BRAID GRAPHS IN COXETER SYSTEMS OF TYPE Λ ARE MEDIAN

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ABSTRACT

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Any two reduced expressions for the same Coxeter group element are related by a sequence of commutation and braid moves. Two reduced expressions are said to be braid equivalent if they are related via a sequence of braid moves. Braid equivalence is an equivalence relation and the corresponding equivalence classes are called braid classes. Each braid class can be encoded in terms of a braid graph in a natural way. In a recent paper, Awik et al. proved that when a Coxeter system is simply-laced and triangle free (i.e., the corresponding Coxeter graph has no three-cycles), the braid graph for a reduced expression is a partial cube (i.e., isometric to a subgraph of a hypercube). In her MS thesis, Barnes provided an alternate proof of this fact and determined the minimal dimension hypercube into which a braid graph can be isometrically embedded. In this thesis, we prove that every braid graph in a simply-laced triangle-free Coxeter system is median, which is a strengthening of previous results. We conjecture that every braid graph of a link corresponds to the Hasse diagram for a distributive lattice.

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Chapter 1

Required graph theory

In this chapter, we introduce the necessary graph theory concepts and terminology. All of the graphs discussed throughout this thesis are assumed to be finite, connected, and simple. We will denote the vertex set of a graph G as $V(G)$ and the edge set as $E(G)$.

Let G be a graph and let $S \subseteq V(G)$. The *subgraph induced by* S, denoted $G[S]$, is the graph whose vertex set is S and whose edges are all the edges of G incident to vertices in S.

Example 1.1. Consider the subgraphs depicted in teal in Figure 1.1. The subgraph in Figure 1.1(a) is induced by $S = \{a, b, c, d, e\}$ while the subgraph in Figure 1.1(b) is not induced since the edge $\{b, e\}$ is absent from the subgraph.

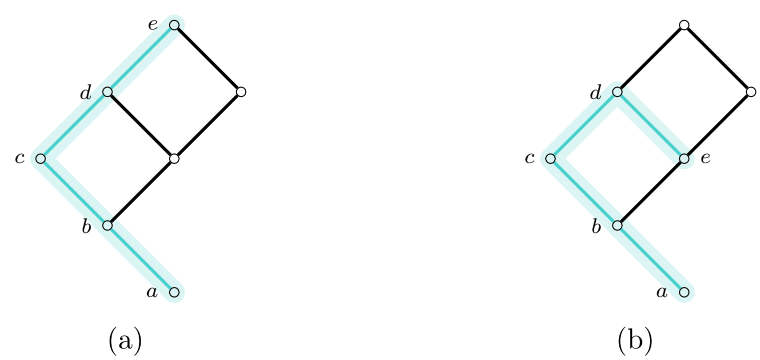


Figure 1.1: An example and non-example of an induced subgraph as described in Example 1.1.

Let G and H be graphs. A *graph map* $f : G \to H$ is a function $f : V(G) \to V(H)$. An injective graph map $f : G \to H$ that satisfies $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(H)$ is called an *embedding* of G into H. That is, an embedding is an injective graph homomorphism. If, in addition, f satisfies $\{f(u), f(v)\} \in E(H)$ implies that $\{u, v\} \in E(G)$, then we say that f is an *induced embedding*. If f is an induced embedding, then G is isomorphic to the subgraph of H induced by the image of f.

Example 1.2. The embedding depicted in Figure 1.2(a) is an induced embedding. However, the map shown in Figure 1.2(b) is not an induced embedding since $\{g(d), g(a)\} \in E(H)$ while $\{d, a\} \notin E(G)$.

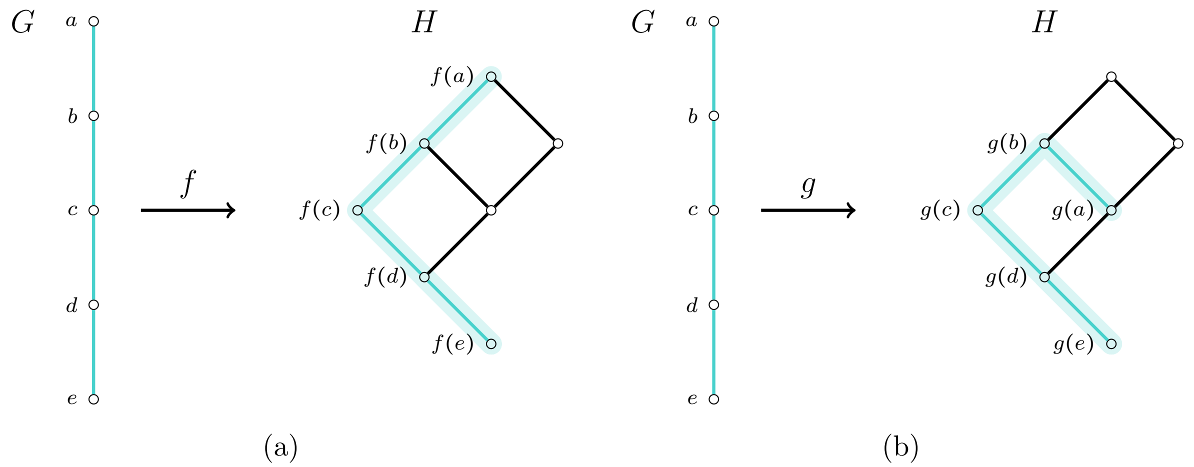


Figure 1.2: An induced embedding and an embedding that is not induced as described in Example 1.2.

We will often use the notion of distance between two vertices of a graph to establish results throughout this thesis. Let G be a graph. A *geodesic* between two vertices u and v is a shortest path between u and v. We define the distance between u and v via

$$d_G(u, v) := \text{the length of any geodesic between } u \text{ and } v.$$

Note that if the context is clear, we will simply write $d(u, v)$ in place of $d_G(u, v)$. Using the given distance metric, we define the *diameter* of G to be

$$\text{diam}(G) := \max\{d(u, v) \mid u, v \in V(G)\}.$$

In other words, $\text{diam}(G)$ is the length of any maximal length geodesic between any two vertices u and v. If $d(u, v) = \text{diam}(G)$, then u and v are said to be *diametrical*.

Let G and H be graphs and let $f : G \to H$ be an embedding. We say that f is an *isometric embedding* if for all $u, v \in V(G)$, $d_G(u, v) = d_H(f(u), f(v))$. In this case, we say that G is *isometric* to the subgraph induced by the image of f. Indeed, preserving distance also preserves adjacency, so every isometric embedding is also an induced embedding. However, the converse is not always true as illustrated in the example below.

Example 1.3. The induced embedding g depicted in Figure 1.3 is not an isometric embedding because $d_G(a, f) = 5$ while $d_H(g(a), g(f)) = 3$.

Let G_1 and G_2 be graphs. The *box product*, denoted $G_1 \square G_2$, is the graph whose vertex set is $V(G_1) \times V(G_2)$ and there is an edge from (x_1, y_1) to (x_2, y_2) provided either:

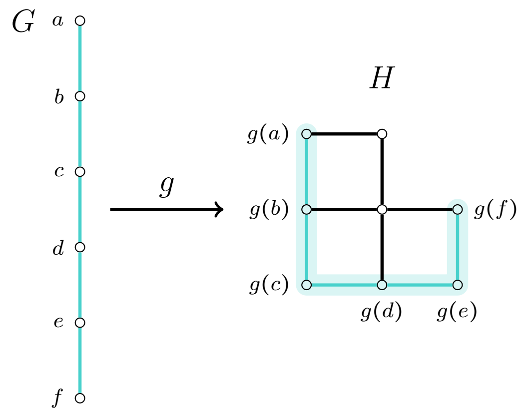
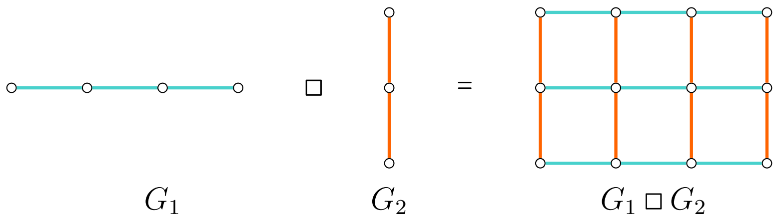


Figure 1.3: An embedding that is induced but not isometric.

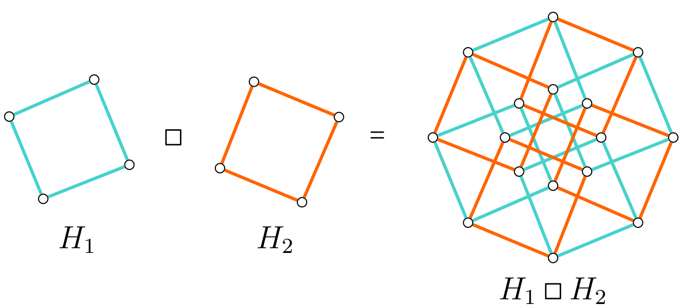
(a) $x_1 = x_2$ and there is an edge from y_1 to y_2 in G_2, or

(b) $y_1 = y_2$ and there is an edge from x_1 to x_2 in G_1.

Example 1.4. Figures 1.4(a) and 1.4(b) depict examples of the box product operator. The colors are provided to illustrate how the two graphs create the box product graph.



(a)



(b)

Figure 1.4: Examples of the box product of graphs.

If $n \in \mathbb{N} \cup \{0\}$, then we define the set of binary strings of length n as:

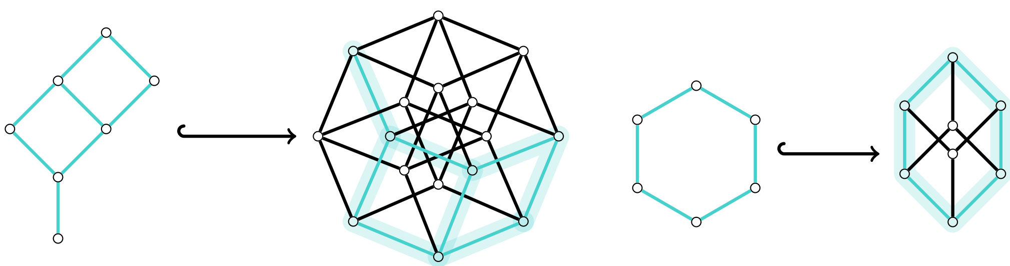
$$\{0,1\}^n := \{a_1a_2\cdots a_n \mid a_k \in \{0,1\}\}.$$

Note that the empty string is the only string of length $n = 0$. For n as defined above, the *hypercube* of dimension n, denoted Q_n, is the graph with vertex set $V(Q_n) = \{0,1\}^n$ and two vertices are adjacent when their corresponding binary strings differ by exactly one digit. We remark that for $n, m \in \mathbb{N} \cup \{0\}$, $Q_n \,\square\, Q_m \cong Q_{n+m}$.

A graph G is a *partial cube* if it can be isometrically embedded in some hypercube Q_n. The *isometric dimension* of a partial cube is defined as the minimum dimension of the hypercube into which the partial cube can be isometrically embedded, and is denoted as

$$\dim_I(G) := \min\{n \in \mathbb{N} \cup \{0\} \mid \text{there exists an isometric embedding of } G \text{ into } Q_n\}.$$

Example 1.5. Figures 1.5(a) and 1.5(b) depict examples of partial cubes together with isometric embeddings into a hypercube. Note that each figure leaves open to interpretation the specific embedding. In particular, Figure 1.5(a) depicts two different isometric embeddings while Figure 1.5(b) has twelve accounting for reflections and rotations of the graph. It turns out that the isometric dimensions are 4 and 3, respectively.



(a) (b)

Figure 1.5: Examples of partial cubes.

The following result from [14] states that the box product of two partial cubes is a partial cube.

Proposition 1.6. If G_1 and G_2 are partial cubes, then $G_1 \,\square\, G_2$ is also a partial cube. Moreover, $\dim_I(G_1 \,\square\, G_2) = \dim_I(G_1) + \dim_I(G_2)$.

The rest of this chapter mimics the development in [13] and [14]. We now define the notion of a semicube, which is an important feature of partial cubes used in multiple results in this thesis. Let G be a graph and let u and v be distinct vertices. Define $W_{uv} \subseteq V(G)$ via

$$W_{uv} := \{w \in V(G) \mid d(w,u) < d(w,v)\}.$$

That is, W_{uv} is the set of vertices in G that are closer to u than v. Symmetrically, W_{vu} is the set of vertices that are closer to v than u. Both the subgraph $G[W_{uv}]$ and the set W_{uv} are referred to as a *semicube* of G. The two semicubes W_{uv} and W_{vu} are referred to as *opposite semicubes*. Note that the definition of semicubes does not require $\{u,v\} \in E(G)$. The next two propositions from [14] state what happens when $\{u,v\} \in E(G)$.

