^{th} braid shadow $[\![i_k, i_{k+1}]\!]$ of $[\boldsymbol{\alpha}]$ simply as S_k.

Example 3.4. Revisiting the reduced expressions in Example 2.6, we can identify which are links. Looking at Example 3.2, we see that $\boldsymbol{\alpha}_1$, $\boldsymbol{\gamma}_1$, $\boldsymbol{\delta}_1$, and $\boldsymbol{\nu}_1$ satisfy the definition of a link while $\boldsymbol{\beta}_1$ does not since $\mathcal{S}([\boldsymbol{\beta}_1]) = \{[\![1,4]\!], [\![4,8]\!], [\![8,10]\!], [\![11,15]\!]\}$ and there is gap between positions 10 and 11. So $\boldsymbol{\beta}_1$ is not a link. It turns out, however, that the factors 2121323243 and 56565 of $\boldsymbol{\beta}_1$ are links in their own right.

The following definition generalizes the same notion that appears in [1] for the simply-laced case. We define $\boldsymbol{\ell}$ to be a *link factor* of a reduced expression $\boldsymbol{\alpha}$ if $\boldsymbol{\ell}$ satisfies the following properties:

(a) $\boldsymbol{\ell}$ is a factor of $\boldsymbol{\alpha}$,

(b) $\boldsymbol{\ell}$ is a link, and

(c) for every factor $\boldsymbol{\gamma}$ of $\boldsymbol{\alpha}$, if $\boldsymbol{\ell}$ is a factor of $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}$ is a link, then $\boldsymbol{\ell} = \boldsymbol{\gamma}$.

It follows from the definition of link factor that each non-identity reduced expression $\boldsymbol{\alpha}$ has a unique link factorization, say $\boldsymbol{\ell}_1\boldsymbol{\ell}_2\cdots\boldsymbol{\ell}_n$, where each $\boldsymbol{\ell}_i$ is a link factor of $\boldsymbol{\alpha}$. For clarity, we will denote the link factorization of a reduced expression $\boldsymbol{\alpha}$ as

$$\boldsymbol{\alpha} = \boldsymbol{\ell}_1 \mid \boldsymbol{\ell}_2 \mid \cdots \mid \boldsymbol{\ell}_n.$$

Similar to how every positive integer not equal to 1 is either a prime or has a non-trivial prime factorization, every non-identity reduced expression is either a link or has a non-trivial link factorization. Despite the fact that the empty word is not a link, we say that the link factorization of the identity is a product consisting of a single copy of the empty word.

Example 3.5. Recall the reduced expression $\boldsymbol{\beta}_1 = 212132324356565$ from Example 2.6(b). This reduced expression is not a link, but it has a non-trivial link factorization as suggested by Example 3.4:

$$\boldsymbol{\beta}_1 = 2121323243 \mid 56565.$$

The following proposition generalizes Proposition 3.9 and Corollary 3.10 in [1], and the proof is more or less identical. It describes the content of $[\boldsymbol{\alpha}]$ and says that the rank of $\boldsymbol{\alpha}$ is nothing more than the sum of the ranks the link factors, and the braid graph of a reduced expression can be decomposed as the box product of the braid graphs of the corresponding link factors.

Proposition 3.6. If $\boldsymbol{\alpha}$ is a reduced expression in a Coxeter system with link factorization $\boldsymbol{\ell}_1 \mid \boldsymbol{\ell}_2 \mid \cdots \mid \boldsymbol{\ell}_n$, then we have the following:

(a) $[\boldsymbol{\alpha}] = \{\boldsymbol{\beta}_1 \mid \boldsymbol{\beta}_2 \mid \cdots \mid \boldsymbol{\beta}_n : \boldsymbol{\beta}_i \in [\boldsymbol{\ell}_i] \text{ for } 1 \leq i \leq n\}$;

(b) The rank of $\boldsymbol{\alpha}$ is given by

$$\operatorname{rank}(\boldsymbol{\alpha}) = \sum_{i=1}^{n} \operatorname{rank}(\boldsymbol{\ell}_i);$$

(c) The cardinality of the braid class of $\boldsymbol{\alpha}$ is given by

$$\operatorname{card}([\boldsymbol{\alpha}]) = \prod_{i=1}^{n} \operatorname{card}([\boldsymbol{\ell}_i]);$$

(d) $\mathcal{B}(\boldsymbol{\alpha}) \cong \mathcal{B}(\boldsymbol{\ell}_1) \square \mathcal{B}(\boldsymbol{\ell}_2) \square \cdots \square \mathcal{B}(\boldsymbol{\ell}_n)$.

Proof. Part (a) follows directly from the definition of link factorization, while Parts (b) and (c) follow by simple counting arguments. To prove Part (d), consider the following bijection on the respective vertex sets

$$\boldsymbol{\beta}_1 \mid \boldsymbol{\beta}_2 \mid \cdots \mid \boldsymbol{\beta}_n \mapsto (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_n),$$

where $\boldsymbol{\beta}_j \in [\boldsymbol{\ell}_j]$. This bijection respects the edges of the corresponding graphs since braid moves on distinct link factors commute. Thus, it is an isomorphism of graphs. ☐

The previous theorem confirms that braid graphs of reduced expressions are decomposable with respect to box product, and we believe that is unique to reduced expressions. That is, we conjecture that for a reduced expression $\boldsymbol{\alpha}$, $\mathcal{B}(\boldsymbol{\alpha})$ is indecomposable with respect to box product if and only if $\boldsymbol{\alpha}$ is a link.

Example 3.7. Using the link factorization of $\boldsymbol{\beta}_1$ as defined by Example 2.6(b) and Proposition 3.6(d), we have $\mathcal{B}(\boldsymbol{\beta}_1) \cong \mathcal{B}(2121323243) \square \mathcal{B}(56565)$. Figure 3.1 illustrates this using colors to help identify the link factors. Also observe that

$$\operatorname{rank}(\boldsymbol{\beta}_1) = 4 = 3 + 1 = \operatorname{rank}(2121323243) + \operatorname{rank}(56565),$$

and

$$\operatorname{card}([\boldsymbol{\beta}_1]) = 8 = (4)(2) = \operatorname{card}([2121323243])\operatorname{card}([56565]).$$

Example 3.8. Consider the Coxeter system of type Γ_4 determined by the Coxeter graph in Figure 3.2. The expression $\boldsymbol{\alpha} = 3231313435656787$ is reduced and has the following link factorization:

$$323131343 \mid 5656 \mid 787.$$

The braid graph and its decomposition are depicted in Figure 3.3. One can also see that

$$\operatorname{rank}(\boldsymbol{\alpha}) = 5 = 3 + 1 + 1 = \operatorname{rank}(3231343) + \operatorname{rank}(5656) + \operatorname{rank}(787),$$

and

$$\operatorname{card}([\boldsymbol{\alpha}]) = 20 = (5)(2)(2) = \operatorname{card}([3231343])\operatorname{card}([5656])\operatorname{card}([787]).$$

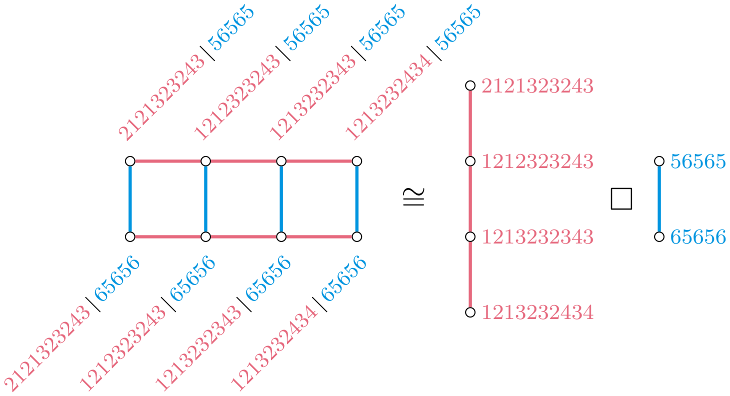


Figure 3.1: The braid graph for the reduced expression in Example 3.7 and its decomposition into a box product of braid graphs for its corresponding link factors.

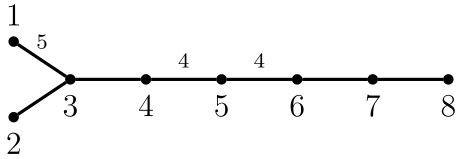


Figure 3.2: The Coxeter graph of type Γ_4.

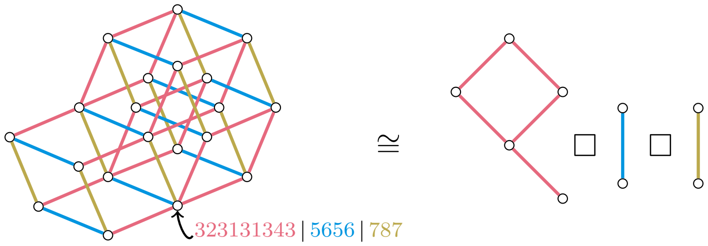


Figure 3.3: The braid graph for the reduced expression in Example 3.8 and its decomposition into a box product of its corresponding factors.

For generators $s, t \in S$, define

$$\langle s,t\rangle_n := \underbrace{st\cdots}_{n}.$$

We say that $\langle s,t\rangle_n$ is an *st-string* of length n. Note that $\langle s,t\rangle_n$ is reduced if $n \leq m(s,t)$. The st-string notation is helpful when we know what letter the word begins with, but sometimes we know what letter it ends with instead. To help in these instances, we define

$$\langle s,t\rangle_n^* := \underbrace{\cdots st}_{n}.$$

We say that $\langle s,t\rangle_n^*$ is a *reverse ts-string* of length n. That is, we now read the word right to left starting with t. Note that $\langle s,t\rangle_n^*$ is reduced if $n \le m(s,t)$.

For a reduced expression $\boldsymbol{\alpha} = s_{x_1}\cdots s_{x_n}$ and an interval $[\![i,j]\!]$, define the factor $\boldsymbol{\alpha}_{[\![i,j]\!]} := s_{x_i}\cdots s_{x_j}$. In other words, $\boldsymbol{\alpha}_{[\![i,j]\!]}$ is the consecutive word occupying the positions $[\![i,j]\!]$ of $\boldsymbol{\alpha}$.

Proposition 3.9. If $\boldsymbol{\alpha}$ is a link of rank $r \ge 1$ in a Coxeter system such that $S_1 \in \mathcal{S}(\boldsymbol{\alpha})$ with $\operatorname{supp}_{S_1}(\boldsymbol{\alpha}) = \{s,t\}$, then

(a) for all $\boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$, $\boldsymbol{\gamma}_{[\![1,m(s,t)-1]\!]} \in \{\langle s,t\rangle_{m(s,t)-1}, \langle t,s\rangle_{m(s,t)-1}\}$, and

(b) for all $\boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$ such that $S_1 \in \mathcal{S}(\boldsymbol{\gamma})$, $\boldsymbol{\gamma}_{S_1} \in \{\langle s,t\rangle_{m(s,t)}, \langle t,s\rangle_{m(s,t)}\}$.

We have analogous statements involving S_r.

Proof. For (a), without loss of generality, let $\boldsymbol{\alpha}_{S_1} = \langle s,t\rangle_{m(s,t)}$ and let $\boldsymbol{\alpha}'$ be the link we obtain after applying the available braid move over S_1 to $\boldsymbol{\alpha}$, so that $\boldsymbol{\alpha}'_{S_1} = \langle t,s\rangle_{m(s,t)}$. By Proposition 3.3, every link braid equivalent to $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ must begin with $\langle s,t\rangle_{m(s,t)-1}$ or $\langle t,s\rangle_{m(s,t)-1}$. Hence, for every $\boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$, $\boldsymbol{\gamma}_{[\![1,m(s,t)-1]\!]}, \in \{\langle s,t\rangle_{m(s,t)-1}, \langle t,s\rangle_{m(s,t)-1}\}$. Part (b) follows immediately from (a). The analogous statements for S_r follow similarly. □

If $\boldsymbol{\alpha}$ is a reduced expression and $S_k = [\![i,j]\!]$ is the k^{th} braid shadow of $[\boldsymbol{\alpha}]$, then we define the interval $[\![i+1, j-1]\!]$ to be the k^{th} *core*, denoted $\operatorname{core}_k(\boldsymbol{\alpha})$. When it is clear which reduced expression we are analyzing, we may denote the k^{th} core as core_k. We can think of the core of a braid shadow as taking the braid shadow and removing one position on either side.

The following result tells us that in a $\underset{m}{\triangle}$-avoiding Coxeter system, the factors occupying the cores of the braid shadows across a braid class may only ever be two things–either an *st*-string or a *ts*-string of length $m(s,t)-2$. Note that this is our first result requiring the hypothesis that the Coxeter system be $\underset{m}{\triangle}$-avoiding.

Proposition 3.10. If $\boldsymbol{\alpha}$ is a link of rank at least one in a $\underset{m}{\triangle}$-avoiding Coxeter system such that $S_k \in \mathcal{S}(\boldsymbol{\alpha})$ with $\operatorname{supp}_{S_k}(\boldsymbol{\alpha}) = \{s,t\}$, then for all $\boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$, $\boldsymbol{\gamma}_{\operatorname{core}_k} \in \{\langle s,t\rangle_{m(s,t)-2}, \langle t,s\rangle_{m(s,t)-2}\}$.

Proof. First, if $S_k = S_1$ or $S_k = S_{\operatorname{rank}(\boldsymbol{\alpha})}$, then the result follows from Proposition 3.9. Next, if $m(s,t) > 3$, the result follows from Proposition 3.3.

Now, we assume $1 < k < \operatorname{rank}(\boldsymbol{\alpha})$ and $m(s,t) = 3$. Since $\operatorname{supp}_{S_k}(\boldsymbol{\alpha}) = \{s,t\}$, we know $\{s,t\} \subseteq \operatorname{supp}_{\operatorname{core}_k}([\boldsymbol{\alpha}])$. For sake of contradiction, assume that there exists a generator $x \in \operatorname{supp}_{\operatorname{core}_k}([\boldsymbol{\alpha}]) \setminus \{s,t\}$. This implies that there exists $\boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$ such that $\boldsymbol{\gamma}_{S_k} = axa$ for some a. Choose such a $\boldsymbol{\gamma}$ so that $d(\boldsymbol{\alpha},\boldsymbol{\gamma}) = n$ is minimal. Let $\boldsymbol{\alpha}, \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n := \boldsymbol{\gamma}$ be a minimal sequence of braid equivalent links, each exactly one braid move apart. By minimality, it follows that $S_{k-1}, S_{k+1} \in \mathcal{S}(\boldsymbol{\alpha})$. Without loss of generality, $\boldsymbol{\alpha}_{S_k} = sts$. By minimality, it must be that

$$(\boldsymbol{\alpha}_{n-1})_{S_k} = xtx,$$

so that $m(x,t) = 3$. Now, in order for x to belong to the support of $\boldsymbol{\alpha}_{n-1}$ over both S_{k-1} and S_{k+1}, s must braid with x. Additionally, S_{k-1} and S_{k+1} must have the same length. Visually, that is

$$(\boldsymbol{\alpha}_i)_{S_{k-1}\cup S_k\cup S_{k+1}} = \underline{\cdots x\overline{s}}\overline{t}\underline{\overline{s}x\cdots},$$

for some $1 \leq i < n-1$. Hence, we have $m(s,t) = 3$, $m(t,x) = 3$, and $m(s,x) \geq 3$, which contradicts our assumption that the Coxeter system was $\underset{m}{\triangle}$-avoiding. Thus, it must be that $\operatorname{supp}_{\mathrm{core}_k}([\boldsymbol{\alpha}]) = \{s,t\}$, so for every $\boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$, $\boldsymbol{\gamma}_{\mathrm{core}_k} \in \{\langle s,t\rangle_{m(s,t)-2}, \langle t,s\rangle_{m(s,t)-2}\}$. ☐

The next example illustrates the need for the $\underset{m}{\triangle}$-avoiding hypothesis in the previous proposition.

Example 3.11. Recall the reduced expression $\boldsymbol{\nu}_1$ from Example 2.6(e), which is in a Coxeter system that is not $\underset{m}{\triangle}$-avoiding. We see that $[\![5,7]\!] \in \mathcal{S}(\boldsymbol{\nu}_1)$ and $\operatorname{supp}_{[\![5,7]\!]}(\boldsymbol{\nu}_1) = \{1,3\}$, while $\operatorname{supp}_{[\![5,7]\!]}([\boldsymbol{\nu}_1]) = \{1,2,3\}$, which violates the conclusion of Proposition 3.10.

The next fact says that the support of any particular braid shadow is constant across each braid equivalent reduced expression. This generalizes Proposition 3.9, which concerns braid shadows on the ends of a link. Our reliance on the $\underset{m}{\triangle}$-avoiding hypothesis is necessitated by the use of Proposition 3.10 in the proof.

Proposition 3.12. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two braid equivalent reduced expressions of length at least three in a $\underset{m}{\triangle}$-avoiding Coxeter system, then for all $S_k \in \mathcal{S}(\boldsymbol{\alpha}) \cap \mathcal{S}(\boldsymbol{\beta})$, $\operatorname{supp}_{S_k}(\boldsymbol{\alpha}) = \operatorname{supp}_{S_k}(\boldsymbol{\beta})$.

Proof. Suppose $S_k \in \mathcal{S}(\boldsymbol{\alpha}) \cap \mathcal{S}(\boldsymbol{\beta})$ such that $\operatorname{supp}_{S_k}(\boldsymbol{\alpha}) = \{s,t\}$. The conclusion is clear if $m(s,t) > 3$ by Proposition 3.10. Now, assume $m(s,t) = 3$. Without loss of generality, suppose $\boldsymbol{\alpha}_{S_k} = sts$. By Proposition 3.10, $\boldsymbol{\beta}_{S_k}$ is equal to either ata or sas for some $a \in S$ with $m(a,t) = 3$ or $m(a,s) = 3$, respectively. Applying the braid move over S_k to $\boldsymbol{\beta}$, we obtain tat or asa, respectively, which violates Proposition 3.10 unless $a = s$ or $a = t$, respectively. Thus, $\operatorname{supp}_{S_k}(\boldsymbol{\beta}) = \{s,t\}$. ☐

The following result asserts that the cores of two consecutive braid shadows across the class of a link share precisely one generator. This result generalizes Corollary 3.16 in [1].

Proposition 3.13. If $\boldsymbol{\alpha}$ is a link of rank at least two in a $\underset{m}{\triangle}$-avoiding Coxeter system, then

$$\operatorname{card}(\operatorname{supp}_{\mathrm{core}_k}([\boldsymbol{\alpha}]) \cap \operatorname{supp}_{\mathrm{core}_{k+1}}([\boldsymbol{\alpha}])) = 1.$$

Proof. By Proposition 3.10, there exists $s,t,u,v \in S$ such that $\operatorname{supp}_{\mathrm{core}_k}([\boldsymbol{\alpha}]) = \{s,t\}$ and $\operatorname{supp}_{\mathrm{core}_{k+1}}([\boldsymbol{\alpha}]) = \{u,v\}$. Since S_k and S_{k+1} are consecutive, it is clear by Propositions 3.3 and 3.10, that $\operatorname{card}(\{s,t\} \cap \{u,v\}) \geq 1$. However, if they were to share two generators, then $\boldsymbol{\alpha}$ would not be reduced. Hence $\operatorname{card}(\{s,t\} \cap \{u,v\}) = 1$. ☐

The following example illustrates that the $\underset{m}{\triangle}$-avoiding hypothesis is indeed required in the two previous propositions.

Example 3.14. Recall the reduced expression $\boldsymbol{\nu}_1$ and its braid class from Example 2.6(e). Notice that $\boldsymbol{\nu}_1$ exists in a Coxeter system which is not $\underset{m}{\triangle}$-avoiding. We can see that $\llbracket 5,7\rrbracket \in \mathcal{S}(\boldsymbol{\nu}_1)\cap\mathcal{S}(\boldsymbol{\nu}_6)$, while $\operatorname{supp}_{\llbracket 5,7\rrbracket}(\boldsymbol{\nu}_1) = \{1,3\}$ and $\operatorname{supp}_{\llbracket 5,7\rrbracket}(\boldsymbol{\nu}_6) = \{2,3\}$, which violates Proposition 3.12. We also see that $\operatorname{supp}_{\mathrm{core}_1}([\boldsymbol{\nu}_1]) = \{1,2\}$ and $\operatorname{supp}_{\mathrm{core}_2}([\boldsymbol{\nu}_1]) = \{1,2,3\}$, violating Proposition 3.13.

Remark 3.15. Let $\boldsymbol{\alpha}$ be a link of rank at least two in a $\underset{m}{\triangle}$-avoiding Coxeter system. Proposition 3.10 allows us to assume that $\operatorname{supp}_{\mathrm{core}_k}([\boldsymbol{\alpha}]) = \{s,t\}$ with $m(s,t) \geq 3$. Moreover, we can utilize Proposition 3.13 to conclude that without loss of generality $\operatorname{supp}_{\mathrm{core}_{k+1}}([\boldsymbol{\alpha}]) = \{t,u\}$ with $m(t,u) \geq 3$. Since we are in a $\underset{m}{\triangle}$-avoiding system, there are cases involving the relationships among s, t, and u:

(a) If $m(s,t) = 3 = m(t,u)$, then $m(s,u) = 2$ (i.e., s and u commute).

(b) If at most one of $m(s,t)$ or $m(t,u)$ is 3, then $m(s,u) \neq 3$.

(c) If both $m(s,t)$ and $m(t,u)$ are larger than 3, then $m(s,u)$ can be anything.