Proposition 1.7. Let G be a graph. If $w \in W_{uv}$ for some edge $\{u,v\} \in E(G)$, then $d(w,v) = d(w,u) + 1$. Moreover, $W_{uv} = \{w \in V(G) \mid d(w,v) = d(w,u) + 1\}$.

That is, if $\{u,v\} \in E(G)$, then all vertices in W_{vu} are exactly one step further from u than v in G. Note that some vertices may not be in either semicube, namely the ones equidistant from u and v.

Proposition 1.8. A graph G is bipartite if and only if W_{uv} and W_{vu} form a partition of $V(G)$ for any edge $\{u,v\} \in E(G)$.

Using the notion of semicubes, we define the *Djoković–Winkler relation* θ on the edges of a graph. If G is a graph, we define $\{x,y\}\,\theta\,\{u,v\}$ if and only if $\{u,v\}$ connects a vertex in W_{xy} to a vertex in W_{yx}. Note that θ is relexive and symmetric, but not necessarily transitive.

Example 1.9. Figure 1.6 provides an example where θ is not transitive. The vertices shaded in teal are in the semicube W_{uv} while the vertices shaded in magenta are in the semicube W_{vu}. If we consider the edges b_1, b_2, and b_3 in Figure 1.6, we see that $b_1\,\theta\,b_2$ and $b_1\,\theta\,b_3$, but $b_2\,\not\theta\,b_3$.

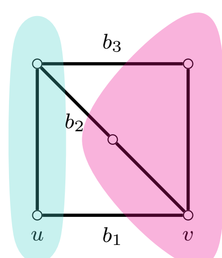


Figure 1.6: An example in which θ is not transitive as described in Example 1.9.

Recall that a subset $S \subseteq V(G)$ is *convex* in G if every geodesic connecting any two vertices in S lies completely in S. The following proposition from [14] describes when θ is transitive, and thus an equivalence relation.

Proposition 1.10. Let G be a graph. The following statements are equivalent:

(i) G is a partial cube.

(ii) G is bipartite and all semicubes are convex.

(iii) G is bipartite and θ is an equivalence relation.

(iv) G is bipartite and for all $\{x,y\},\{u,v\} \in E(G)$, if $\{x,y\}\,\theta\,\{u,v\}$, then $\{W_{xy},W_{yx}\} = \{W_{uv},W_{vu}\}$.

(v) G is bipartite and for any pair of adjacent vertices of G, there is a unique pair of opposite semicubes separating these two vertices.

Notice that if G is a partial cube, then G is bipartite and θ is an equivalence relation on $E(G)$. If G is a partial cube and $\{u,v\} \in E(G)$, we denote the equivalence class of $\{u,v\}$ under θ as F_{uv}. That is

$$F_{uv} \coloneqq \{\{a,b\} \in E(G) \mid \{u,v\}\,\theta\,\{a,b\}\} = \{\{a,b\} \in E(G) \mid a \in W_{uv}, b \in W_{vu}\}.$$

Notice that F_{uv} is the set of edges joining W_{uv} and W_{vu}. We will refer to the edges in F_{uv} as *F-edges*. Note that while $F_{uv} = F_{vu}$, $W_{uv} \neq W_{vu}$. As a special case of Proposition 1.10, if $x,y \in W_{uv}$ are endpoints of edges in F_{uv}, then any geodesic connecting x and y is entirely contained in W_{uv}.

Figure 1.7 gives a rough illustration of the semicubes W_{uv} and W_{vu} for an edge $\{u,v\}$ in a partial cube, where all vertices contained within the teal box are closer to u than v and all the vertices contained in the magenta box are closer to v than u. The edges in black are the edges in F_{uv}. Notice that the magenta box is smaller than the teal box, illustrating that opposite semicubes need not be the same cardinality.

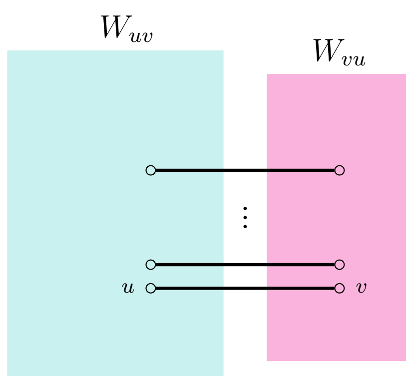


Figure 1.7: A rough illustration of two semicubes for a partial cube together with the corresponding class of F-edges.

Example 1.11. Consider the partial cube in Figure 1.8. The semicube W_{uv} is highlighted in teal while the opposite semicube W_{vu} is highlighted in magenta. The corresponding F-edges are colored black. To aid the reader, the colors are chosen to match the colors in Figure 1.7.

The next proposition is also from [14].

Proposition 1.12. If G is a partial cube, then $\dim_I(G)$ is equal to the number of equivalence classes induced by the Djoković–Winkler relation θ.

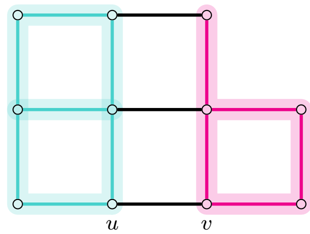


Figure 1.8: Example of semicubes from Example 1.11 and the corresponding F-edges for a partial cube.

Example 1.13. The graph given in Figure 1.9 is the same partial cube as shown in Figure 1.8, but with all five equivalence classes indicated by the five different colors. It follows that the isometric dimension of this graph is 5 by Proposition 1.12.

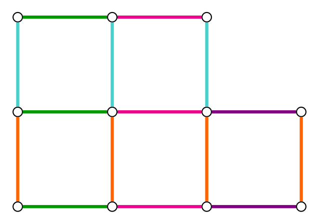


Figure 1.9: Example of the various equivalence classes of F-edges induced by θ from Example 1.13.

The next result is likely well known, but we were unable to find a reference, so we provide a proof here.

Proposition 1.14. Let G be a partial cube with $\{u, v\} \in E(G)$. If $\{u_1, v_1\}, \{u_2, v_2\} \in F_{uv}$ with $u_i \in W_{uv}$ and $v_i \in W_{vu}$, then $d(u_1, u_2) = d(v_1, v_2)$.

Proof. By Proposition 1.7 applied to W_{vu} $d(u_1, v_2) = d(v_1, v_2) + 1$. On the other hand, $d(u_1, v_2) = d(u_1, u_2) + 1$ by Proposition 1.7 applied to W_{uv}. Thus, $d(u_1, u_2) = d(v_1, v_2)$. □

Now, we turn our attention to median graphs, a prominent idea in this thesis. Let G be a graph. The *interval* between vertices u and v, denoted $I(u, v)$, is the union of vertices on all geodesics between u and v. A graph G is *median* if

$$|I(u, v) \cap I(u, w) \cap I(v, w)| = 1$$

for all $u, v, w \in V(G)$. In other words, G is median if there is a unique vertex x that simultaneously lies on a geodesic between u and v, a geodesic between u and w, and a geodesic between v and w for all triples u, v, w. If G is a median graph, then we will let $\operatorname{med}(u, v, w)$ be equal to the unique vertex in $I(u, v) \cap I(u, w) \cap I(v, w)$.

Example 1.15. The shading in Figures 1.10(a) and 1.10(b) depicts $I(u,v)$ in red, $I(v,w)$ in blue, and $I(u,w)$ in green. In Figure 1.10(a), we see that all three colors overlap at the vertex x, illustrating that $|I(u,v) \cap I(u,w) \cap I(v,w)| = 1$. It turns out that the same is true for any three vertices in this graph. Thus, the graph given in Figure 1.10(a) is median. On the other hand, in Figure 1.10(b), we see that there is no vertex common to all of these intervals. Hence, $I(u,v) \cap I(u,w) \cap I(v,w) = \varnothing$, and so the graph in Figure 1.10(b) is not median.

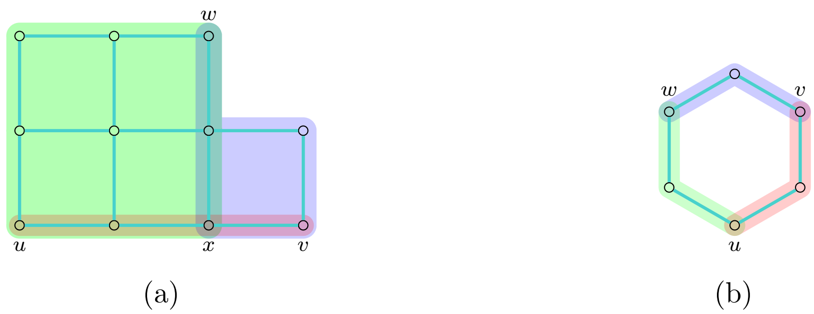


Figure 1.10: Examples of a median graph and non-median graph from Example 1.15.

The next proposition from [15] connects partial cubes and median graphs.

Proposition 1.16. If a graph G is median, then G is a partial cube.

Example 1.17. As seen in Example 1.5, the cycle graph with six vertices from Figure 1.5(b) can be isometrically embedded into a hypercube, and is therefore a partial cube. However, as shown in Example 1.17, this graph is not a median graph, so the converse of the previous proposition does not hold.

The following result is commonly known and states that, like the collection of partial cubes, the collection of median graphs is closed under the box product operation.

Proposition 1.18. If graphs G_1 and G_2 are median, then $G_1 \square G_2$ is also median.

To conclude this introductory chapter on graphs, we discuss convex expansions and their relationship to median graphs. Given a graph G and a convex set $C \subseteq V(G)$, we define the *expanded graph relative to* C, denoted $\exp(G,C)$, as follows:

- Start with the graph G;
- Make an isomorphic copy of $G[C]$, denoted G'_C, where each $u \in C$ corresponds to $u' \in C' := V(G'_C)$;
- For each $u \in C$, join u and u' with an edge.

The illustration given in Figure 1.11 shows a rough depiction of the process described above. The vertices in $G[C]$ mirror the vertices in the magenta G'_C, and each pair of vertices u and u' are connected by a black edge. The rectangles $G[C]$ and G'_C are drawn the same size to indicate the isomorphism between the two graphs: $\{u, v\} \in E(G[C])$ if and only if $\{u', v'\} \in E(G'_C)$. When G is a partial cube, the teal G and magenta G'_C are opposite semicubes and the black edges joining each u and u' pair are all part of the same F-class of edges.

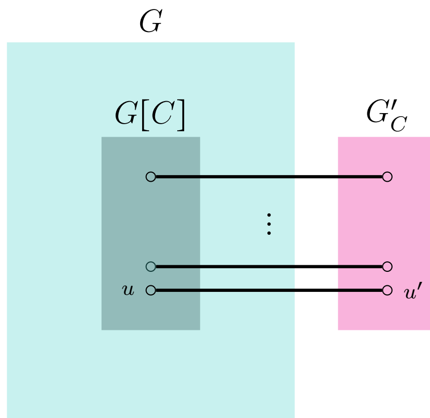


Figure 1.11: A rough illustration of the convex expansion process.

Example 1.19. Figure 1.12 illustrates a sequence of convex expansions. The grey highlighted portion of each subfigure shows which subgraph is playing the role of $G[C]$. Each subsequent graph represents the graph obtained when the convex expansion is performed on the grey portion.

The following theorem from [13], sometimes referred to as Mulder's Theorem, states that a median graph can always be obtained through a sequence of convex expansions that begin from a single vertex.

Proposition 1.20. A graph G is median if and only if it can be obtained from a single vertex by a sequence of convex expansions.

Example 1.21. Proposition 1.20 implies that each graph in Figure 1.12 is median.