Remark 3.16. Let $\boldsymbol{\alpha}$ be a link of rank at least two in a $\underset{m}{\triangle}$-avoiding Coxeter system. The above propositions and remark allow us to describe exactly what the overlap between two braid shadows can be. Let $S_{k-1} = \llbracket i_{k-1}, i_k\rrbracket$, $S_k = \llbracket i_k, i_{k+1}\rrbracket$, $\operatorname{supp}_{\mathrm{core}_{k-1}}([\boldsymbol{\alpha}]) = \{s,t\}$, and $\operatorname{supp}_{\mathrm{core}_k}([\boldsymbol{\alpha}]) = \{t,u\}$ according to Remark 3.15. Then we know the following:

(a) $\operatorname{supp}_{i_k}([\boldsymbol{\alpha}]) = \{s,t,u\}$,

(b) for all $\boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$, $\boldsymbol{\gamma}_{i_k-1} \neq \boldsymbol{\gamma}_{i_k+1}$ and

(c) for all $\boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$, $\boldsymbol{\gamma}_{i_k} \in \{s,t,u\} \setminus \{\boldsymbol{\gamma}_{i_k-1}, \boldsymbol{\gamma}_{i_k+1}\}$.

In light of the above, for each $\boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$ there are three possible forms that the subword $\boldsymbol{\gamma}_{\llbracket i_k-1, i_k+1\rrbracket}$ can take:

(i) $\boldsymbol{\gamma}_{S_{k-1}\cup S_k} = \underline{\cdots s\overline{t}}\,\overline{u\cdots}$

(ii) $\boldsymbol{\gamma}_{S_{k-1}\cup S_k} = \underline{\cdots s\overline{u}}\,\overline{t\cdots}$

(iii) $\boldsymbol{\gamma}_{S_{k-1}\cup S_k} = \underline{\cdots t\overline{s}}\,\overline{u\cdots}$,

where the three displayed letters appear over $\llbracket i_k - 1, i_k + 1\rrbracket$.

Note that we will often invoke the facts stated in Remarks 3.15 and 3.16 without explicit reference to them. Moreover, we can extend these facts to arbitrary reduced expressions by applying the results to the corresponding link factors.

Remark 3.17. If $\boldsymbol{\alpha}$ is a reduced expression in a $\underset{m}{\triangle}$-avoiding Coxeter system, then a consequence of Proposition 3.10 and Remark 3.16 is that

$$\operatorname{card}([\boldsymbol{\alpha}]) \leq 2^{\operatorname{rank}(\boldsymbol{\alpha})}.$$

Note that it is easy to construct examples where equality is obtained for any rank.

Recall that the core is defined as an interval, but we are also interested in the subword occupying those positions. For a link $\boldsymbol{\alpha}$ with rank $r \geq 1$, define the *i^{th} signature of* $\boldsymbol{\alpha}$ via $\operatorname{sig}_i(\boldsymbol{\alpha}) := \boldsymbol{\alpha}_{\mathrm{core}_i}$ and the *signature of* $\boldsymbol{\alpha}$ via $\operatorname{sig}(\boldsymbol{\alpha}) = (\operatorname{sig}_1(\boldsymbol{\alpha}), \ldots, \operatorname{sig}_r(\boldsymbol{\alpha}))$. That is, $\operatorname{sig}(\boldsymbol{\alpha})$ is the ordered list of subwords that appear in the cores of the braid shadows in $\boldsymbol{\alpha}$. In light of Proposition 3.13, consecutive signature entries share at most one generator but must be distinct. Note that we could extend the definition of signature to arbitrary reduced expressions by concatenating the signatures for the corresponding link factors. However, we only require that it be defined for links in this thesis.

For two braid equivalent links $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of rank at least one, we define $\Delta(\operatorname{sig}(\boldsymbol{\alpha}), \operatorname{sig}(\boldsymbol{\beta}))$ to be the number of entries that differ between the signatures of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Example 3.18. Recalling the reduced expressions from Example 2.6(a) and (e), we can find their signatures, and we can find the number of entries that differ between their signatures:

(a) Since $\operatorname{sig}(\boldsymbol{\alpha}_1) = (12, 323)$ and $\operatorname{sig}(\boldsymbol{\alpha}_3) = (21, 232)$, $\Delta(\operatorname{sig}(\boldsymbol{\alpha}), \operatorname{sig}(\boldsymbol{\beta})) = 2$;

(b) Since $\operatorname{sig}(\boldsymbol{\nu}_1) = (212, 3, 212)$ and $\operatorname{sig}(\boldsymbol{\nu}_4) = (212, 3, 121)$, $\Delta(\operatorname{sig}(\boldsymbol{\nu}_1), \operatorname{sig}(\boldsymbol{\nu}_4)) = 1$.

Loosely speaking, the following result tells us that signature determines the expression of a link. This generalizes Lemma 5.6 in [1].

Proposition 3.19. Suppose $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two braid equivalent links of rank at least one in a $\underset{m}{\triangle}$-avoiding Coxeter system. Then $\boldsymbol{\alpha} = \boldsymbol{\beta}$ if and only if $\operatorname{sig}(\boldsymbol{\alpha}) = \operatorname{sig}(\boldsymbol{\beta})$.

Proof. The forward direction is immediate. To prove the backward direction, we suppose $\operatorname{sig}(\boldsymbol{\alpha}) = \operatorname{sig}(\boldsymbol{\beta})$, so that $\boldsymbol{\alpha}_{\mathrm{core}_k} = \boldsymbol{\beta}_{\mathrm{core}_k}$ for all $1 \leq k \leq r$, where $r = \operatorname{rank}(\boldsymbol{\alpha})$. Now, if each $S_k = \llbracket i_k, i_{k+1} \rrbracket$, then by Remark 3.16, $\boldsymbol{\alpha}_{\llbracket i_{k+1} \rrbracket} = \boldsymbol{\beta}_{\llbracket i_{k+1} \rrbracket}$ for all $1 \leq k \leq r-1$. Thus, we have $\boldsymbol{\alpha}_{\llbracket 2, \ell(\boldsymbol{\alpha})-1 \rrbracket} = \boldsymbol{\beta}_{\llbracket 2, \ell(\boldsymbol{\alpha})-1 \rrbracket}$. Now we apply Proposition 3.9 and we get the desired result. $\square$

We conjecture that the conclusion of Proposition 3.19 does not require the assumption that the Coxeter system is $\underset{m}{\triangle}$-avoiding, but our proof does utilize this hypothesis since we rely on Remark 3.16, which applies to $\underset{m}{\triangle}$-avoiding systems.

We now introduce some additional notation that will be helpful when discussing sequences of braid moves. We will use $b_k^{j_k}$ to denote the k^{th} braid move in a minimal sequence of braid moves occurring in the j_k^{th} braid shadow S_{j_k}. We will simply write b^{j_k} to denote applying a braid move over S_{j_k}. If $\boldsymbol{\alpha}$ is a reduced expression such that $S_{j_k} \in \mathcal{S}(\boldsymbol{\alpha})$, then $b^{j_k}(\boldsymbol{\alpha})$ is the reduced expression resulting from performing the braid move on S_{j_k}.

The following proposition says that each braid shadow is only used once along a geodesic in the corresponding braid graph between two reduced expressions, and that every geodesic between two reduced expressions uses the same set of braid shadows. This generalizes Proposition 5.1 in [2].

Proposition 3.20. Suppose $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two braid equivalent reduced expressions of rank at least one in a $\underset{m}{\triangle}$-avoiding Coxeter system. A braid sequence $b_1^{j_1}, b_2^{j_2}, \ldots, b_n^{j_n}$ from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ is minimal if and only if each j_i appears exactly once. Moreover, if $b_1^{j_1}, b_2^{j_2}, \ldots, b_n^{j_n}$ and $b_1^{l_1}, b_2^{l_2}, \ldots, b_n^{l_n}$ are minimal braid sequences from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$, then $\{j_1, \ldots, j_n\} = \{l_1, \ldots, l_n\}$.

Proof. Towards a contradiction, suppose there exists a minimal braid sequence $b_1^{j_1}, b_2^{j_2}, \ldots, b_n^{j_n}$ from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ such that $j_i = j_{i^*}$ for some $i \neq i^*$. Choose $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that n is minimal among all such pairs. Since n is minimal, we can assume that $j_1 = j_n$, and this is the only repetition that occurs. Two cases arise.

First, assume that the first two braid shadows S_{j_1} and S_{j_2} are disjoint. Clearly, $b_1^{j_1}$ and $b_2^{j_2}$ could be applied in either order, which contradicts the minimality of n.

Now, assume that S_{j_1} and S_{j_2} overlap by one position. Without loss of generality, let S_{j_2} fall on the right hand side of S_{j_1}, and let $\operatorname{supp}_{S_{j_1}}(\boldsymbol{\alpha}) = \{s, t\}$ and $\operatorname{supp}_{S_{j_2}}(\boldsymbol{\alpha}) = \{t, u\}$. Then by Remark 3.16,

$$\begin{aligned}
\boldsymbol{\alpha}_{S_{j_1} \cup S_{j_2}} &= \langle s, t\rangle^*_{m(s,t)-1} s \langle u, t\rangle_{m(u,t)-1} \\
&\xrightarrow{b_1^{j_1}} \langle t, s\rangle^*_{m(s,t)-1} t \langle u, t\rangle_{m(u,t)-1} \\
&\xrightarrow{b_2^{j_2}} \langle t, s\rangle^*_{m(s,t)-1} u \langle t, u\rangle_{m(u,t)-1}.
\end{aligned}$$

Let $\boldsymbol{\alpha}'$ denote the link obtained by applying all the braid moves in the sequence but $b_n^{j_n}$. Since j_1 and j_2 are distinct from $j_3, j_4, \ldots, j_{n-1}$,

$$\boldsymbol{\alpha}'_{S_{j_1}} = \langle t, s\rangle^*_{m(s,t)-1} u.$$

Hence $S_{j_1} \notin \mathcal{S}(\boldsymbol{\alpha}')$. This contradicts our assumption that $b_n^{j_1}$ is the final move because it is not available in the second to last word in the sequence. So we have proved the forward implication.

For the converse, suppose each j_i appears exactly once in a braid sequence $b_1^{j_1}, b_2^{j_2}, \ldots, b_n^{j_n}$ from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$. Clearly, $d(\boldsymbol{\alpha}, \boldsymbol{\beta}) \leq n$. Certainly, given the hypotheses $n = \Delta(\operatorname{sig}(\boldsymbol{\alpha}), \operatorname{sig}(\boldsymbol{\beta}))$. Also note that a geodesic from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ must include an odd number of braid moves for each signature change, so $\Delta(\operatorname{sig}(\boldsymbol{\alpha}), \operatorname{sig}(\boldsymbol{\beta})) \leq d(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Then we have that $n \leq d(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Hence $d(\boldsymbol{\alpha}, \boldsymbol{\beta}) = n$, which implies that $b_1^{j_1}, b_2^{j_2}, \ldots, b_n^{j_n}$ is a minimal braid sequence.