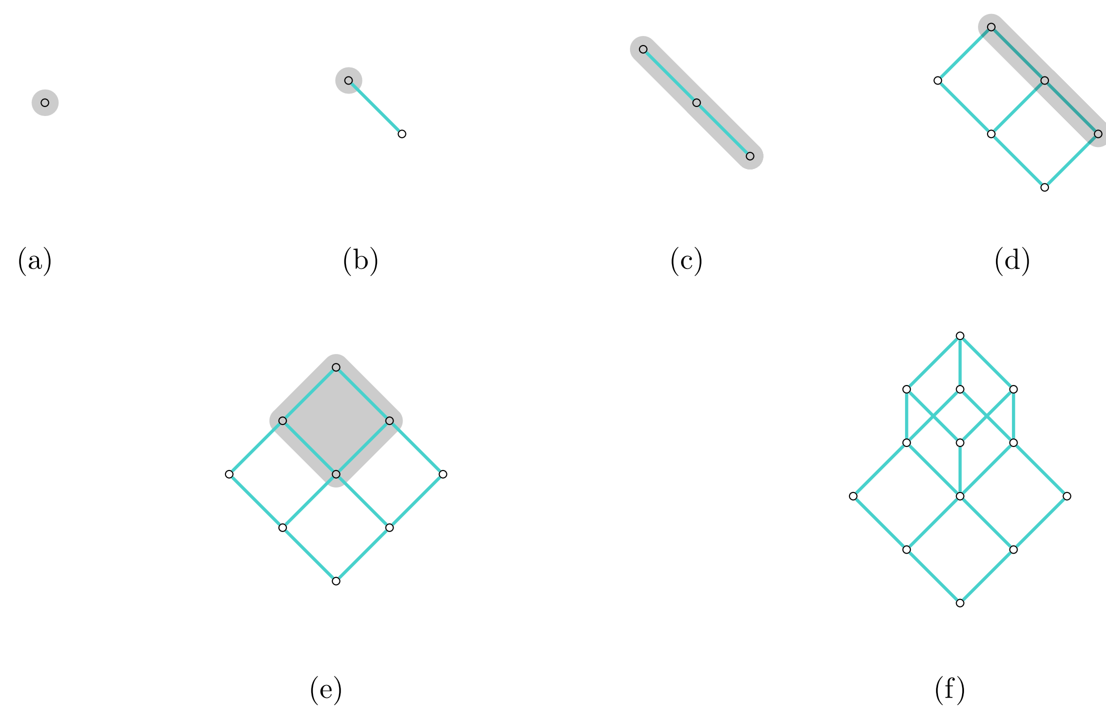


Figure 1.12: A sequence of convex expansions starting from a single vertex.

Chapter 2

Braid classes in simply-laced Coxeter systems

This chapter will introduce Coxeter systems and braid classes. As we develop the ideas involved in illustrating the overall structure of braid classes, we will discuss braid shadows and links, which are concepts introduced in [2].

A *Coxeter matrix* is an $n \times n$ symmetric matrix $M = (m_{ij})$ with entries $m_{ij} \in \{1, 2, 3, \ldots, \infty\}$ such that $m_{ii} = 1$ for all $1 \leq i \leq n$ and $m_{ij} \geq 2$ for $i \neq j$. A *Coxeter system* is a pair (W, S) where $S = \{s_1, s_2, \ldots, s_n\}$ is a set of generators and W is a group, called a *Coxeter group*, with presentation

$$W = \langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m(s_i, s_j)} = e \rangle,$$

where $m(s_i, s_j) := m_{ij}$ for some $n \times n$ Coxeter matrix $M = (m_{ij})$. For $s, t \in S$, the condition $m(s,t) = \infty$ means that there is no relation imposed between s and t. It turns out that the elements of S are distinct as group elements and for $s \neq t$, $m(s,t)$ is the order of st as shown in [11]. Since elements of S have order two, the relation $(st)^{m(s,t)} = e$ can be written as

$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)}$$

with $m(s,t) \geq 2$ letters. When $m(s,t) = 2$, $st = ts$ is called a *commutation relation* and when $m(s,t) \geq 3$, the corresponding relation is called a *braid relation*. For $m(s,t) < \infty$, the replacement

$$\underbrace{sts\cdots}_{m(s,t)} \longmapsto \underbrace{tst\cdots}_{m(s,t)}$$

is called a *commutation move* if $m(s,t) = 2$ and a *braid move* if $m(s,t) \geq 3$.

We can visually encode the information given in a Coxeter system into a *Coxeter graph*, Γ, having vertex set S and edges $\{s,t\}$ for each $m(s,t) \geq 3$. Moreover, each edge is labeled with the corresponding $m(s,t)$, although typically the labels of 3 are omitted because they are the most common. We say that (W,S), or just W, is of type Γ, and we may denote the Coxeter group as $W(\Gamma)$ and the generating set as $S(\Gamma)$ for emphasis.

In this thesis, we will direct our focus to a special class of Coxeter systems. A Coxeter system is *simply laced* if for all $s, t \in S, m(s,t) \leq 3$. That is, a Coxeter system is said to be simply laced if the generators have either a commutation relation or a braid relation of length 3 imposed upon them. If a Coxeter graph Γ contains no three-cycles, we say that the corresponding Coxeter system (W, S) is *triangle free*. A Coxeter system that is both simply laced and triangle free is said to be of type Λ.

Example 2.1. The Coxeter graphs given in Figure 2.1 correspond to four common simply-laced Coxeter systems. Using the Coxeter graphs, we can determine the defining relations between the generators of these Coxeter systems. The Coxeter system of type A_n is given by the Coxeter graph in Figure 2.1(a). The Coxeter group $W(A_n)$ has generating set $S(A_n) = \{s_1, s_2, \ldots, s_n\}$ with defining relations

- $s_i^2 = e$ for all i;
- $s_i s_j = s_j s_i$ when $|i-j| > 1$;
- $s_i s_j s_i = s_j s_i s_j$ when $|i-j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group S_{n+1} under the mapping that sends s_i to the adjacent transposition $(i, i+1)$.

The Coxeter system of type D_n is given by the Coxeter graph in Figure 2.1(b). The Coxeter group $W(D_n)$ has generating set $S(D_n) = \{s_1, s_2, \ldots, s_n\}$ and has defining relations

- $s_i^2 = e$ for all i;
- $s_i s_j = s_j s_i$ if $|i-j| > 1$ and $i, j \neq 1$, or if $i = 1$ and $j \neq 3$;
- $s_1 s_3 s_1 = s_3 s_1 s_3$ and $s_i s_j s_i = s_j s_i s_j$ if $|i-j| = 1$.

The Coxeter group $W(D_n)$ is isomorphic to the index two subgroup of the group of signed permutations on n letters having an even number of sign changes.

The Coxeter systems of types $\widetilde{A}_n$ and $\widetilde{D}_n$ depicted in Figures 2.1(c) and 2.1(d), respectively, turn out to yield infinite Coxeter groups. All of these Coxeter systems are of type Λ except type $\widetilde{A}_2$ since its Coxeter graph is a 3-cycle.

Consider a Coxeter system (W, S). Define S^* to be the free monoid on S. We call $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ a *word* while a *factor* of $\boldsymbol{\alpha}$ is a word of the form $s_{x_i} s_{x_{i+1}} \cdots s_{x_{j-1}} s_{x_j}$ for $1 \leq i \leq j \leq m$. Now, let $w \in W$. If $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is equal to w when considered as an element of the group W, we say that $\boldsymbol{\alpha}$ is an *expression* for w. If m is minimal among all possible expressions for w, we say that $\boldsymbol{\alpha}$ is a *reduced expression* for w. We define the *length* of w, denoted $\ell(w)$, to be the number of letters in a reduced expression. We will also say that any reduced expression for w has length $\ell(w)$. Note that any factor of a reduced expression is also reduced. We denote the set of all reduced expressions for a group element

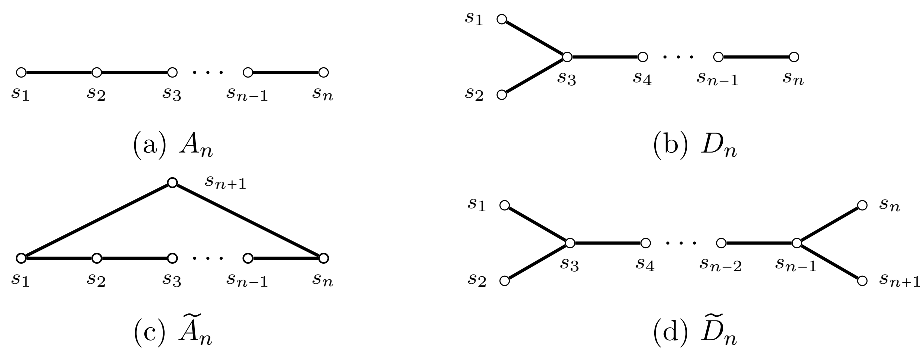


Figure 2.1: Examples of common simply-laced Coxeter graphs.

$w \in W$ by $\mathcal{R}(w)$. For brevity, if we are considering a particular labeling of a Coxeter graph, we will often replace s_i with i.

The following proposition, commonly referred to as Matsumoto's Theorem, characterizes the relationships between reduced expressions for a single group element. Matsumoto's Theorem appears in [9].

Proposition 2.2. In a Coxeter system (W, S), any two reduced expressions for the same group element differ by a sequence of commutation and braid moves.

Take (W, S) to be a Coxeter system and let $w \in W$. In light of Matsumoto's Theorem, we define two equivalence relations on the set of a reduced expressions for some element $w \in W$. For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{R}(w)$, we define $\sim_c$ via $\boldsymbol{\alpha} \sim_c \boldsymbol{\beta}$ if $\boldsymbol{\alpha}$ may be obtained from $\boldsymbol{\beta}$ by performing a single commutation move of the form $st \mapsto ts$ with $m(s,t) = 2$. The equivalence relation $\approx_c$ is defined by taking the reflexive and transitive closure of $\sim_c$ (i.e., $\approx_c$ is the smallest equivalence relation containing $\sim_c$). The corresponding equivalence classes under $\approx_c$ are referred to as *commutation classes*, denoted $[\boldsymbol{\alpha}]_c$. Appropriately, we say that two reduced expressions are *commutation equivalent* if they are in the same commutation class. In the case of Coxeter systems of type A_n, Elnitsky [7] showed that the set of commutation classes for a given permutation w is in one-to-one correspondence with the set of rhombic tilings of a certain polygon determined by w. Meng [12] studied the number of commutation classes and their relationships via braid moves, and Bédard [4] developed recursive formulas for the number of reduced expressions in each commutation class.

Analogously, we define $\sim_b$ via $\boldsymbol{\alpha} \sim_b \boldsymbol{\beta}$ if $\boldsymbol{\alpha}$ may be obtained from $\boldsymbol{\beta}$ by applying a single braid move of the form

$$\underbrace{sts\cdots}_{m(s,t)} \longmapsto \underbrace{tst\cdots}_{m(s,t)}$$

with $m(s,t) \geq 3$. We define the equivalence relation $\approx_b$ by taking the reflexive and transitive closure of $\sim_b$, and call each equivalence class under $\approx_b$ a *braid class*, denoted $[\boldsymbol{\alpha}]_b$. If two reduced expressions are in the same braid class, we say that these expressions are *braid* *equivalent.* Braid classes have appeared in the work of Bergeron, Ceballos, and Labbé [5] while Zollinger [16] provided formulas for the cardinality of braid classes in the case of Coxeter systems of type A_n. Fishel et al. [8] provided upper and lower bounds on the number of reduced expressions for a fixed permutation in Coxeter systems of type A_n by studying the commutation classes and braid classes in tandem. However, unlike commutation classes, braid classes have received very little attention.

Example 2.3. Consider the expression $\boldsymbol{\alpha} = 1321434$ for some w in the Coxeter system of type D_4. It turns out that $\boldsymbol{\alpha}$ is reduced, so $\ell(w) = 7$. The set of 15 reduced expressions is partitioned into five commutation classes:

$$[1321434]_c = \{1321434, 1324134, 1342134, 1341234, 1314234, 1312434\}$$
$$[3123243]_c = \{3123243, 3213243, 3213423, 3123423\}$$
$$[3134234]_c = \{3134234, 3132434\}$$
$$[1321343]_c = \{1321343, 1312343\}$$
$$[3132343]_c = \{3132343\}$$

and nine braid classes:

$$[1312343]_b = \{1312343, 1312434, 3132434, 3132343, 3123243\}$$
$$[1321343]_b = \{1321343, 1321434\}$$
$$[1314234]_b = \{1314234, 3134234\}$$
$$[1324134]_b = \{1324134\}$$
$$[1342134]_b = \{1342134\}$$
$$[1341234]_b = \{1341234\}$$
$$[3213243]_b = \{3213243\}$$
$$[3213423]_b = \{3213423\}$$
$$[3123423]_b = \{3123423\}$$

The focus of this thesis will primarily be on braid classes and their structure. Accordingly, we will write $[\boldsymbol{\alpha}] \coloneqq [\boldsymbol{\alpha}]_b$ for the remainder of this thesis.

Example 2.4. Below we describe three different braid classes. We have used underlines and overlines to indicate where braid moves may occur.

(a) In the Coxeter system of type A_6, the expression $\boldsymbol{\alpha}_1 = 1213243565$ is reduced. Its braid class consists of the following reduced expressions:

$$\boldsymbol{\alpha}_1 = \underline{121}3243\underline{565},\ \boldsymbol{\alpha}_2 = \underline{21\overline{2}}\overline{32}43\underline{565},\ \boldsymbol{\alpha}_3 = 21\underline{32\overline{3}}\overline{43}\underline{565},\ \boldsymbol{\alpha}_4 = 2132\overline{434}\underline{565},$$

$$\boldsymbol{\alpha}_5 = \underline{121}3243\underline{656},\ \boldsymbol{\alpha}_6 = \underline{21\overline{2}}\overline{32}43\underline{656},\ \boldsymbol{\alpha}_7 = 21\underline{32\overline{3}}\overline{43}\underline{656},\ \boldsymbol{\alpha}_8 = 2132\overline{434}\underline{656}.$$

(b) In the Coxeter system of type D_4, the expression $\boldsymbol{\beta}_1 = 4341232$ is reduced and its braid class consists of the following reduced expressions:

$$\boldsymbol{\beta}_1 = \underline{434}1\underline{232},\ \boldsymbol{\beta}_2 = \underline{343}1\underline{232},\ \boldsymbol{\beta}_3 = \underline{434}1\underline{323},\ \boldsymbol{\beta}_4 = \underline{34\overline{3}}\overline{1}\underline{\overline{3}23},\ \boldsymbol{\beta}_5 = 34\underline{131}23.$$

(c) In the Coxeter system of type D_4, the expression $\boldsymbol{\gamma}_1 = 343132343$ is reduced and its braid class consists of the following reduced expressions:

$$\boldsymbol{\gamma}_1 = \underline{34\overline{3}}\overline{1}\underline{\overline{3}2\overline{3}}\overline{43},\ \boldsymbol{\gamma}_2 = 34\underline{131}2\underline{343},\ \boldsymbol{\gamma}_3 = \underline{434}1\underline{32\overline{3}}\overline{43},\ \boldsymbol{\gamma}_4 = \underline{343}1\underline{232}43,$$

$$\boldsymbol{\gamma}_5 = \underline{434}1\underline{232}43,\ \boldsymbol{\gamma}_6 = \underline{343}132\underline{434},\ \boldsymbol{\gamma}_7 = 34\underline{131}2\underline{434},\ \boldsymbol{\gamma}_8 = \underline{434}132\underline{434}.$$

Throughout the remainder of this chapter, we will assume (W, S) is simply laced. The following terminology and definitions allow us to introduce the notions of braid shadow and link. For $i, j \in \mathbb{N}$ with $i \leq j$, we define the interval $[\![i, j]\!] := \{i, i+1, \ldots, j-1, j\}$. That is, $[\![i, j]\!]$ denotes the set of all natural numbers between i and j. It follows that $[\![i, i]\!] := \{i\}$. The intervals $[\![i, j]\!]$ will be used to denote the positions between i and j in a reduced expression.

For a single reduced expression $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m}$, the *local support* of $\boldsymbol{\alpha}$ over the interval $[\![i, j]\!]$ is defined via

$$\operatorname{supp}_{[\![i,j]\!]}(\boldsymbol{\alpha}) := \{s_{x_k} \mid k \in [\![i, j]\!]\}.$$

We define the *local support of the braid class* $[\boldsymbol{\alpha}]$ over the interval $[\![i, j]\!]$ via

$$\operatorname{supp}_{[\![i,j]\!]}([\boldsymbol{\alpha}]) := \bigcup_{\boldsymbol{\beta} \in [\boldsymbol{\alpha}]} \operatorname{supp}_{[\![i,j]\!]}(\boldsymbol{\beta}).$$

That is, the set $\operatorname{supp}_{[\![i,j]\!]}(\boldsymbol{\alpha})$ contains the generators that appear in positions $i, i+1, \ldots, j$ of a single reduced expression $\boldsymbol{\alpha}$, while $\operatorname{supp}_{[\![i,j]\!]}([\boldsymbol{\alpha}])$ contains the generators that appear in positions $i, i+1, \ldots, j$ of any reduced expression in the braid class $[\boldsymbol{\alpha}]$. In the special case of the degenerate interval $[\![i, i]\!]$, we write $\operatorname{supp}_i(\boldsymbol{\alpha}) := \operatorname{supp}_{[\![i,i]\!]}(\boldsymbol{\alpha})$ and $\operatorname{supp}_i([\boldsymbol{\alpha}]) := \operatorname{supp}_{[\![i,i]\!]}([\boldsymbol{\alpha}])$. Further, we let $\boldsymbol{\alpha}_{[\![i,j]\!]}$ denote the factor of $\boldsymbol{\alpha}$ appearing in positions $i, i+1, \ldots, j$ of $\boldsymbol{\alpha}$.

Let $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m}$ be a reduced expression for $w \in W$. From [2], we say $[\![i-1, i+1]\!]$ is a *braid shadow* for $\boldsymbol{\alpha}$ if $\boldsymbol{\alpha}_{[\![i-1,i+1]\!]} = sts$ with $m(s, t) = 3$. We denote the collection of braid shadows for $\boldsymbol{\alpha}$ by $\mathcal{S}(\boldsymbol{\alpha})$. The set of braid shadows for the braid class $[\boldsymbol{\alpha}]$ is aptly defined as

$$\mathcal{S}([\boldsymbol{\alpha}]) := \bigcup_{\boldsymbol{\beta} \in [\boldsymbol{\alpha}]} \mathcal{S}(\boldsymbol{\beta}).$$

The *rank* of a reduced expression $\boldsymbol{\alpha}$, denoted $\operatorname{rank}(\boldsymbol{\alpha})$, is defined to be the cardinality of $\mathcal{S}([\boldsymbol{\alpha}])$.

In short, a braid shadow for a reduced expression $\boldsymbol{\alpha}$ is an interval of positions where a braid move may be applied in $\boldsymbol{\alpha}$. The set $\mathcal{S}(\boldsymbol{\alpha})$ is the collection of all braid shadows for a specific reduced expression $\boldsymbol{\alpha}$, while $\mathcal{S}([\boldsymbol{\alpha}])$ is the set of all braid shadows for all reduced expressions braid equivalent to $\boldsymbol{\alpha}$. If $[\![i-1,i+1]\!]$ is a braid shadow for $[\boldsymbol{\alpha}]$, then the position i in any reduced expression in $[\boldsymbol{\alpha}]$ is called the *center* of the braid shadow.

Example 2.5. Consider the reduced expressions given in Example 2.4. We see that:

(a) $\mathcal{S}(\boldsymbol{\alpha}_1) = \{[\![1,3]\!],[\![8,10]\!]\}$ and $\mathcal{S}([\boldsymbol{\alpha}_1]) = \{[\![1,3]\!],[\![3,5]\!],[\![5,7]\!],[\![8,10]\!]\}$,

(b) $\mathcal{S}(\boldsymbol{\beta}_1) = \{[\![1,3]\!],[\![5,7]\!]\}$ and $\mathcal{S}([\boldsymbol{\beta}_1]) = \{[\![1,3]\!],[\![3,5]\!],[\![5,7]\!]\}$,

(c) $\mathcal{S}(\boldsymbol{\gamma}_1) = \{[\![1,3]\!],[\![3,5]\!],[\![5,7]\!],[\![7,9]\!]\}$ and $\mathcal{S}([\boldsymbol{\gamma}_1]) = \{[\![1,3]\!],[\![3,5]\!],[\![5,7]\!],[\![7,9]\!]\}$.

The following result from [2] states that for any reduced expression $\boldsymbol{\alpha}$, any distinct pair of braid shadows across $[\boldsymbol{\alpha}]$ must either be disjoint or overlap by exactly one position.

Proposition 2.6. Suppose (W,S) is a simply-laced Coxeter system. If $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$ with $[\![i-1,i+1]\!] \in \mathcal{S}([\boldsymbol{\alpha}])$, then $[\![i-2,i]\!],[\![i,i+2]\!] \notin \mathcal{S}([\boldsymbol{\alpha}])$.

This result gives ground to the following definition from [2]. If $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$ with $\ell(w) = m \geq 1$, we define $\boldsymbol{\alpha}$ to be a *link* if either $m = 1$ or m is odd and

$$\mathcal{S}([\boldsymbol{\alpha}]) = \{[\![1,3]\!],[\![3,5]\!],\ldots,[\![m-4,m-2]\!],[\![m-2,m]\!]\}.$$

If $\boldsymbol{\alpha}$ is a link, we say that the braid class $[\boldsymbol{\alpha}]$ is a *braid chain*. Note that if $\boldsymbol{\alpha}$ is a link and $\boldsymbol{\beta} \in [\boldsymbol{\alpha}]$, then $\boldsymbol{\beta}$ is also a link. Furthermore, if $\boldsymbol{\alpha}$ is a link of rank at least 1, then the centers occur in the even-index positions.

Example 2.7. Consider the reduced expressions given in Example 2.4. Since $[\![7,9]\!] \notin \mathcal{S}([\boldsymbol{\alpha}_1])$ and m is not odd, $\boldsymbol{\alpha}_1$ is a not a link, and hence $[\boldsymbol{\alpha}_1]$ is not a braid chain. It turns out that the factors 1213243 and 565 of $\boldsymbol{\alpha}_1$ are links. However, since $\mathcal{S}([\boldsymbol{\beta}_1]) = \{[\![1,3]\!],[\![3,5]\!],[\![5,7]\!]\}$, $\boldsymbol{\beta}_1$ is a link and $[\boldsymbol{\beta}_1]$ is a braid chain. Lastly, since $\mathcal{S}([\boldsymbol{\gamma}_1]) = \{[\![1,3]\!],[\![3,5]\!],[\![5,7]\!],[\![7,9]\!]\}$, $\boldsymbol{\gamma}_1$ is a link and $[\boldsymbol{\gamma}_1]$ is a braid chain.

Let $\boldsymbol{\alpha}$ be a reduced expression for $w \in W$ such that $\ell(w) \geq 1$. Then $\boldsymbol{\beta}$ is said to be a *link factor* of $\boldsymbol{\alpha}$, denoted $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$, if and only if

(a) $\boldsymbol{\beta}$ is a factor of $\boldsymbol{\alpha}$,

(b) $\boldsymbol{\beta}$ is a link, and

(c) If $\boldsymbol{\beta} < \boldsymbol{\gamma} \leq \boldsymbol{\alpha}$, then $\boldsymbol{\gamma}$ is not a link.

That is, the link factors of a reduced expression are the largest factors of that expression that are also links. It follows that we may uniquely write each reduced expression $\boldsymbol{\alpha}$ for a nonidentity group element as a product of link factors $\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\cdots\boldsymbol{\alpha}_k$, where each $\boldsymbol{\alpha}_i$ is a link factor. This product is called the *link factorization* of $\boldsymbol{\alpha}$. We may denote the link factorization as $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 \mid \boldsymbol{\alpha}_2 \mid \cdots \mid \boldsymbol{\alpha}_k$. For convenience, we say that the link factorization of the identity is a product of a single copy of the empty word, but it is important to note that the empty word is not actually a link. The next result appears in [2].

Proposition 2.8. Suppose (W, S) is a simply-laced Coxeter system. If $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$ with link factorization $\boldsymbol{\alpha}_1 \mid \boldsymbol{\alpha}_2 \mid \cdots \mid \boldsymbol{\alpha}_k$, then

(a) $[\boldsymbol{\alpha}] = \{\boldsymbol{\beta}_1 \mid \boldsymbol{\beta}_2 \mid \cdots \mid \boldsymbol{\beta}_k : \boldsymbol{\beta}_i \in [\boldsymbol{\alpha}_i] \text{ for } 1 \leq i \leq k\}$,

(b) The cardinality of the braid class for $\boldsymbol{\alpha}$ is given by $\operatorname{card}([\boldsymbol{\alpha}]) = \prod_{i=1}^{k} \operatorname{card}([\boldsymbol{\alpha}_i])$,

(c) The rank of $\boldsymbol{\alpha}$ is given by $\operatorname{rank}(\boldsymbol{\alpha}) = \sum_{i=1}^{k} \operatorname{rank}(\boldsymbol{\alpha}_i)$.

We will continue to develop the local structure of links in Coxeter systems of type Λ. The next proposition from [2] tells us that if two braid equivalent links have a common braid shadow, then the support in the positions covered by this braid shadow in these two links are equal.

Proposition 2.9. Suppose that (W, S) is of type Λ. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two braid equivalent links of rank at least one, then for all $[\![2i-1, 2i+1]\!] \in \mathcal{S}(\boldsymbol{\alpha}) \cap \mathcal{S}(\boldsymbol{\beta}), \operatorname{supp}_{[\![2i-1,2i+1]\!]}(\boldsymbol{\alpha}) = \operatorname{supp}_{[\![2i-1,2i+1]\!]}(\boldsymbol{\beta})$.

This result may not hold if (W, S) is not triangle free. Consider the following example.