Now, assume $b_1^{j_1}, b_2^{j_2}, \ldots, b_n^{j_n}$ and $b_1^{l_1}, b_2^{l_2}, \ldots, b_n^{l_n}$ are both minimal braid sequences from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$. Let $1 \leq i \leq n$. By the above, $\operatorname{sig}_{j_i}(\boldsymbol{\alpha}) \neq \operatorname{sig}_{j_i}(\boldsymbol{\beta})$, so there must exist $1 \leq m \leq n$ such that $l_m = j_i$. We conclude that $\{j_1, \ldots, j_n\} = \{l_1, \ldots, l_n\}$. □

When examining braid graphs, it is often useful to assign colors to braid shadow locations. If one has done so, then Proposition 3.20 may be understood as there is no repetition of colors on a geodesic between two reduced expressions, and the same color set is used on each geodesic between two reduced expressions. The following example helps visualize this connection.

Example 3.21. Consider the Coxeter system Γ_4 determined by the Coxeter graph in Figure 3.2. The reduced expression $\boldsymbol{\alpha} = \underline{343}131\underline{232}43545$ is a link. Figure 3.4 depicts the braid graph for $\boldsymbol{\alpha}$. Note that we have used colors to depict the braid shadow over which the braid move was performed. The blue edges correspond to S_1, the black edges correspond to S_2, the pink edges correspond to S_3, the yellow edges correspond to S_4, and the green edges correspond to S_5. One can verify that each geodesic between any pair of vertices uses the same color set with each color appearing exactly once.

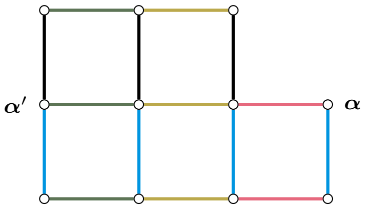


Figure 3.4: The braid graph for the reduced expression in Example 3.21, where the edges are colored according to the corresponding location of the braid move.

The next example illustrates the necessity of the $\underset{m}{\triangle}$-avoiding hypothesis in Proposition 3.20.

Example 3.22. Recall the reduced expression $\boldsymbol{\nu}_1$ from Example 2.6(e) and its braid graph. Figure 3.5 depicts $\mathcal{B}(\boldsymbol{\nu}_1)$, where the edges are colored according to the corresponding location of the braid move. The blue edges correspond to S_1, the pink edges correspond to S_2, and the yellow edges correspond to S_3. Note that pink appears twice on any geodesic between $\boldsymbol{\nu}_2$ and $\boldsymbol{\nu}_6$.

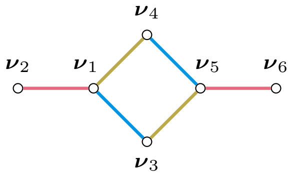


Figure 3.5: The braid graph from Figure 2.6(e), where the edges are now colored according to the corresponding location of the braid move.

The next result follows directly from Proposition 3.20. It says that the length of a geodesic between two links on a braid graph is equal to the number of changes in their respective signatures and the diameter of a braid graph for a given link is at most the rank of the link.

Corollary 3.23. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are braid equivalent links in a $\underset{m}{\triangle}$-avoiding Coxeter system, then $d(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \Delta(\mathrm{sig}(\boldsymbol{\alpha}), \mathrm{sig}(\boldsymbol{\beta}))$ and $\mathrm{diam}(\mathcal{B}(\boldsymbol{\alpha})) \leq \mathrm{rank}(\boldsymbol{\alpha})$.

Example 3.24. Recall the reduced expression $\boldsymbol{\alpha} = 34313123243545$ from Example 3.21. Consider $\boldsymbol{\alpha}'$ in Figure 3.4, which is the reduced expression we obtain by performing the third, fourth, and then fifth braid moves to $\boldsymbol{\alpha}$. That is,

$$b^5(b^4(b^3(\boldsymbol{\alpha}))) = \underline{34\overline{3}}\overline{131}3243\underline{5454} = \boldsymbol{\alpha}'.$$

One can see that $\mathrm{sig}(\boldsymbol{\alpha}) = (4, 131, 3, 4, 5)$ while $\mathrm{sig}(\boldsymbol{\alpha}') = (4, 131, 2, 3, 4)$, so that we have $\Delta(\mathrm{sig}(\boldsymbol{\alpha}), \mathrm{sig}(\boldsymbol{\alpha}')) = 3$. Looking at Figure 3.4, one can verify that $d(\boldsymbol{\alpha}, \boldsymbol{\alpha}') = 3$, as expected. Moreover, $\mathrm{diam}(\mathcal{B}(\boldsymbol{\alpha})) = 5 = \mathrm{rank}(\boldsymbol{\alpha})$.

We conjecture that $\mathrm{diam}(\mathcal{B}(\boldsymbol{\alpha})) = \mathrm{rank}(\boldsymbol{\alpha})$ for any reduced expression in a $\underset{m}{\triangle}$-avoiding Coxeter system. The next example illustrates the need for the $\underset{m}{\triangle}$-avoiding hypothesis in both Corollary 3.23 and the previous conjecture.

Example 3.25. Recall the link $\boldsymbol{\nu}_1$ from Example 2.6(e) and its braid graph. Notice that $\mathrm{diam}(\mathcal{B}(\boldsymbol{\nu}_1)) = 4 > 3 = \mathrm{rank}(\boldsymbol{\nu}_1)$, which shows that Corollary 3.23 fails if the Coxeter system is not $\underset{m}{\triangle}$-avoiding.

Generalizing a conjecture in [2], we hypothesize that for a link $\boldsymbol{\alpha}$ in a $\underset{m}{\triangle}$-avoiding Coxeter system, there is a unique diametrical pair that determines the diameter of $\mathcal{B}(\boldsymbol{\alpha})$. As the next example illustrates, the conjecture fails for arbitrary reduced expressions.

Example 3.26. Recall the reduced expression $\boldsymbol{\beta}_1$ from Example 2.6(b). Considering $\mathcal{B}(\boldsymbol{\beta}_1)$ in Figure 2.6, it is clear that $\mathrm{diam}(\mathcal{B}(\boldsymbol{\beta}_1)) = 4$, and that there are two diametrical pairs, namely the pair $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_8$ and the pair $\boldsymbol{\beta}_5$ and $\boldsymbol{\beta}_4$.

Chapter 4

Median structure of braid graphs for links

We will now work towards our main result. For a link $\boldsymbol{\alpha}$, we begin by defining the set

$$\overline{\mathrm{sig}}_i(\boldsymbol{\alpha}) := \{\boldsymbol{\beta} \in [\boldsymbol{\alpha}] : \mathrm{sig}_i(\boldsymbol{\beta}) = \mathrm{sig}_i(\boldsymbol{\alpha})\}.$$

That is, $\overline{\mathrm{sig}}_i(\boldsymbol{\alpha})$ is the set of all links braid equivalent to $\boldsymbol{\alpha}$ that have the same i^{th} signature entry as $\boldsymbol{\alpha}$. Later, we will be specifically interested in the set $\overline{\mathrm{sig}}_r(\boldsymbol{\alpha})$, where $r = \mathrm{rank}(\boldsymbol{\alpha})$.

Example 4.1. Consider the links $\boldsymbol{\alpha}_1$ and $\boldsymbol{\gamma}_1$ from Example 2.6(a) and (c). Note that $\mathrm{rank}(\boldsymbol{\alpha}_1) = 2$ and $\mathrm{rank}(\boldsymbol{\gamma}_1) = 3$. We have the following sets for these reduced expressions:

(a) $\overline{\mathrm{sig}}_1(\boldsymbol{\alpha}_1) = \{\boldsymbol{\alpha}_1\}$;

(b) $\overline{\mathrm{sig}}_2(\boldsymbol{\alpha}_1) = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}$;

(c) $\overline{\mathrm{sig}}_1(\boldsymbol{\gamma}_1) = \{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_3\}$;

(d) $\overline{\mathrm{sig}}_3(\boldsymbol{\gamma}_1) = \{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2\}$.

Let $\boldsymbol{\alpha}$ be a link of rank $r \geq 2$. If $\mathrm{supp}_{\mathrm{core}_r}([\boldsymbol{\alpha}]) = \{s, t\}$, then we define $\hat{\boldsymbol{\alpha}}$ to be the reduced expression obtained by deleting the rightmost $m(s,t) - 1$ letters. If we choose a random link $\boldsymbol{\alpha}$, then $\hat{\boldsymbol{\alpha}}$ is not necessarily a link. However, if we choose a special link, then this construction yields a link. The following proposition guarantees that this specific link always exists. This proposition generalizes Lemma 5.4 in [1].

Proposition 4.2. If $\boldsymbol{\alpha}$ is a link of rank $r \geq 2$ in a $\underset{m}{\triangle}$-avoiding Coxeter system, then there exists $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ such that $S_{r-1}, S_r \in \mathcal{S}(\boldsymbol{\sigma})$.

Proof. Let $\boldsymbol{\alpha}$ be a link of rank $r \geq 2$. Suppose $\mathrm{supp}_{\mathrm{core}_{r-1}}(\boldsymbol{\alpha}) = \{u, t\}$ and $\mathrm{supp}_{\mathrm{core}_r}(\boldsymbol{\alpha}) = \{s, t\}$, where $m(u,t), m(s,t) \geq 3$. Since $\boldsymbol{\alpha}$ is a link, we may choose $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ such that $S_{r-1} \in \mathcal{S}(\boldsymbol{\sigma})$ and $\boldsymbol{\sigma}_{S_{r-1}} = \langle u, t\rangle^*_{m(u,t)}$. Then by Proposition 3.10, $\mathrm{sig}_r(\boldsymbol{\sigma}) = \langle s, t\rangle_{m(s,t)-2}$. Proposition 3.9 implies that $S_r \in \mathcal{S}(\boldsymbol{\sigma})$. □

The following example is provided to help the reader gain intuition about the next proposition.