Example 2.10. Consider the Coxeter system of type $\widetilde{A}_2$, which is determined by the Coxeter graph in Figure 2.1(c). Given the reduced expression $\boldsymbol{\alpha} = 1213121$, it is clear that $\boldsymbol{\beta} = 2123212 \in [\boldsymbol{\alpha}]$. However, $\operatorname{supp}_{[\![3,5]\!]}(\boldsymbol{\alpha}) = \{1, 3\}$ while $\operatorname{supp}_{[\![3,5]\!]}(\boldsymbol{\beta}) = \{2, 3\}$ despite $[\![3, 5]\!] \in \mathcal{S}(\boldsymbol{\alpha}) \cap \mathcal{S}(\boldsymbol{\beta})$. Hence Proposition 2.9 does not necessarily hold if the Coxeter system is not triangle free.

The next proposition from [2] states that if a braid shadow exists in a link, the support of that braid shadow determines which generators can appear at the center of that shadow across the entire braid class. Further, if there exists overlapping braid shadows in a link, then their supports intersect at a single generator.

Proposition 2.11. Suppose (W, S) is type Λ. If $\boldsymbol{\alpha}$ is a link for $w \in W$, then $[\![2i-1, 2i+1]\!] \in \mathcal{S}(\boldsymbol{\alpha})$ if and only if $[\![2i-1, 2i+1]\!] \in \mathcal{S}([\boldsymbol{\alpha}])$ and $\operatorname{supp}_{[\![2i-1,2i+1]\!]}(\boldsymbol{\alpha}) = \operatorname{supp}_{2i}([\boldsymbol{\alpha}])$. Moreover, if $[\![2i-1, 2i+1]\!], [\![2i+1, 2i+3]\!] \in \mathcal{S}([\boldsymbol{\alpha}])$, then $\operatorname{card}(\operatorname{supp}_{2i}([\boldsymbol{\alpha}]) \cap \operatorname{supp}_{[\![2i+2]\!]}([\boldsymbol{\alpha}])) = 1$.

In light of the previous proposition, if $[\![2i-1, 2i+1]\!] \in \mathcal{S}(\boldsymbol{\alpha})$, we may assume that $\operatorname{supp}_{[\![2i-1,2i+1]\!]}(\boldsymbol{\alpha}) = \{s, t\}$ and $\operatorname{supp}_{2i}([\boldsymbol{\alpha}]) = \{s, t\}$ with $m(s, t) = 3$. That is, the cardinality of the center of any braid shadow in $[\boldsymbol{\alpha}]$ is 2. Additionally, if $[\![2i+1, 2i+3]\!] \in \mathcal{S}(\boldsymbol{\alpha})$, then we can conclude that $\operatorname{supp}_{[\![2i+1,2i+3]\!]}(\boldsymbol{\alpha}) = \{t, u\}$ and $\operatorname{supp}_{2i+2}([\boldsymbol{\alpha}]) = \{t, u\}$ with $m(t, u) = 3$ and $m(s, u) = 2$. We will frequently use these facts throughout this thesis without directly mentioning the proposition.

Example 2.12. Consider the reduced expression $\boldsymbol{\delta}_1 = 1213121$ in type $\widetilde{A}_2$. The braid class for $\boldsymbol{\delta}_1$ consists of:

$$\boldsymbol{\delta}_1 = \underline{12\overline{1}}\overline{3}\underline{\overline{1}21} \;\; \boldsymbol{\delta}_2 = 12\overline{313}21 \;\; \boldsymbol{\delta}_3 = \underline{212}3\underline{121},$$

$$\boldsymbol{\delta}_4 = \underline{12\overline{1}}\overline{3}\underline{\overline{2}12}, \;\; \boldsymbol{\delta}_5 = \underline{21\overline{2}}\overline{3}\underline{\overline{2}12}, \;\; \boldsymbol{\delta}_6 = 21\overline{323}12.$$

Notice that $\mathrm{supp}_{[\![3,5]\!]}(\boldsymbol{\delta}_1) = \{1,3\} \neq \{2,3\} = \mathrm{supp}_{[\![3,5]\!]}(\boldsymbol{\delta}_5)$, so without the triangle free assumption, Proposition 2.11 does not hold.

The next two propositions from [2] continue to establish facts about the local structure of links. The next result establishes that the ends of a link are determined by the centers of the first and last braid shadows.

Proposition 2.13. Suppose that (W,S) is of type Λ and let $\boldsymbol{\alpha}$ be a link of rank $r \geq 1$.

(a) If $\mathrm{supp}_2([\boldsymbol{\alpha}]) = \{s,t\}$ with $m(s,t) = 3$, then $\boldsymbol{\alpha}_{[\![1,2]\!]} = st$ or $\boldsymbol{\alpha}_{[\![1,2]\!]} = ts$.

(b) If $\mathrm{supp}_{2r}([\boldsymbol{\alpha}]) = \{s,t\}$ with $m(s,t) = 3$, then $\boldsymbol{\alpha}_{[\![2r,2r+1]\!]} = st$ or $\boldsymbol{\alpha}_{[\![2r,2r+1]\!]} = ts$.

The following result tells us that if there are two overlapping braid shadows in a link, then there are three possible generators in the common position.

Proposition 2.14. Suppose (W,S) is a Coxeter system of type Λ. If $\boldsymbol{\alpha}$ is a link of rank $r \geq 2$ such that for $1 \leq i \leq r-1$, $\mathrm{supp}_{2i}([\boldsymbol{\alpha}]) = \{s,t\}$ and $\mathrm{supp}_{2i+2}([\boldsymbol{\alpha}]) = \{t,u\}$ with $m(s,t) = 3 = m(t,u)$, then $\mathrm{supp}_{2i+1}([\boldsymbol{\alpha}]) = \{s,t,u\}$, $\boldsymbol{\alpha}_{2i} \neq \boldsymbol{\alpha}_{2i+2}$, and $\boldsymbol{\alpha}_{2i+1} \in \{s,t,u\} \setminus \{\boldsymbol{\alpha}_{2i}, \boldsymbol{\alpha}_{2i+2}\}$.

One consequence of the previous propositions is that for any two overlapping braid shadows in a braid chain $[\boldsymbol{\alpha}]$, there are three possible forms that $\boldsymbol{\alpha}_{[\![2i,2i+2]\!]}$ may take:

(a) $$\underbrace{\cdots \underset{2i-1}{?}\;\underset{2i}{s}\;\underset{2i+1}{u}\;\underset{2i+2}{t}\;\underset{2i+3}{?} \cdots}_{\boldsymbol{\alpha}}$$

(b) $$\underbrace{\cdots \underset{2i-1}{?}\;\underset{2i}{s}\;\underset{2i+1}{t}\;\underset{2i+2}{u}\;\underset{2i+3}{?} \cdots}_{\boldsymbol{\alpha}}$$

(c) $$\underbrace{\cdots \underset{2i-1}{?}\;\underset{2i}{t}\;\underset{2i+1}{s}\;\underset{2i+2}{u}\;\underset{2i+3}{?} \cdots}_{\boldsymbol{\alpha}}$$

where $m(s,t) = 3 = m(t,u)$ and $m(s,u) = 2$. Note that positions $2i$ and $2i+2$ are centers, while position $2i+1$ is the location where the two braid shadows in $[\boldsymbol{\alpha}]$ overlap.

The previous propositions outline how one can use the generator in the center of a braid shadow to determine the generators around that center. The significance of the center of a braid shadow influences the following definition, which appeared in [3]. If (W, S) is of type Λ and $\boldsymbol{\alpha}$ is a link of rank $r \geq 1$, the *signature* of $\boldsymbol{\alpha}$, denoted $\mathrm{sig}(\boldsymbol{\alpha})$, is the ordered list of generators appearing in the centers of the braid shadows of $\boldsymbol{\alpha}$. That is, $\mathrm{sig}(\boldsymbol{\alpha})$ is the ordered list of generators appearing in the even positions. We use $\mathrm{sig}_i(\boldsymbol{\alpha})$ to represent the ith entry of $\mathrm{sig}(\boldsymbol{\alpha})$. In other words, the ith entry of the signature corresponds to the generator appearing at the ith center.

The following result originally appeared in [2], but was rephrased in terms of signature in [3].

Proposition 2.15. Suppose (W, S) is type Λ and let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two braid equivalent links of rank at least one. Then $\boldsymbol{\alpha} = \boldsymbol{\beta}$ if and only if $\mathrm{sig}(\boldsymbol{\alpha}) = \mathrm{sig}(\boldsymbol{\beta})$.

The conclusion of this chapter will focus on a special partition of $[\boldsymbol{\alpha}]$. We define two special sets, which were first introduced in [2]. Let $\boldsymbol{\alpha}$ be a link of rank $r \geq 1$. We define:

$$X_{\boldsymbol{\alpha}} := \{\boldsymbol{\beta} \in [\boldsymbol{\alpha}] \mid \mathrm{sig}_r(\boldsymbol{\beta}) = \mathrm{sig}_r(\boldsymbol{\alpha})\}$$

$$Y_{\boldsymbol{\alpha}} := \{\boldsymbol{\beta} \in [\boldsymbol{\alpha}] \mid \mathrm{sig}_r(\boldsymbol{\beta}) \neq \mathrm{sig}_r(\boldsymbol{\alpha})\}.$$

That is, $X_{\boldsymbol{\alpha}}$ is the set of links in $[\boldsymbol{\alpha}]$ that share the same penultimate generator as $\boldsymbol{\alpha}$, while $Y_{\boldsymbol{\alpha}}$ is the set of links that do not. Specifically, if $\mathrm{supp}_{2r}([\boldsymbol{\alpha}]) = \{s, t\}$ and $\mathrm{supp}_{2r}(\boldsymbol{\alpha}) = \{s\}$, then every link in $X_{\boldsymbol{\alpha}}$ has s in position $2r$ while every link in $Y_{\boldsymbol{\alpha}}$ has t in position $2r$. It is clear to see that $Y_{\boldsymbol{\alpha}}$ is the complement of $X_{\boldsymbol{\alpha}}$ in $[\boldsymbol{\alpha}]$.

Example 2.16. Consider the link $\boldsymbol{\alpha}_1$ in part (a) of Example 2.4. Then one can see that $X_{\boldsymbol{\alpha}_1} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4\}$ and $Y_{\boldsymbol{\alpha}_1} = \{\boldsymbol{\alpha}_5, \boldsymbol{\alpha}_6, \boldsymbol{\alpha}_7, \boldsymbol{\alpha}_8\}$.

The next two propositions from [2] follow from Proposition 2.14 and state that if $\boldsymbol{\alpha}$ is a link, then for all pairs of overlapping braid shadows in $[\boldsymbol{\alpha}]$, there is a link in $[\boldsymbol{\alpha}]$ where overlapping braid shadows occur simultaneously.

Proposition 2.17. If (W, S) is of type Λ and $\boldsymbol{\alpha}$ is a link of rank $r \geq 2$, then for all $2 \leq i \leq r$, there exists $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ with the property that $[\![2i-3, 2i-1]\!], [\![2i-1, 2i+1]\!] \in \mathcal{S}(\boldsymbol{\sigma})$.

If we choose $\boldsymbol{\sigma}$ according to Proposition 2.17, then by Proposition 2.13, if $\boldsymbol{\beta} \in X_{\boldsymbol{\sigma}}$, then $\boldsymbol{\beta}_{[\![2r,2r+1]\!]} = \boldsymbol{\sigma}_{[\![2r,2r+1]\!]}$. Similarly, if $\boldsymbol{\beta} \in Y_{\boldsymbol{\sigma}}$, then $\boldsymbol{\beta}_{[\![2r,2r+1]\!]} \neq \boldsymbol{\sigma}_{[\![2r,2r+1]\!]}$. In other words, all links in $X_{\boldsymbol{\sigma}}$ have the same final two generators and all links in $Y_{\boldsymbol{\sigma}}$ have the same final two generators.

Let $\boldsymbol{\alpha}$ be a link consisting of at least two generators. We define $\hat{\boldsymbol{\alpha}}$ to be the reduced expression obtained by deleting the two rightmost letters of $\boldsymbol{\alpha}$. Certainly, $\hat{\boldsymbol{\alpha}}$ is reduced, but it is important to note that just because $\boldsymbol{\alpha}$ is a link does not necessarily guarantee that $\hat{\boldsymbol{\alpha}}$ will be a link. Before introducing the next proposition, we need some notation. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two braid equivalent links in a Coxeter system of type Λ. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related by a single braid move that occurs in the jth braid shadow (i.e., only the jth entry of the signature differs between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$), we denote this braid move as b^j and accordingly write $b^j(\boldsymbol{\alpha}) = \boldsymbol{\beta}$ to represent that applying the braid move b^j to $\boldsymbol{\alpha}$ yields $\boldsymbol{\beta}$. The subsequent proposition, which combines multiple results from [2], first establishes the special condition on a link $\boldsymbol{\alpha}$ such that $\hat{\boldsymbol{\alpha}}$ is also a link. This result then outlines properties imposed on the braid class $[\hat{\boldsymbol{\alpha}}]$ and the relationship between $X_{\boldsymbol{\sigma}}$ and $Y_{\boldsymbol{\sigma}}$ in this scenario.

Proposition 2.18. Suppose (W, S) is of type Λ and $\boldsymbol{\alpha}$ is a link of rank $r \geq 2$. Let $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ such that $[\![2r-3, 2r-1]\!], [\![2r-1, 2r+1]\!] \in \mathcal{S}(\boldsymbol{\sigma})$ as described in Proposition 2.17. Then:

(a) $\{X_\sigma, Y_\sigma\}$ is a partition of $[\boldsymbol{\alpha}]$;

(b) $\hat{\boldsymbol{\sigma}}$ is a link of rank $r-1$;

(c) If $\boldsymbol{\beta} \in X_{\boldsymbol{\sigma}}$, then $\hat{\boldsymbol{\beta}} \in [\hat{\boldsymbol{\sigma}}]$;

(d) Every element of $[\hat{\boldsymbol{\sigma}}]$ is of the form $\hat{\boldsymbol{\beta}}$ for some $\boldsymbol{\beta} \in X_{\boldsymbol{\sigma}}$;

(e) If $\boldsymbol{\beta} \in Y_{\boldsymbol{\sigma}}$, then $[\![2r-1, 2r+1]\!] \in \mathcal{S}(\boldsymbol{\beta})$ and $(b^r(\boldsymbol{\beta}))_{[\![1,2r-1]\!]} \in [\hat{\boldsymbol{\sigma}}]$.

Example 3.9 in the following chapter will provide a concrete example of the previous proposition in terms of its manifestation in the context of braid graphs.

Chapter 3

Braid graphs in Coxeter systems

In this chapter, we will discuss how the braid relations between reduced expressions may be graphically represented. Let (W,S) be a Coxeter system and let $w \in W$. The *Matsumoto graph* $\mathcal{G}(w)$ is defined to be the graph whose vertex set is $\mathcal{R}(w)$, where two vertices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are connected by an edge if and only if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related via a single commutation or braid move. Temporarily, we will color an edge orange if it corresponds to a commutation move and we will color an edge teal if it corresponds to a braid move. Matsumoto's Theorem implies that $\mathcal{G}(w)$ is connected. In [5], Bergeron, Ceballos, and Labbé proved that every cycle in a Matsumoto graph for finite Coxeter groups is of even length. This result was extended to arbitrary Coxeter systems in [10]. As a result of this fact, we get the following proposition.

Proposition 3.1. If (W,S) is a Coxeter system and $w \in W$, then $\mathcal{G}(w)$ is bipartite.

Example 3.2. Recall the reduced expression $\boldsymbol{\alpha} = 1321434$ from Example 2.3 in the Coxeter system of type D_4. There are 15 reduced expressions in $\mathcal{R}(w)$ and the corresponding Matsumoto graph is given in Figure 3.1. The edges of $\mathcal{G}(w)$ show how pairs of reduced expressions are related via commutation or braid moves. Notice that the braid classes correspond to the connected components of the teal subgraphs obtained by deleting the orange edges of the Matsumoto graph given in Figure 3.1 while the singleton braid classes correspond to the six vertices that are not incident to any teal edges. A similar structure holds for the commutation classes.

If we focus on the maximal teal connected components of a Matsumoto graph, we obtain graphical representations of the corresponding braid class. Each maximal teal connected component defines a braid graph for a braid class. More formally, for a reduced expression $\boldsymbol{\alpha}$, the *braid graph* of $\boldsymbol{\alpha}$, denoted $\mathcal{B}(\boldsymbol{\alpha})$, is the graph whose vertex set is $[\boldsymbol{\alpha}]$ and $\boldsymbol{\beta}, \boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$ are connected by an edge if and only if $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are related by a single braid move. Braid graphs are defined with respect to a fixed reduced expression (or braid class) as opposed to the corresponding group element. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are braid equivalent, then $\mathcal{B}(\boldsymbol{\alpha}) = \mathcal{B}(\boldsymbol{\beta})$. On the other hand, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related via a commutation move, then $\mathcal{B}(\boldsymbol{\alpha}) \neq \mathcal{B}(\boldsymbol{\beta})$ but they might be isomorphic.

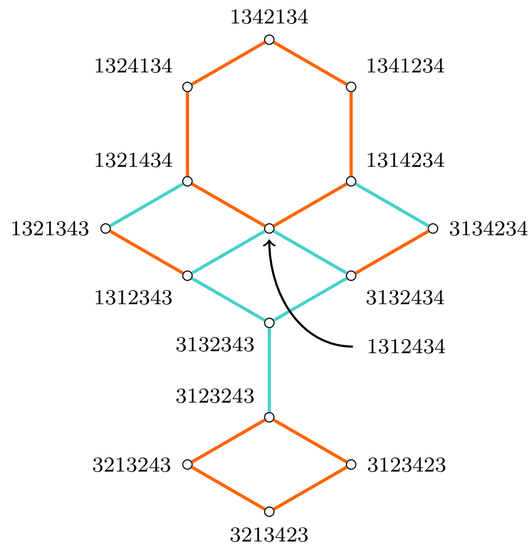


Figure 3.1: Example of a Matsumoto graph for the reduced expression given in Example 3.2 in the Coxeter system of type D_4.

Example 3.3. Figure 3.2 depicts the braid graphs for the reduced expressions given in Example 2.4. The vertices are labeled with the corresponding reduced expressions.

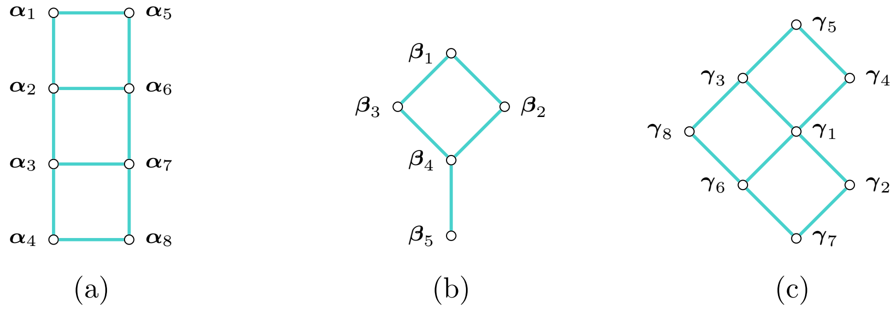


Figure 3.2: Braid graphs corresponding to Examples 2.4 and 3.3.

The next proposition is a direct result of Proposition 3.1.

Proposition 3.4. If (W, S) is a Coxeter system and $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$, then $\mathcal{B}(\boldsymbol{\alpha})$ is bipartite.

The following proposition from [2] describes how the braid graph for some reduced expression is obtained from the braid graphs of its link factors in simply-laced Coxeter systems.

Proposition 3.5. If (W, S) is a simply-laced Coxeter system and $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$ with link factorization $\boldsymbol{\alpha}_1 \mid \boldsymbol{\alpha}_2 \mid \cdots \mid \boldsymbol{\alpha}_k$, then

$$\mathcal{B}(\boldsymbol{\alpha}) \cong \mathcal{B}(\boldsymbol{\alpha}_1) \,\square\, \mathcal{B}(\boldsymbol{\alpha}_2) \,\square \cdots \square\, \mathcal{B}(\boldsymbol{\alpha}_k).$$

Example 3.6. Consider the reduced expression $\boldsymbol{\alpha} = 121324356576$ in the Coxeter system of type A_7. The link factorization for $\boldsymbol{\alpha}$ is $1213243 \mid 56576$. The decomposition $\mathcal{B}(1213243) \,\square\, \mathcal{B}(56576) \cong \mathcal{B}(\boldsymbol{\alpha})$ is illustrated in Figure 3.3. We have utilized addtional colors beyond teal to help distinguish the link factors.

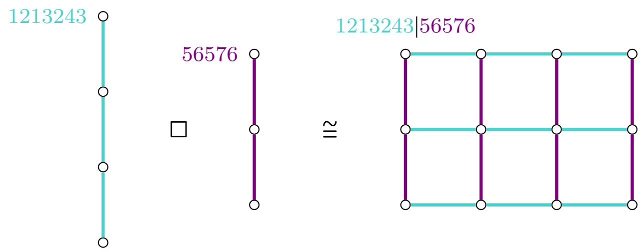


Figure 3.3: Decomposition of the braid graph for the reduced expression in Example 3.6.

Example 3.7. Consider the reduced expression $\boldsymbol{\alpha} = 3231343565787$ in the Coxeter system of type D_7. The link factorization for $\boldsymbol{\alpha}$ is $3231343 \mid 565 \mid 787$. The braid graph for the first link factor is isomorphic to the braid graph in Figure 3.2(b). The braid graph for the entire reduced expression and its decomposition are shown in Figure 3.4. Again, we have utilized colors to help distinguish the link factors.

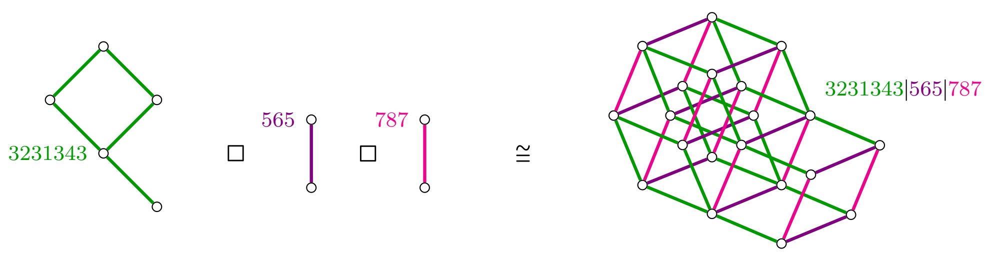


Figure 3.4: Braid graph for the reduced expression from Example 3.7 and its decomposition into a box product of braid graphs for the corresponding link factors.

The rest of this chapter is dedicated to establishing structural results pertaining to braid graphs in Coxeter systems of type Λ. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be links that are related by a single braid move that occurs in the jth braid shadow. Recall from Chapter 2 that the corresponding braid move is denoted by b^j. We accordingly label the edge connecting $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with a j. The next proposition from [2] establishes the structure of the subgraphs of $\mathcal{B}(\boldsymbol{\alpha})$

induced by the sets $X_{\boldsymbol{\sigma}}$ and $Y_{\boldsymbol{\sigma}}$ where $\boldsymbol{\sigma}$ is chosen according to Proposition 2.17 such that $[\![2r-3, 2r-1]\!], [\![2r-1, 2r+1]\!] \in \mathcal{S}(\boldsymbol{\sigma})$. This proposition applies the results established in Proposition 2.18 to braid graphs.

Proposition 3.8. Suppose (W, S) is of type Λ and $\boldsymbol{\alpha}$ is a link of rank $r \geq 2$ and choose $\boldsymbol{\sigma}$ according to Proposition 2.17 such that $[\![2r-3, 2r-1]\!], [\![2r-1, 2r+1]\!] \in \mathcal{S}(\boldsymbol{\sigma})$. Then:

(a) There exists an isometric embedding from $\mathcal{B}(\hat{\boldsymbol{\sigma}})$ into $\mathcal{B}(\boldsymbol{\alpha})$ whose image is $\mathcal{B}(\boldsymbol{\alpha})[X_{\boldsymbol{\sigma}}]$;

(b) The induced subgraph $B(\boldsymbol{\alpha})[Y_{\boldsymbol{\sigma}}]$ is an isometric subgraph of $B(\boldsymbol{\alpha})$;

(c) If $\boldsymbol{\beta} \in X_{\boldsymbol{\sigma}}$ and $\boldsymbol{\gamma} \in Y_{\boldsymbol{\sigma}}$, then $d(\boldsymbol{\beta}, \boldsymbol{\gamma}) = d(\boldsymbol{\beta}, b^r(\boldsymbol{\gamma})) + 1$.

Example 3.9. Consider the link $\boldsymbol{\alpha} = 32313435464$ in the Coxeter system of type $\widetilde{D}_5$. One possible choice for a link satisfying the conditions in Proposition 2.17 with braid shadows $[\![2r-3, 2r-1]\!]$ and $[\![2r-1, 2r+1]\!]$ is $\boldsymbol{\sigma} = 32314345464$. The braid graph for $\boldsymbol{\alpha}$ is given in Figure 3.5. We have highlighted $\mathcal{B}(\boldsymbol{\alpha})[X_{\boldsymbol{\sigma}}]$ in teal and $\mathcal{B}(\boldsymbol{\alpha})[Y_{\boldsymbol{\sigma}}]$ in magenta. In this case, $\hat{\boldsymbol{\sigma}} = 323143454$, and $\mathcal{B}(\hat{\boldsymbol{\sigma}}) \cong \mathcal{B}(\boldsymbol{\alpha})[X_{\boldsymbol{\sigma}}]$. Each of the edges joining $\mathcal{B}(\boldsymbol{\alpha})[X_{\boldsymbol{\sigma}}]$ and $\mathcal{B}(\boldsymbol{\alpha})[Y_{\boldsymbol{\sigma}}]$ correspond to the braid move applied in the rightmost braid shadow and are shown in black. These black edges constitute the equivalence class of F-edges containing $\{\boldsymbol{\alpha}, b^5(\boldsymbol{\alpha})\}$ as stated in Proposition 3.14.