Example 4.3. Consider the link $\boldsymbol{\alpha} = 323131343435464$ in the Coxeter system of type Γ_5 determined by Figure 4.1. We can choose $\boldsymbol{\sigma} = 323131434345464$ according to Proposition 4.2. Now, set $\boldsymbol{\tau} = b^5(\boldsymbol{\sigma}) = 323131434345646$. Figure 4.2 depicts the braid graph for $\boldsymbol{\alpha}$. Note that we have highlighted $\mathcal{B}(\boldsymbol{\alpha})[\overline{\mathrm{sig}}_5(\boldsymbol{\sigma})]$ in blue and $\mathcal{B}(\boldsymbol{\alpha})[\overline{\mathrm{sig}}_5(\boldsymbol{\tau})]$ in pink. We have that $\hat{\boldsymbol{\sigma}} = 3231314343454$ and $\mathcal{B}(\hat{\boldsymbol{\sigma}}) \cong \mathcal{B}(\boldsymbol{\alpha})[\overline{\mathrm{sig}}_5(\boldsymbol{\sigma})]$. Each of the edges joining $\mathcal{B}(\boldsymbol{\alpha})[\overline{\mathrm{sig}}_5(\boldsymbol{\sigma})]$ and $\mathcal{B}(\boldsymbol{\alpha})[\overline{\mathrm{sig}}_5(\boldsymbol{\tau})]$ correspond to b^5 and are colored black. Foreshadowing to the proof of Theorem 4.9, notice that $V(\mathcal{B}(\boldsymbol{\alpha})[\overline{\mathrm{sig}}_5(\boldsymbol{\sigma})]) = W_{\boldsymbol{\sigma},\boldsymbol{\tau}} = \overline{\mathrm{sig}}_5(\boldsymbol{\sigma})$, $V(\mathcal{B}(\boldsymbol{\alpha})[\overline{\mathrm{sig}}_5(\boldsymbol{\tau})]) = U_{\boldsymbol{\tau},\boldsymbol{\sigma}}$, and the vertices highlighted in grey make up the set $U_{\boldsymbol{\sigma},\boldsymbol{\tau}}$.

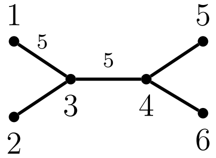


Figure 4.1: The Coxeter graph of type Γ_5.

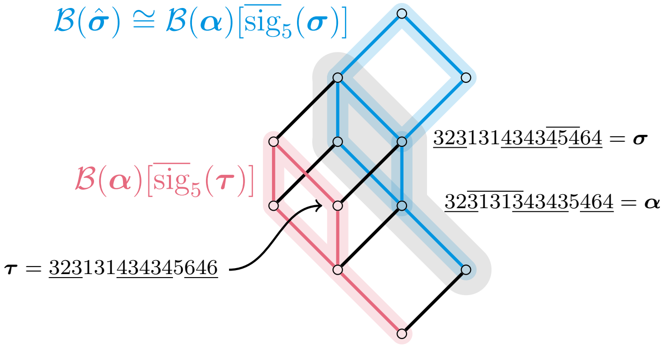


Figure 4.2: The braid graph for the reduced expression in Example 4.3 along with a partition of the vertices according to Proposition 4.4.

The following proposition generalizes Lemmas 5.5 and 5.8 from [1], which handled simply-laced Coxeter systems.

Proposition 4.4. Let $\boldsymbol{\alpha}$ be a link of rank $r \geq 2$ in a $\underset{m}{\triangle}$-avoiding Coxeter system. If we choose $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ such that $S_{r-1}, S_r \in \mathcal{S}(\boldsymbol{\sigma})$ according to Proposition 4.2, then we have:

(a) $\hat{\boldsymbol{\sigma}}$ is a link of rank $r - 1$;

(b) If $\boldsymbol{\beta} \in \overline{\mathrm{sig}}_r(\boldsymbol{\sigma})$, then $\hat{\boldsymbol{\beta}} \in [\hat{\boldsymbol{\sigma}}]$;

(c) Every element of $[\hat{\boldsymbol{\sigma}}]$ is of the form $\hat{\boldsymbol{\beta}}$ for some $\boldsymbol{\beta} \in \overline{\operatorname{sig}}_r(\boldsymbol{\sigma})$;

(d) $\{\overline{\operatorname{sig}}_r(\boldsymbol{\sigma}), \overline{\operatorname{sig}}_r(b^r(\boldsymbol{\sigma}))\}$ is a partition of $[\boldsymbol{\alpha}]$;

(e) If $\boldsymbol{\beta} \in \overline{\operatorname{sig}}_r(b^r(\boldsymbol{\sigma}))$, then $S_r \in \mathcal{S}(\boldsymbol{\beta})$ and $(b^r(\boldsymbol{\beta}))_{[\![1,\ell(\boldsymbol{\beta})-m(s,t)+1]\!]} \in [\hat{\boldsymbol{\sigma}}]$.

Proof. Parts (a), (b), and (e) are immediate.

Now, suppose $\boldsymbol{\sigma}_{[\![\ell(\boldsymbol{\sigma})-m(u,t)+2,\ell(\boldsymbol{\sigma})]\!]} = \langle u, t\rangle_{m(u,t)-1}$ and let $\boldsymbol{\gamma} \in [\hat{\boldsymbol{\sigma}}]$. Then there must be a sequence of braid moves that transforms $\hat{\boldsymbol{\sigma}}$ into $\boldsymbol{\gamma}$. Now, set $\boldsymbol{\beta} := \boldsymbol{\gamma}\langle u, t\rangle_{m(u,t)-1}$ so that $\hat{\boldsymbol{\beta}} = \boldsymbol{\gamma}$. Because the rightmost braid move is never performed in the previously mentioned sequence, applying the sequence to $\boldsymbol{\sigma}$ instead of $\hat{\boldsymbol{\sigma}}$ will yield $\boldsymbol{\beta}$. Clearly, $\operatorname{sig}_r(\boldsymbol{\beta}) = \operatorname{sig}_r(\boldsymbol{\sigma})$, so $\boldsymbol{\beta} \in \overline{\operatorname{sig}}_r(\boldsymbol{\sigma})$. This proves Part (c).

Lastly, note that $\overline{\operatorname{sig}}_r(\boldsymbol{\sigma})$ is non-empty since it contains $\boldsymbol{\sigma}$. Applying the rightmost braid move to $\boldsymbol{\sigma}$ will yield an element of $\overline{\operatorname{sig}}_r(b^r(\boldsymbol{\sigma}))$, so $\overline{\operatorname{sig}}_r(b^r(\boldsymbol{\sigma}))$ is also non-empty. Clearly, $\overline{\operatorname{sig}}_r(\boldsymbol{\sigma})$ and $\overline{\operatorname{sig}}_r(b^r(\boldsymbol{\sigma}))$ are disjoint, and $\overline{\operatorname{sig}}_r(\boldsymbol{\sigma}) \cup \overline{\operatorname{sig}}_r(b^r(\boldsymbol{\sigma})) = [\boldsymbol{\alpha}]$. Hence we have proved Part (d). $\square$

We now define a graph homomorphism. If $\boldsymbol{\alpha}$ is a link of rank $r \geq 2$ in a $\underset{m}{\triangle}$-avoiding Coxeter system, choose $\boldsymbol{\sigma}$ such that $S_{r-1}, S_r \in \mathcal{S}(\boldsymbol{\sigma})$ according to Proposition 4.2. By Proposition 4.4, every element in $[\hat{\boldsymbol{\sigma}}]$ is of the form $\hat{\boldsymbol{\beta}}$ where $\boldsymbol{\beta} \in \overline{\operatorname{sig}}_r(\boldsymbol{\sigma})$. That is, if $\operatorname{sig}_r(\boldsymbol{\sigma}) = \langle s, t\rangle_{m(s,t)-2}$, then $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}\langle s, t\rangle_{m(s,t)-1}$. Now, define the graph homomorphism $\Omega\colon \mathcal{B}(\hat{\boldsymbol{\sigma}}) \to \mathcal{B}(\boldsymbol{\alpha})[\overline{\operatorname{sig}}_r(\boldsymbol{\sigma})]$ via $\Omega(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$. That is, the map Ω appends the suffix that was deleted when forming $\hat{\boldsymbol{\beta}}$. Note that Ω is well defined by Proposition 4.4.

Example 4.5. Consider the link $\boldsymbol{\sigma} = 545343131323431313$ in the Coxeter system Γ_4 determined by the Coxeter graph in Figure 3.2. Note that $\operatorname{rank}(\boldsymbol{\sigma}) = 6$ and $S_5, S_6 \in \mathcal{S}(\boldsymbol{\sigma})$, so $\boldsymbol{\sigma}$ is chosen according to Proposition 4.2. Then $\hat{\boldsymbol{\sigma}} = 54534313132343$, and

$$\Omega(\hat{\boldsymbol{\sigma}}) = \Omega(545\underline{34\overline{3}}\overline{131}\underline{\overline{3}2\overline{3}}\overline{43}) = 545\underline{34\overline{3}}\overline{131}\underline{\overline{3}2\overline{3}}\overline{4}\underline{\overline{3}1313} = \boldsymbol{\sigma}$$

Figure 4.3 illustrates the map Ω. Notice that Ω is an isometric embedding.

The following proposition establishes that Ω is an isometric embedding as suggested by Example 4.5. The next two results generalize Corollary 5.10 from [1].

Proposition 4.6. Let $\boldsymbol{\alpha}$ be a link of rank $r \geq 2$ in a $\underset{m}{\triangle}$-avoiding Coxeter system. If we choose $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ such that $S_{r-1}, S_r \in \mathcal{S}(\boldsymbol{\sigma})$, then Ω is an isometric embedding with $\operatorname{im}(\Omega) = \overline{\operatorname{sig}}_r(\boldsymbol{\sigma})$.

Proof. Assume that $\boldsymbol{\sigma}_{[\![\ell(\boldsymbol{\sigma})-m(s,t)+2,\ell(\boldsymbol{\sigma})]\!]} = \langle s, t\rangle_{m(s,t)-1}$. Choose $\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}} \in [\hat{\boldsymbol{\sigma}}]$ such that $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \overline{\operatorname{sig}}_r(\boldsymbol{\sigma})$ according to Proposition 4.4(c). We will show that

$$d_{\mathcal{B}(\hat{\boldsymbol{\sigma}})}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}}) = d_{\mathcal{B}(\boldsymbol{\alpha})}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = d_{\mathcal{B}(\boldsymbol{\alpha})[\overline{\operatorname{sig}}_r(\boldsymbol{\sigma})]}(\boldsymbol{\beta}, \boldsymbol{\gamma}).$$

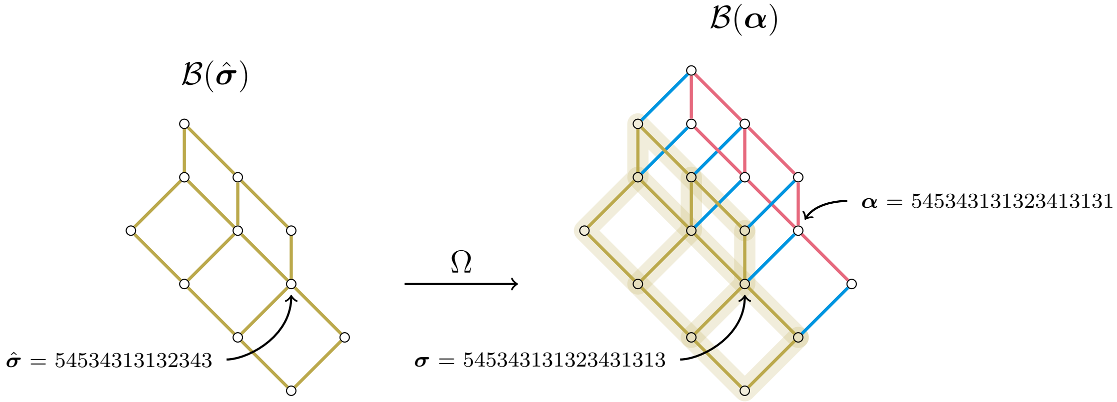


Figure 4.3: An illustration of the map $\Omega\colon \mathcal{B}(\hat{\boldsymbol{\sigma}}) \to \mathcal{B}(\boldsymbol{\alpha})$ from Example 4.5.