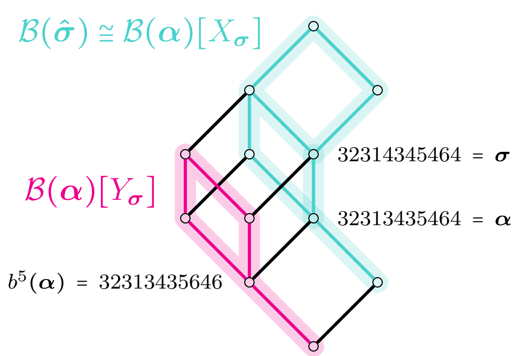


Figure 3.5: Braid graph for the reduced expression in Example 3.9 together with a partition of the vertices according to Propositions 2.18 and 3.8.

We denote a minimal sequence of braid moves from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ as $b_1^{j_1}, b_2^{j_2}, \ldots, b_k^{j_k}$, where $b_i^{j_i}$ is the ith braid move in the sequence that occurs in the j_ith shadow. A minimal braid sequence corresponds to a geodesic in $\mathcal{B}(\boldsymbol{\alpha})$ consisting of edges labeled consecutively with $j_1, j_2, \ldots, j_k$. The next proposition cobbles together results from [3].

Proposition 3.10. Suppose (W, S) is type Λ and let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two braid equivalent links of rank at least one.

(a) If $b_1^{j_1}, b_2^{j_2}, ..., b_k^{j_k}$ is a minimal braid sequence from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$, then each j_i appears exactly once.

(b) If $b_1^{j_1}, b_2^{j_2}, \dots, b_k^{j_k}$ and $b_1^{l_1}, b_2^{l_2}, \dots, b_k^{l_k}$ are minimal braid sequences from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$, then $\{j_1, ..., j_k\} = \{l_1, ..., l_k\}$.

It follows from part (a) of the previous proposition that each braid move in a minimal sequence is unique. From part (b), each minimal sequence of braid moves between two reduced expressions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ contains the same set of braid shadows. Graphically this means that each geodesic between two reduced expressions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ utilizes the same set of edge labels and each label appears once. Instead of labeling edges with the corresponding braid shadow location, we will often assign colors to edges in a braid graph to represent which braid shadow that edge corresponds to.

Example 3.11. Figure 3.6 depicts a braid graph for the link $\boldsymbol{\alpha}$ = 3431232324354 in the Coxeter system of type D_5. One can verify that each geodesic between any pair of vertices utilizes the same set of colors with each color appearing exactly once. The green edges correspond to the braid move b^5, the pink edges correspond to b^4, the purple edges correspond to b^3, the teal edges correspond to b^2, and finally the orange edges correspond to b^1.

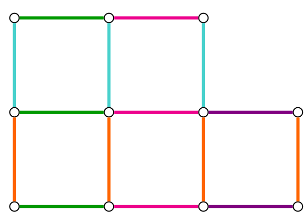


Figure 3.6: Braid graph for the link in Example 3.11. Edges are colored according to which braid shadow they correspond to.

Recall that $\text{sig}(\boldsymbol{\alpha})$ is the ordered list of generators appearing in the even positions of a link. Following [3], for braid equivalent links, we define $\Delta(\text{sig}(\boldsymbol{\alpha}), \text{sig}(\boldsymbol{\beta}))$ to be the number of generators that differ between the signatures of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The following corollaries from [3] are immediate consequences of Proposition 3.10.

Corollary 3.12. If (W, S) is type Λ and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two braid equivalent links, then $d(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \Delta(\text{sig}(\boldsymbol{\alpha}), \text{sig}(\boldsymbol{\beta}))$.

Corollary 3.13. If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link of rank r, then $\text{diam}(\mathcal{B}(\boldsymbol{\alpha})) \leq r$.

To state the next proposition, we need another definition involving signature. If $\boldsymbol{\alpha}$ is a link in a Coxeter system of type Λ, we define

$$\overline{\text{sig}}_i(\boldsymbol{\alpha}) := \{\boldsymbol{x} \in [\boldsymbol{\alpha}] \mid \text{sig}_i(\boldsymbol{x}) = \text{sig}_i(\boldsymbol{\alpha})\}$$

to be the set of all reduced expressions that have the same generator in the center of the ith braid shadow as $\boldsymbol{\alpha}$. Note that if $\boldsymbol{\alpha}$ is a link of rank $r \geq 2$ and we choose $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ according to Proposition 2.17 with $[\![2r-3, 2r-1]\!], [\![2r-1, 2r+1]\!] \in \mathcal{S}(\boldsymbol{\sigma})$, then $\overline{\text{sig}}_r(\boldsymbol{\sigma}) = X_{\boldsymbol{\sigma}}$. The next proposition is another result from [3].

Proposition 3.14. If (W, S) is type Λ, $\boldsymbol{\alpha}$ is a link of rank at least one, and $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ is an edge in $\mathcal{B}(\boldsymbol{\alpha})$, then:

(a) There exists a unique i such that $\text{sig}_i(\boldsymbol{\alpha}) \neq \text{sig}_i(\boldsymbol{\beta})$ and $W_{\boldsymbol{\alpha\beta}} = \overline{\text{sig}}_i(\boldsymbol{\alpha})$; and

(b) $\{\boldsymbol{x}, \boldsymbol{y}\} \in F_{\boldsymbol{\alpha\beta}}$ if and only if $\{\boldsymbol{x}, \boldsymbol{y}\}$ is labeled by i.

That is, every semicube is uniquely determined by the generator appearing in the ith entry of the signature. Moreover, all edges in $F_{\boldsymbol{\alpha\beta}}$ correspond to the ith braid shadow and if two edges in $\mathcal{B}(\boldsymbol{\alpha})$ have the same label, then both are in the same F-class.

Example 3.15. Consider the links $\boldsymbol{\alpha} = \underline{434}132\underline{43\overline{4}}\overline{54}$ and $\boldsymbol{\beta} = \underline{434}1\overline{32\underline{3}}\underline{43}54$ in the Coxeter system of type D_5. Note that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related by a single braid move in the fourth braid shadow, and hence $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ is an edge in $\mathcal{B}(\boldsymbol{\alpha})$. The vertices in the subgraph highlighted in teal correspond to the reduced expressions in $\overline{\text{sig}}_4(\boldsymbol{\alpha})$ and the vertices in the subgraph highlighted in magenta correspond to the reduced expressions in $\overline{\text{sig}}_4(\boldsymbol{\beta})$. As expected from Proposition 3.14, the vertices in the teal subgraph correspond to $W_{\boldsymbol{\alpha\beta}}$, while the vertices in the magenta subgraph correspond to $W_{\boldsymbol{\beta\alpha}}$. Moreover, the black edges connecting $W_{\boldsymbol{\alpha\beta}}$ and $W_{\boldsymbol{\beta\alpha}}$ are the corresponding F-edges, all of which correspond to the same braid shadow, $[\![7, 9]\!]$. We have labeled a few vertices to aid the reader.

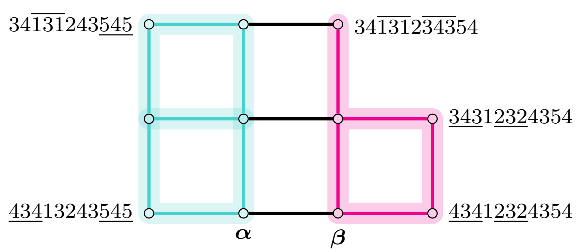


Figure 3.7: Example of semicubes and the corresponding F-edges for the braid graph of the link in Example 3.15.

Recall that according to Proposition 3.4, every braid graph is bipartite. It follows from Propositions 1.10 and 3.14 that every braid graph for a link in a Coxeter system of type Λ is a partial cube. This was the main result in [2]. In [3], Barnes provided an alternative proof using Propositions 1.10 and 3.14, and also verified a conjecture in [2] that the isometric dimension is equal to the rank of the link.

Proposition 3.16. If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link, then $\mathcal{B}(\boldsymbol{\alpha})$ is a partial cube with $\dim_I(\mathcal{B}(\boldsymbol{\alpha})) = \operatorname{rank}(\boldsymbol{\alpha})$.

The following was first conjectured in [3].

Conjecture 3.17. If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link, then $\operatorname{diam}(\mathcal{B}(\boldsymbol{\alpha})) = \operatorname{rank}(\boldsymbol{\alpha})$.

Example 3.18. Recall the braid class for $\boldsymbol{\beta}_1$ in the Coxeter system type D_4 from Examples 2.4(b) and 3.3(b). Notice that $\operatorname{rank}(\boldsymbol{\beta}_1) = 3$. Figure 3.8 depicts an embedding of $\mathcal{B}(\boldsymbol{\beta}_1)$ into a hypercube of dimension 3. It is clear that we cannot embed into a lower dimension hypercube, and so $\dim_I(\mathcal{B}(\boldsymbol{\beta}_1)) = 3$. This illustrates Proposition 3.16. Moreover, $\operatorname{diam}(\mathcal{B}(\boldsymbol{\beta}_1)) = 3$, which confirms Conjecture 3.17 in this example.

$$\boldsymbol{\beta}_1 = \underline{434}1\underline{232},\ \boldsymbol{\beta}_2 = \underline{343}1\underline{232},\ \boldsymbol{\beta}_3 = \underline{434}1\underline{323},\ \boldsymbol{\beta}_4 = \underline{34\overline{3}}\overline{1}\underline{\overline{3}23},\ \boldsymbol{\beta}_5 = 34\underline{1312}3.$$

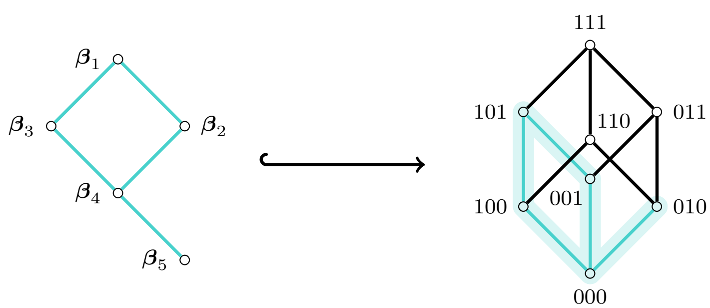


Figure 3.8: A braid graph as a partial cube with isometric dimension, rank, and diameter equal to 3 as described in Example 3.18.

The following is a consequence of Propositions 1.6, 2.8, 3.5, and 3.16 and also appears in [3].

Corollary 3.19. If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a reduced expression with link factorization $\boldsymbol{\alpha}_1 \mid \boldsymbol{\alpha}_2 \mid \cdots \mid \boldsymbol{\alpha}_k$, then $\mathcal{B}(\boldsymbol{\alpha})$ is a partial cube with

$$\dim_I(\mathcal{B}(\boldsymbol{\alpha})) = \sum_{i=1}^{k} \operatorname{rank}(\boldsymbol{\alpha}_i).$$

The next result follows immediately from Proposition 1.10 and Corollary 3.19. To our knowledge, this result does not appear elsewhere.

Corollary 3.20. If (W, S) is type Λ, $\boldsymbol{\alpha}$ is a link of rank r, and we choose $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ such that $[\![2r-3, 2r-1]\!], [\![2r-1, 2r+1]\!] \in \mathcal{S}(\boldsymbol{\sigma})$ according to Proposition 2.17, then $X_{\boldsymbol{\sigma}}$ and $Y_{\boldsymbol{\sigma}}$ are convex.

Chapter 4

New results on braid graphs

This chapter outlines several new results pertaining to braid graphs. To start, we need a definition utilized in the proofs of some of these results. Recall that every cycle in a Matsumoto graph is of even length, so as a result every cycle in a braid graph is of even length. For an even length cycle in a graph, we define *opposite edges* to be the pair of opposite edges when a cycle is interpreted as a regular polygon. Collectively, the next few results describe the local cycle-structure of a braid graph.

Theorem 4.1. If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link such that $\mathcal{B}(\boldsymbol{\alpha})$ contains a 4-cycle, then opposite edges in that 4-cycle correspond to the same braid move.