A geodesic in $\mathcal{B}(\hat{\boldsymbol{\sigma}})$ between $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\gamma}}$ yields a corresponding path between $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ in $\mathcal{B}(\boldsymbol{\alpha})$, and hence $d_{\mathcal{B}(\boldsymbol{\alpha})}(\boldsymbol{\beta}, \boldsymbol{\gamma}) \leq d_{\mathcal{B}(\hat{\boldsymbol{\sigma}})}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}})$. Since $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \overline{\operatorname{sig}_r}(\boldsymbol{\sigma})$, by Proposition 3.20 no geodesic between $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ in $\mathcal{B}(\boldsymbol{\alpha})$ involves an edge labeled by S_r. This implies that every geodesic between $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ is contained in $\overline{\operatorname{sig}_r}(\boldsymbol{\sigma})$, so that every geodesic between $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ yields a path between $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\gamma}}$ in $\mathcal{B}(\hat{\boldsymbol{\sigma}})$. Therefore, we have $d_{\mathcal{B}(\hat{\boldsymbol{\sigma}})}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}}) \leq d_{\mathcal{B}(\boldsymbol{\alpha})}(\boldsymbol{\beta}, \boldsymbol{\gamma})$. It follows that $d_{\mathcal{B}(\hat{\boldsymbol{\sigma}})}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}}) = d_{\mathcal{B}(\boldsymbol{\alpha})}(\boldsymbol{\beta}, \boldsymbol{\gamma})$. By Proposition 3.20, $d_{\mathcal{B}(\boldsymbol{\alpha})}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = d_{\mathcal{B}(\boldsymbol{\alpha})[\overline{\operatorname{sig}_r}(\boldsymbol{\sigma})]}(\boldsymbol{\beta}, \boldsymbol{\gamma})$. Therefore, Ω is an isometric embedding.

It is clear that $\operatorname{im}(\Omega) \subseteq \overline{\operatorname{sig}_r}(\boldsymbol{\sigma})$ since the r^{th} signature of every element of $\operatorname{im}(\Omega)$ agrees with the r^{th} signature of $\boldsymbol{\sigma}$. Now let $\boldsymbol{x} \in \overline{\operatorname{sig}_r}(\boldsymbol{\sigma})$. Then by Proposition 4.4(b) $\hat{\boldsymbol{x}} \in [\hat{\boldsymbol{\sigma}}]$. So $\Omega(\hat{\boldsymbol{x}}) = \boldsymbol{x}$, which implies $\boldsymbol{x} \in \operatorname{im}(\Omega)$. Thus, $\overline{\operatorname{sig}_r}(\boldsymbol{\sigma}) \subseteq \operatorname{im}(\Omega)$. Hence, $\operatorname{im}(\Omega) = \overline{\operatorname{sig}_r}(\boldsymbol{\sigma})$. ☐

The next corollary follows immediately from Proposition 4.6.

Corollary 4.7. Let $\boldsymbol{\alpha}$ be a link of rank $r \geq 2$ in a $\underset{m}{\triangle}$-avoiding Coxeter system. If we choose $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ such that $S_{r-1}, S_r \in \mathcal{S}(\boldsymbol{\sigma})$, then $\mathcal{B}(\hat{\boldsymbol{\sigma}}) \cong \mathcal{B}(\boldsymbol{\alpha})[\overline{\operatorname{sig}_r}(\boldsymbol{\sigma})]$.

The following result generalizes Proposition 6.1 in [2]. Our proof is identical, but we include it here for completeness.

Proposition 4.8. If $\boldsymbol{\alpha}$ is a link of rank $r \geq 1$ in a $\underset{m}{\triangle}$-avoiding Coxeter system and $\{\boldsymbol{\beta}, \boldsymbol{\gamma}\} \in E(\mathcal{B}(\boldsymbol{\alpha}))$, then there exists a unique i with $1 \leq i \leq r$ such that $\operatorname{sig}_i(\boldsymbol{\beta}) \neq \operatorname{sig}_i(\boldsymbol{\gamma})$ and $W_{\boldsymbol{\beta},\boldsymbol{\gamma}} = \overline{\operatorname{sig}_i}(\boldsymbol{\beta})$.

Proof. By Corollary 3.23, there exists a unique i such that $\operatorname{sig}_i(\boldsymbol{\beta}) \neq \operatorname{sig}_i(\boldsymbol{\gamma})$.

Note that both $W_{\boldsymbol{\beta},\boldsymbol{\gamma}}$ and $\overline{\operatorname{sig}_i}(\boldsymbol{\beta})$ are non-empty because they both contain $\boldsymbol{\beta}$. For the forward containment, let $\boldsymbol{x} \in W_{\boldsymbol{\beta},\boldsymbol{\gamma}}$. Towards contradiction, assume $\operatorname{sig}_i(\boldsymbol{x}) \neq \operatorname{sig}_i(\boldsymbol{\beta})$. By assumption and Corollary 3.23, we obtain

$$\begin{aligned} d(\boldsymbol{x},\boldsymbol{\beta}) &= \Delta(\operatorname{sig}(\boldsymbol{x}),\operatorname{sig}(\boldsymbol{\beta})) \\ &= \Delta(\operatorname{sig}(\boldsymbol{x}),\operatorname{sig}(\boldsymbol{\gamma}))+1 \\ &= d(\boldsymbol{x},\boldsymbol{\gamma})+1, \end{aligned}$$

where the second equality holds because we assumed $\operatorname{sig}_i(\boldsymbol{\beta}) \neq \operatorname{sig}_i(\boldsymbol{\gamma})$ and $\operatorname{sig}_i(\boldsymbol{x}) \neq \operatorname{sig}_i(\boldsymbol{\beta})$, which implies $\operatorname{sig}_i(\boldsymbol{x}) = \operatorname{sig}_i(\boldsymbol{\gamma})$. However $d(\boldsymbol{x},\boldsymbol{\beta}) = d(\boldsymbol{x},\boldsymbol{\gamma})+1$ contradicts Proposition 1.6. Thus, $\operatorname{sig}_i(\boldsymbol{x}) = \operatorname{sig}_i(\boldsymbol{\beta})$, so $\boldsymbol{x} \in \overline{\operatorname{sig}}_i(\boldsymbol{\beta})$ and hence $W_{\boldsymbol{\beta},\boldsymbol{\gamma}} \subseteq \overline{\operatorname{sig}}_i(\boldsymbol{\beta})$.

For the backward containment, let $\boldsymbol{x} \in \overline{\operatorname{sig}}_i(\boldsymbol{\beta})$. That is, $\operatorname{sig}_i(\boldsymbol{x}) = \operatorname{sig}_i(\boldsymbol{\beta})$. Then by Corollary 3.23,

$$\begin{aligned} d(\boldsymbol{x},\boldsymbol{\gamma}) &= \Delta(\operatorname{sig}(\boldsymbol{x}),\operatorname{sig}(\boldsymbol{\gamma})) \\ &= \Delta(\operatorname{sig}(\boldsymbol{x}),\operatorname{sig}(\boldsymbol{\beta}))+1 \\ &= d(\boldsymbol{x},\boldsymbol{\gamma})+1, \end{aligned}$$

where the second equality holds because we assumed $\operatorname{sig}_i(\boldsymbol{x}) = \operatorname{sig}_i(\boldsymbol{\beta})$ and $\operatorname{sig}_i(\boldsymbol{\beta}) \neq \operatorname{sig}_i(\boldsymbol{\gamma})$, so $\operatorname{sig}_i(\boldsymbol{x}) \neq \operatorname{sig}_i(\boldsymbol{\gamma})$. By Proposition 1.6, we have $\boldsymbol{x} \in W_{\boldsymbol{\beta},\boldsymbol{\gamma}}$, so $\overline{\operatorname{sig}}_i(\boldsymbol{\beta}) \subseteq W_{\boldsymbol{\beta},\boldsymbol{\gamma}}$.

Hence, $W_{\boldsymbol{\beta},\boldsymbol{\gamma}} = \overline{\operatorname{sig}}_i(\boldsymbol{\beta})$. $\square$

The next theorem states that in $\underset{m}{\triangle}$-avoiding Coxeter systems braid graphs for links are median. We can consider this to be our main result, and it generalizes the main result in [2], which handles the simply-laced case.

Theorem 4.9. If $\boldsymbol{\alpha}$ is a link in a $\underset{m}{\triangle}$-avoiding Coxeter system, then $\mathcal{B}(\boldsymbol{\alpha})$ is median.

Proof. Suppose $\boldsymbol{\alpha}$ is a link of rank r. We proceed by induction on r. For the base steps, we consider $r = 0$ and $r = 1$. If $r = 0$, then $\mathcal{B}(\boldsymbol{\alpha})$ is a single vertex, which is clearly median. If $r = 1$, then $\mathcal{B}(\boldsymbol{\alpha})$ is two vertices with one edge connecting them, which is also median.

Now, we move to the inductive step. Let $r \geq 2$ and assume all braid graphs for links of rank $r-1$ are median. Suppose $\boldsymbol{\alpha}$ is a link of rank r. Now, choose $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ such that $S_{r-1}, S_r \in \mathcal{S}(\boldsymbol{\sigma})$ according to Proposition 4.2. Then $\boldsymbol{\sigma}$ is of the form

$$\boldsymbol{\sigma} = \boldsymbol{\varphi}\langle t,u\rangle^*_{m(u,t)-1}t\langle s,t\rangle_{m(s,t)-1},$$

where $s,t,u \in S$, $m(s,t), m(t,u) \geq 3$, and the product on the right is reduced. By Proposition 4.4(a), $\hat{\boldsymbol{\sigma}}$ is a link of rank $r-1$, so $\mathcal{B}(\hat{\boldsymbol{\sigma}})$ is median by the inductive hypothesis. By Proposition 4.6, $\mathcal{B}(\hat{\boldsymbol{\sigma}})$ is isomorphic to the induced subgraph $\mathcal{B}(\boldsymbol{\alpha})[\overline{\operatorname{sig}}_r(\boldsymbol{\sigma})]$. Thus, it suffices to show that $\mathcal{B}(\boldsymbol{\alpha})$ can be obtained by performing one peripheral expansion on $\mathcal{B}(\hat{\boldsymbol{\sigma}})$.

Towards that goal, (noting that by Proposition 4.8 we know that $W_{\boldsymbol{\sigma},b^r(\boldsymbol{\sigma})} = \overline{\operatorname{sig}}_r(\boldsymbol{\sigma})$) we consider the following set

$$U := U_{\boldsymbol{\sigma},b^r(\boldsymbol{\sigma})} = \{\boldsymbol{\beta} \in \overline{\operatorname{sig}}_r(\boldsymbol{\sigma}) : \{\boldsymbol{\beta},\boldsymbol{\gamma}\} \in E(\mathcal{B}(\boldsymbol{\sigma})) \text{ for some } \boldsymbol{\gamma} \in \overline{\operatorname{sig}}_r(b^r(\boldsymbol{\sigma}))\}.$$

We will show that U is convex and that we can obtain $\mathcal{B}(\boldsymbol{\alpha})$ by doing a peripheral expansion to $\mathcal{B}(\boldsymbol{\alpha})[\overline{\mathrm{sig}}_r(\boldsymbol{\sigma})]$ relative to U. Note that every $\boldsymbol{\beta} \in U$ ends in $\langle t, s\rangle_{m(s,t)}$. Also, by Proposition 4.4(e), if $\boldsymbol{\psi} \in \overline{\mathrm{sig}}_r(b^r(\boldsymbol{\sigma}))$, then $\boldsymbol{\psi}$ ends in $\langle s, t\rangle_{m(s,t)}$. This implies that

$$U = \{\boldsymbol{\beta} \in \overline{\mathrm{sig}}_r(\boldsymbol{\sigma}) : \boldsymbol{\beta}_{S_r} = \langle t, s\rangle_{m(s,t)}\}.$$

We show that U is convex by contradiction. Let $\boldsymbol{\varphi}, \boldsymbol{\tau} \in U$ and suppose that there is some geodesic between $\boldsymbol{\varphi}$ and $\boldsymbol{\tau}$ that contains $\boldsymbol{\nu} \in \overline{\mathrm{sig}}_r(\boldsymbol{\sigma}) \setminus U$. Since $\boldsymbol{\nu} \in \overline{\mathrm{sig}}_r(\boldsymbol{\sigma}) \setminus U$, it must have the same r^{th} signature as $\boldsymbol{\sigma}$ but it cannot have the same subword in the r^{th} braid shadow. Thus, b^{r-1} must have been applied during the sequence of braid moves that takes $\boldsymbol{\varphi}$ to $\boldsymbol{\nu}$, so $\boldsymbol{\nu}_{S_r} = \langle u, s\rangle_{m(s,u)}$. However,

$$\boldsymbol{\varphi}_{S_r} = \langle t, s\rangle_{m(s,t)} = \boldsymbol{\tau}_{S_r},$$

so b^{r-1} must have been applied twice, which contradicts Proposition 3.20, so U must be convex.

Now we consider the following set

$$U_{b^r(\boldsymbol{\sigma}),\boldsymbol{\sigma}} = \{\boldsymbol{\beta} \in \overline{\mathrm{sig}}_r(b^r(\boldsymbol{\sigma})) : \{\boldsymbol{\beta}, \boldsymbol{\gamma}\} \in E(\mathcal{B}(\boldsymbol{\sigma})) \text{ for some } \boldsymbol{\gamma} \in \overline{\mathrm{sig}}_r(\boldsymbol{\sigma})\}.$$

We will now show that this set is actually equal to $\overline{\mathrm{sig}}_r(b^r(\boldsymbol{\sigma}))$. We have $U_{b^r(\boldsymbol{\sigma}),\boldsymbol{\sigma}} \subseteq \overline{\mathrm{sig}}_r(b^r(\boldsymbol{\sigma}))$ because $W_{b^r(\boldsymbol{\sigma}),\boldsymbol{\sigma}} = \overline{\mathrm{sig}}_r(b^r(\boldsymbol{\sigma}))$ by Proposition 4.8, and $U_{b^r(\boldsymbol{\sigma}),\boldsymbol{\sigma}} \subseteq W_{b^r(\boldsymbol{\sigma}),\boldsymbol{\sigma}}$ by definition. If $\boldsymbol{x} \in \overline{\mathrm{sig}}_r(b^r(\boldsymbol{\sigma}))$, then $\boldsymbol{x}_{S_r} = \langle s, t\rangle_{m(s,t)}$ so that $b^r(\boldsymbol{x})_{S_r} = \langle t, s\rangle_{m(s,t)}$. We also know that $\{\boldsymbol{x}, \boldsymbol{\gamma}\} \in E(\mathcal{B}(\boldsymbol{\sigma}))$ for some $\boldsymbol{\gamma} \in \overline{\mathrm{sig}}_r(\boldsymbol{\sigma})$, so we have $\overline{\mathrm{sig}}_r(b^r(\boldsymbol{\sigma})) \subseteq U_{b^r(\boldsymbol{\sigma}),\boldsymbol{\sigma}}$.

Lastly, we must show that $\mathcal{B}(\boldsymbol{\alpha})[U] \cong \mathcal{B}(\boldsymbol{\alpha})[\overline{\mathrm{sig}}_r(b^r(\boldsymbol{\sigma}))]$. Suppose that $\{\boldsymbol{x}, \boldsymbol{\nu}\}$ is an edge in $\mathcal{B}(\boldsymbol{\alpha})[U]$. That means $\boldsymbol{x}$ and $\boldsymbol{\nu}$ are related by a single braid move, so there must be some $1 \leq i \leq r$ such that $b^i(\boldsymbol{x}) = \boldsymbol{\nu}$. However, since $\boldsymbol{x}, \boldsymbol{\nu} \in U$, $\boldsymbol{x}_{S_r} = \boldsymbol{\nu}_{S_r}$ so S_i must be disjoint from S_r. This means that $S_i \in \mathcal{S}(b^r(\boldsymbol{x}))$, so $b^i(b^r(\boldsymbol{x})) = b^r(\boldsymbol{\nu})$. This means that $b^r(\boldsymbol{x})$ and $b^r(\boldsymbol{\nu})$ are related by the same braid move b^i, so they are connected by an edge in $\mathcal{B}(\boldsymbol{\alpha})[\overline{\mathrm{sig}}_r(b^r(\boldsymbol{\sigma}))]$. Thus for every edge in $\mathcal{B}(\boldsymbol{\alpha})[U]$, there is a corresponding edge in $\mathcal{B}(\boldsymbol{\alpha})[\overline{\mathrm{sig}}_r(b^r(\boldsymbol{\sigma}))]$. Similarly, for every edge in $\mathcal{B}(\boldsymbol{\alpha})[\overline{\mathrm{sig}}_r(b^r(\boldsymbol{\sigma}))]$, there is a corresponding edge in $\mathcal{B}(\boldsymbol{\alpha})[U]$. Thus the two induced subgraphs are isomorphic. Therefore, $\mathcal{B}(\boldsymbol{\alpha})$ can be obtained by performing one peripheral expansion on $\mathcal{B}(\boldsymbol{\alpha})[\overline{\mathrm{sig}}_r(\boldsymbol{\sigma})]$. Thus, $\mathcal{B}(\boldsymbol{\alpha})$ is median by Proposition 1.14. □

Combining the above theorem with Propositions 1.10 and 3.6(d) yields the following fact.

Corollary 4.10. If $\boldsymbol{\alpha}$ is a reduced expression in a $\underset{m}{\triangle}$-avoiding Coxeter system, then $\mathcal{B}(\boldsymbol{\alpha})$ is median.

It is important to note that the converse of Corollary 4.10 is not true. That is, not every median graph arises as a braid graph of a reduced expression in a $\underset{m}{\triangle}$-avoiding Coxeter system.

Example 4.11. Figure 4.4 depicts a median graph that the authors of [2] argue cannot be realized as a braid graph in a simply-laced and triangle-free Coxeter system. Their argument can easily be generalized to arbitrary $\underset{m}{\triangle}$-avoiding Coxeter systems.

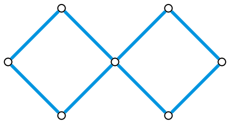


Figure 4.4: A median graph that does not arise as a braid graph in $\underset{m}{\triangle}$-avoiding Coxeter system.

Corollary 4.10 states that every braid graph is median in a $\underset{m}{\triangle}$-avoiding Coxeter system, while the next example suggests that braid graphs are median in arbitrary Coxeter systems. We conjecture every braid graph in any Coxeter system is median. However, a new approach must be implemented to prove this.

Example 4.12. Recall the reduced expression $\boldsymbol{\nu}_1$ from Example 2.6(e) and its braid graph from Figure 2.7. Although $\boldsymbol{\nu}_1$ is in a Coxeter system whose Coxeter graph contains a three-cycle with the labels $3, 3, 5$, its braid graph is median.

By Proposition 1.8 we get the following result, which was proved in the simply-laced case in [1], and again in [2] using a different approach.

Corollary 4.13. If $\boldsymbol{\alpha}$ is a reduced expression in a $\underset{m}{\triangle}$-avoiding Coxeter system, then $\mathcal{B}(\boldsymbol{\alpha})$ is a partial cube.

In [2], the authors prove that if a Coxeter system is of type Λ and $\boldsymbol{\alpha}$ is a reduced expression, then the isometric dimension of $\mathcal{B}(\boldsymbol{\alpha})$ is equal to the rank of $\boldsymbol{\alpha}$. That is, $\dim_I(\mathcal{B}(\boldsymbol{\alpha})) = \operatorname{rank}(\boldsymbol{\alpha})$. We conjecture that this same result holds when the Coxeter system is $\underset{m}{\triangle}$-avoiding. However, this result does not hold when the Coxeter system is not $\underset{m}{\triangle}$-avoiding, as the next example illustrates.

Example 4.14. Recall the reduced expression $\boldsymbol{\nu}_1 =$ from Example 2.6(e) and its braid graph. Clearly, $\operatorname{rank}(\boldsymbol{\nu}_1) = 3$. However, it is easy to see that $\dim_I(\mathcal{B}(\boldsymbol{\nu}_1)) > 3$. It turns out that the isometric dimension of $\mathcal{B}(\boldsymbol{\nu}_1)$ is 4. It follows that $\dim_I(\mathcal{B}(\boldsymbol{\nu}_1)) = 4 > 3 = \operatorname{rank}(\boldsymbol{\nu}_1)$, so the conjecture concerning isometric dimension fails if the Coxeter system is not $\underset{m}{\triangle}$-avoiding.

We now define another set dealing with signatures of links. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are braid equivalent links, we define

$$\overline{\operatorname{sig}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \{\boldsymbol{\gamma} \in [\boldsymbol{\alpha}] : \operatorname{sig}_i(\boldsymbol{\gamma}) = \operatorname{sig}_i(\boldsymbol{\alpha}) \text{ whenever } \operatorname{sig}_i(\boldsymbol{\alpha}) = \operatorname{sig}_i(\boldsymbol{\beta})\}.$$

That is, $\overline{\operatorname{sig}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the set of links in the class of $\boldsymbol{\alpha}$ whose signature entries agree with the signature entries that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have in common. Note that if $\operatorname{sig}_i(\boldsymbol{\alpha}) \neq \operatorname{sig}_i(\boldsymbol{\beta})$ for all $1 \leq i \leq \operatorname{rank}(\boldsymbol{\alpha})$, then the qualifying statement is vacuously true, and $\overline{\operatorname{sig}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = [\boldsymbol{\alpha}]$. The following proposition generalizes Proposition 7.5 in [2], and our proof is nearly identical.

Proposition 4.15. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are braid equivalent links in a $\underset{m}{\triangle}$-avoiding Coxeter system, then $I(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \overline{\operatorname{sig}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Proof. The containment $I(\boldsymbol{\alpha},\boldsymbol{\beta}) \subseteq \overline{\mathrm{sig}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ follows from Proposition 3.20. For the reverse containment, suppose $\boldsymbol{\gamma} \in \overline{\mathrm{sig}}(\boldsymbol{\alpha},\boldsymbol{\beta})$. Then there is a minimal braid sequence $b_1^{j_1},\dots,b_k^{j_k}$ from $\boldsymbol{\alpha}$ to $\boldsymbol{\gamma}$ and a minimal braid sequence $b_{k+1}^{j_{k+1}},\dots,b_n^{j_n}$ from $\boldsymbol{\gamma}$ to $\boldsymbol{\beta}$. We will show that $b_1^{j_1},\dots,b_k^{j_k},b_{k+1}^{j_{k+1}},\dots,b_n^{j_n}$ is a minimal braid sequence from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$. By Proposition 3.20, $j_1,\dots,j_k$ are pairwise distinct and $j_{k+1},\dots,j_n$ are pairwise distinct. So, using Proposition 3.20 again, it suffices to show that $\{j_1,\dots,j_k\} \cap \{j_{k+1},\dots,j_n\} = \emptyset$. Towards a contradiction, suppose that there exists $j_l = j_m$ for some $1 \le l \le k$ and $k+1 \le m \le n$. Let $j := j_l = j_m$. Notice that $j_l = j_m$ is the only repeat in the sequence. Since S_j is applied at some point between $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$, it must be that $\mathrm{sig}_j(\boldsymbol{\alpha}) \neq \mathrm{sig}_j(\boldsymbol{\gamma})$. Similarly, $\mathrm{sig}_j(\boldsymbol{\beta}) \neq \mathrm{sig}_j(\boldsymbol{\gamma})$. This implies that $\mathrm{sig}_j(\boldsymbol{\alpha}) = \mathrm{sig}_j(\boldsymbol{\beta})$. However, since $\boldsymbol{\gamma} \in \overline{\mathrm{sig}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ and $\mathrm{sig}_j(\boldsymbol{\alpha}) \neq \mathrm{sig}_j(\boldsymbol{\gamma})$, it must be that $\mathrm{sig}_j(\boldsymbol{\alpha}) \neq \mathrm{sig}_j(\boldsymbol{\beta})$. This is a contradiction, so there must not exist such $j_m = j_l$. Thus, the braid sequence $b_1^{j_1},\dots,b_k^{j_k},b_{k+1}^{j_{k+1}},\dots,b_n^{j_n}$ from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ is minimal, so by Proposition 3.20 we have $I(\boldsymbol{\alpha},\boldsymbol{\beta}) \supseteq \overline{\mathrm{sig}}(\boldsymbol{\alpha},\boldsymbol{\beta})$. ☐

We now wish to construct a way to calculate the median $\mathrm{med}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ for three braid equivalent links $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ in a $\underset{m}{\triangle}$-avoiding Coxeter system. It turns out that there is often a relationship between median graphs and the so-called majority rule. In light of Proposition 3.10, for $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ braid equivalent links of rank at least one, we define the i^{th} *majority* of $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ via

$$\mathrm{maj}_i(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) := \begin{cases} \mathrm{sig}_i(\boldsymbol{\alpha}), & \text{if } \mathrm{sig}_i(\boldsymbol{\alpha}) = \mathrm{sig}_i(\boldsymbol{\beta}) \text{ or } \mathrm{sig}_i(\boldsymbol{\alpha}) = \mathrm{sig}_i(\boldsymbol{\gamma}) \\ \mathrm{sig}_i(\boldsymbol{\beta}), & \text{otherwise.} \end{cases}$$

That is, given a triple of braid equivalent links, when at least two of the links contain the same subword in the i^{th} core, we record that subword. For braid equivalent links $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ of rank $r \ge 1$, we define the *majority* of $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ via

$$\mathrm{maj}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) := (\mathrm{maj}_1(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),\dots,\mathrm{maj}_r(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})).$$

Similar to the signature of a link, the majority of three braid equivalent links is an ordered list of subwords.

Example 4.16. Recall the links below in the Coxeter system of type Γ_3 from Example 2.6(d):

$$\boldsymbol{\delta}_3 = \underline{3232}1\underline{343}54,\ \boldsymbol{\delta}_5 = 232\underline{131}4354, \text{ and } \boldsymbol{\delta}_7 = \underline{2323}143\underline{545}.$$

The first step in calculating the majority $\mathrm{maj}(\boldsymbol{\delta}_3,\boldsymbol{\delta}_5,\boldsymbol{\delta}_7)$ is to find the signature of each link:

$$\begin{aligned} \mathrm{sig}(\boldsymbol{\delta}_3) &= (23,1,4,5) \\ \mathrm{sig}(\boldsymbol{\delta}_5) &= (32,3,4,5) \\ \mathrm{sig}(\boldsymbol{\delta}_7) &= (32,1,3,4). \end{aligned}$$

According to the definition of the i^{th} majority, $\mathrm{maj}_1(\boldsymbol{\delta}_3,\boldsymbol{\delta}_5,\boldsymbol{\delta}_7) = 32$, $\mathrm{maj}_2(\boldsymbol{\delta}_3,\boldsymbol{\delta}_5,\boldsymbol{\delta}_7) = 1$, $\mathrm{maj}_3(\boldsymbol{\delta}_3,\boldsymbol{\delta}_5,\boldsymbol{\delta}_7) = 4$, and $\mathrm{maj}_4(\boldsymbol{\delta}_3,\boldsymbol{\delta}_5,\boldsymbol{\delta}_7) = 5$. Collecting these in an ordered list, numbered left to right, yields $\mathrm{maj}(\boldsymbol{\delta}_3,\boldsymbol{\delta}_5,\boldsymbol{\delta}_7) = (32,1,4,5)$.

The next proposition generalizes Lemma 7.6 in [2]. It says that the intersection of the intervals for each pair in the triple of links is nothing more than the links in the class whose signature is the majority of the triple.

Proposition 4.17. If $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are braid equivalent links in a $\underset{m}{\triangle}$-avoiding Coxeter system, then

$$\overline{\mathrm{sig}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cap \overline{\mathrm{sig}}(\boldsymbol{\beta}, \boldsymbol{\gamma}) \cap \overline{\mathrm{sig}}(\boldsymbol{\gamma}, \boldsymbol{\alpha}) = \{\boldsymbol{x} \in [\boldsymbol{\alpha}] : \mathrm{sig}(\boldsymbol{x}) = \mathrm{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})\}.$$

Proof. The result follows immediately from the definitions of the set $\overline{\mathrm{sig}}$ for two links and majority of three links. ☐

We know by Theorem 4.9 that the braid graph for a link in a $\underset{m}{\triangle}$-avoiding Coxeter system is a median graph. This tells us that for any three braid equivalent links $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ of rank at least one in a $\underset{m}{\triangle}$-avoiding Coxeter system, the median $\mathrm{med}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ exists. The next result generalizes Proposition 7.7 in [2], and it makes the connection between median and majority clear. In particular, it states that the median of three braid equivalent links is the link in their class determined by the signature that is equal to the majority of the triple.

Proposition 4.18. If $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are braid equivalent links in a $\underset{m}{\triangle}$-avoiding Coxeter system and $\boldsymbol{x} \in [\boldsymbol{\alpha}]$ is the unique link satisfying $\mathrm{sig}(\boldsymbol{x}) = \mathrm{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, then $\mathrm{med}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = \boldsymbol{x}$.

Proof. Using Propositions 4.15 and 4.17 together, we have

$$\begin{aligned} I(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cap I(\boldsymbol{\beta}, \boldsymbol{\gamma}) \cap I(\boldsymbol{\gamma}, \boldsymbol{\alpha}) &= \overline{\mathrm{sig}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cap \overline{\mathrm{sig}}(\boldsymbol{\beta}, \boldsymbol{\gamma}) \cap \overline{\mathrm{sig}}(\boldsymbol{\gamma}, \boldsymbol{\alpha}) \\ &= \{\boldsymbol{x} \in [\boldsymbol{\alpha}] : \mathrm{sig}(\boldsymbol{x}) = \mathrm{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})\}. \end{aligned}$$

By Theorem 4.9, this set consists of one unique link, namely $\mathrm{med}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. By Proposition 3.19, $\mathrm{med}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is the unique link $\boldsymbol{x} \in [\boldsymbol{\alpha}]$ satisfying $\mathrm{sig}(\boldsymbol{x}) = \mathrm{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. ☐

Example 4.19. Recall from Example 2.6(d) the braid equivalent links in the Coxeter system of type Γ_3:

$$\boldsymbol{\delta}_3 = \underline{3232}1\underline{343}54, \boldsymbol{\delta}_4 = \underline{232\overline{3}}\overline{1}\underline{\overline{3}43}54, \boldsymbol{\delta}_5 = 232\underline{131}4354, \text{ and } \boldsymbol{\delta}_7 = \underline{2323}143\underline{545}.$$

And recall from Example 4.16 that the majority of these three links is $\mathrm{maj}(\boldsymbol{\delta}_3, \boldsymbol{\delta}_5, \boldsymbol{\delta}_7) = (32, 1, 4, 5)$, which is $\mathrm{sig}(\boldsymbol{\delta}_4)$. By Proposition 4.18, $\mathrm{med}(\boldsymbol{\delta}_3, \boldsymbol{\delta}_5, \boldsymbol{\delta}_7)$ must be $\boldsymbol{\delta}_4$. Figure 4.5 depicts $\mathcal{B}(\boldsymbol{\delta}_3)$, and the intervals $I(\boldsymbol{\delta}_3, \boldsymbol{\delta}_5)$, $I(\boldsymbol{\delta}_5, \boldsymbol{\delta}_7)$, and $I(\boldsymbol{\delta}_7, \boldsymbol{\delta}_3)$ are highlighted in blue, yellow, and pink, respectively. One can visually check that the intervals overlap at $\boldsymbol{\delta}_4$, as expected.

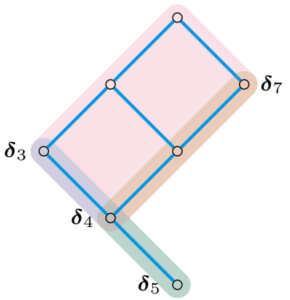


Figure 4.5: Median computation for Example 4.19.

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