Proof. Without loss of generality, suppose that $\boldsymbol{\alpha}$ appears in a 4-cycle and let $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ be an edge in this cycle. Consider the semicubes $W_{\boldsymbol{\alpha\beta}}$ and $W_{\boldsymbol{\beta\alpha}}$ as visualized in Figure 1.7. By Proposition 3.14, the edge $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ and its opposite edge, must correspond to the same braid move since both edges are certainly in the same F-class. □

The next four results further investigate when a 4-cycle appears in a braid graph.

Theorem 4.2. Suppose (W, S) is type Λ and let $\boldsymbol{\alpha}$ be a link. Then $\boldsymbol{\beta} \in [\boldsymbol{\alpha}]$ has two non-overlapping braid shadows $[\![2i-1, 2i+1]\!]$ and $[\![2j-1, 2j+1]\!]$ if and only if $\mathcal{B}(\boldsymbol{\alpha})$ has a 4-cycle with the opposite edges labeled by i and j.

Proof. The forward implication is clear. Now, suppose that $\mathcal{B}(\boldsymbol{\alpha})$ has a 4-cycle. By Theorem 4.1, there exist braid shadows $[\![2i-1, 2i+1]\!]$ and $[\![2j-1, 2j+1]\!]$ such that the pairs of opposite edges are labeled with i and j. It follows that each reduced expression in this 4-cycle has at least two available braid moves, namely b^i and b^j. It remains to show that $[\![2i-1, 2i+1]\!]$ and $[\![2j-1, 2j+1]\!]$ are non-overlapping. Let $\boldsymbol{\beta} \in [\boldsymbol{\alpha}]$ be a reduced expression corresponding to a vertex on this 4-cycle so that $[\![2i-1, 2i+1]\!], [\![2j-1, 2j+1]\!] \in \mathcal{S}(\boldsymbol{\beta})$. For sake of contradiction, suppose that $[\![2i-1, 2i+1]\!]$ and $[\![2j-1, 2j+1]\!]$ are overlapping, and without loss of generality suppose that $2i+1 = 2j-1$, so that $i+1 = j$. Further, suppose that $\boldsymbol{\beta}_{[\![2i-1,2i+1]\!]} = sts$ and $\boldsymbol{\beta}_{[\![2i+1,2i+3]\!]} = sus$ where $m(s,t) = 3 = m(s,u)$ and $m(t,u) = 2$. Then

$b^i(\boldsymbol{\beta})_{[\![2i+1,2i+3]\!]} = tus$, so that $[\![2j-1,2j+1]\!] = [\![2i+1,2i+3]\!] \notin \mathcal{S}(b^i(\boldsymbol{\beta}))$. This is a contradiction since $b^i(\boldsymbol{\beta})$ lies on this 4-cycle. Therefore, the braid moves corresponding to the edges incident to each vertex are disjoint. □

Theorem 4.3. If (W,S) is type Λ and $\boldsymbol{\alpha}$ is a link such that $\mathcal{B}(\boldsymbol{\alpha})$ has a vertex $\boldsymbol{\lambda}$ of degree 3 or more, then two of the edges incident to $\boldsymbol{\lambda}$ are involved in a 4-cycle.

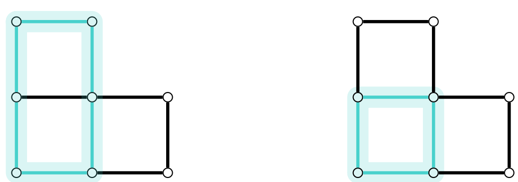
Proof. Let $\boldsymbol{\lambda}$ correspond to a vertex of degree 3 or more. Then there exists $[\![i-1,i+1]\!], [\![j-1,j+1]\!], [\![k-1,k+1]\!] \in \mathcal{S}(\boldsymbol{\lambda})$. It is clear that in any arrangement of these three braid moves, there are at least two that are disjoint, which yields a 4-cycle by Theorem 4.2. □

Note that the opposite edges in the 4-cycle in Theorem 4.3 correspond to the same braid move by Theorem 4.1. The next result is immediate from Theorem 4.3.

Corollary 4.4. If (W,S) is type Λ and $\boldsymbol{\alpha}$ is a link such that $\mathcal{B}(\boldsymbol{\alpha})$ is a tree, then $\mathcal{B}(\boldsymbol{\alpha})$ is a path.

In order to state the following results clearly, we first need a definition. A *primitive cycle*[1] in a graph G is a cycle that is an isometric subgraph of G. That is, a cycle is primitive if the distance in G between vertices on the cycle is the same as their distance along that cycle. This concept likely exists in graph theory, but we were unable to find a reference, so the phrase "primitive cycle" is of our own choosing.

Example 4.5. Figure 4.1(a) depicts a graph and a highlighted cycle that is not a primitive cycle while Figure 4.1(b) depicts the same graph with a highlighted cycle that is a primitive cycle.



(a) (b)

Figure 4.1: A non-primitive cycle and a primitive cycle.

The following theorem extends the result in Theorem 4.1 and is a stepping stone to Theorem 4.7.

Theorem 4.6. If (W,S) is type Λ and $\boldsymbol{\alpha}$ is a reduced expression, then the opposite edges of an even-length **new:**isometric**old:**primitive cycle in $\mathcal{B}(\boldsymbol{\alpha})$ correspond to the same braid move.

[1] *Note:* "Primitive cycle" is not the correct concept. Instead, we need "convex cycle", which is a cycle whose corresponding subgraph is convex. The appropriate changes have been indicated in blue versus red.

Proof. Let P be a **new:**isometric**old:**primitive cycle of even length in $\mathcal{B}(\boldsymbol{\alpha})$ and let $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ and $\{\boldsymbol{\alpha}', \boldsymbol{\beta}'\}$ be opposite edges of P. Without loss of generality, assume $d(\boldsymbol{\alpha}, \boldsymbol{\alpha}') < d(\boldsymbol{\alpha}, \boldsymbol{\beta}')$. Since P is of even length, there is a geodesic along P from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}'$ that passes through $\boldsymbol{\alpha}'$ and a geodesic (of the same length) from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}'$ that passes through $\boldsymbol{\beta}$. Now, suppose that the edge $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ is labeled with i. This means that the geodesic from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}'$ passing through $\boldsymbol{\beta}$ contains an edge labeled with i, and hence the geodesic from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}'$ passing through $\boldsymbol{\alpha}'$ must also contain an edge labeled with i by Proposition 3.10. For sake of contradiction, suppose that there exists an edge $\{\boldsymbol{\alpha}'', \boldsymbol{\beta}''\} \neq \{\boldsymbol{\alpha}', \boldsymbol{\beta}'\}$ along P with $\{\boldsymbol{\alpha}'', \boldsymbol{\beta}''\}$ labeled with i as depicted in Figure 4.2. Now consider the geodesic along P from $\boldsymbol{\beta}$ to $\boldsymbol{\alpha}'$ passing through $\boldsymbol{\alpha}$. Then we have two edges on this geodesic labeled by i, which contradicts Proposition 3.10. Therefore, the opposite edge of $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ is labeled with i. □

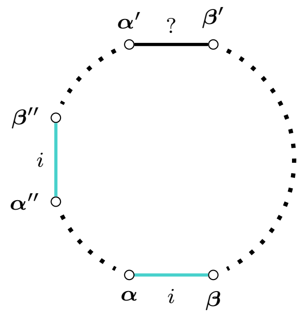


Figure 4.2: A **new:**convex**old:**primitive cycle that illustrates the contradiction given in the proof of Theorem 4.6.

Theorem 4.7. If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link, then every **new:**convex**old:**primitive cycle in $\mathcal{B}(\boldsymbol{\alpha})$ is of length 4.

Proof. Suppose that P is a **new:**convex**old:**primitive cycle on $\mathcal{B}(\boldsymbol{\alpha})$ of length greater than 4 and let $\boldsymbol{\alpha}$ be a link on P, where $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ and $\{\boldsymbol{\alpha}', \boldsymbol{\beta}'\}$ are opposite edges with $\boldsymbol{\alpha}'$ closer to $\boldsymbol{\alpha}$ than $\boldsymbol{\beta}$. It follows that there are two available braid moves, say b^i and b^j, in $\boldsymbol{\alpha}$, so that $[\![2i-1, 2i+1]\!], [\![2j-1, 2j+1]\!] \in \mathcal{S}(\boldsymbol{\alpha})$ and the edges of P incident to $\boldsymbol{\alpha}$ are labeled with i and j. Suppose that $b^j(\boldsymbol{\beta}) = \boldsymbol{\alpha}$. If $[\![2i-1, 2i+1]\!]$ and $[\![2j-1, 2j+1]\!]$ are disjoint, then the **new:**convex**old:**primitive cycle is of length 4 by Proposition 4.2. Otherwise, these braid shadows must overlap. Without loss of generality, assume that $2i+1 = 2j-1$, so that $i+1 = j$. Then, without loss of generality, $\boldsymbol{\alpha}$ is of the form:

$$\underbrace{\cdots \underset{2i-1}{\underline{t}}\ \underset{2i}{\underline{s}}\ \underset{2i+1}{\underline{t}}\ \underset{2i+2}{\underline{u}}\ \underset{2i+3}{\underline{t}} \cdots}_{\boldsymbol{\alpha}},$$

where $m(s,t) = 3 = m(t,u)$ and $m(s,u) = 2$. Then $b^i(\boldsymbol{\alpha})$ is of the form:

$$\underbrace{\cdots\underset{2i-1}{\underline{\ s\ }}\,\underset{2i}{\underline{\ t\ }}\,\underset{2i+1}{\underline{\ s\ }}\,\underset{2i+2}{\underline{\ u\ }}\,\underset{2i+3}{\underline{\ t\ }}\cdots}_{b^i(\boldsymbol{\alpha})}$$

It is important to note that there is no longer a braid move available in the $(i+1)$st shadow of $b^i(\boldsymbol{\alpha})$. Moreover, by Proposition 3.10, there is not another edge labeled with $i+1$ in a geodesic between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}'$ passing through $b^i(\boldsymbol{\alpha})$ and $\boldsymbol{\alpha}'$. But, this means that there will not be an available braid move in the $(i+1)$st shadow of $\boldsymbol{\alpha}'$. However, by Theorem 4.6, $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ and $\{\boldsymbol{\alpha}', \boldsymbol{\beta}'\}$ must both be labeled with $i+1$, which is a contradiction. Therefore, every **new:**convex**old:**primitive cycle in a braid graph $\mathcal{B}(\boldsymbol{\alpha})$ must be length 4. □

The next result verifies a conjecture proposed in [3]. This result together with Corollary 4.9 constitute the main result of this thesis.

Theorem 4.8. If (W,S) is type Λ and $\boldsymbol{\alpha}$ is a link, then $\mathcal{B}(\boldsymbol{\alpha})$ is median.

Proof. We will proceed by induction on rank. If $\text{rank}(\boldsymbol{\alpha}) = 0$, then $\mathcal{B}(\boldsymbol{\alpha})$ is a single vertex, which is clearly median. If $\text{rank}(\boldsymbol{\alpha}) = 1$, then $\mathcal{B}(\boldsymbol{\alpha})$ consists of two vertices connected by a single edge. Since this graph is a convex expansion of a single vertex, $\mathcal{B}(\boldsymbol{\alpha})$ is median by Mulder's Theorem (Proposition 1.20). This verifies the base cases.

Now, assume that for $r \geq 2$, every braid graph for a link of rank $r-1$ is median. Let $\boldsymbol{\alpha}$ be a link of rank r. According to Proposition 2.17, choose $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ such that $[\![2r-3, 2r-1]\!], [\![2r-1, 2r+1]\!] \in \mathcal{S}(\boldsymbol{\sigma})$. Explicitly, suppose $\boldsymbol{\sigma}_{[\![2r-3,2r+1]\!]} = tutst$ where $m(u,t) = 3 = m(s,t)$ and $m(u,s) = 2$. Consider $X_{\boldsymbol{\sigma}} = \{\boldsymbol{\beta} \in [\boldsymbol{\alpha}] \mid \text{sig}_r(\boldsymbol{\beta}) = \text{sig}_r(\boldsymbol{\sigma})\}$. We have that, for all $\boldsymbol{\beta} \in X_{\boldsymbol{\sigma}}, \boldsymbol{\beta}_{[\![2r,2r+1]\!]} = st$ by Proposition 2.13. It follows from Proposition 2.18 and the definition of $Y_{\boldsymbol{\sigma}}$ that for all $\boldsymbol{\gamma} \in Y_{\boldsymbol{\sigma}}, \boldsymbol{\gamma}_{[\![2r-1,2r+1]\!]} = sts$. By Proposition 2.18, $\hat{\boldsymbol{\sigma}}$ is a link of rank $r-1$ and every element of $[\hat{\boldsymbol{\sigma}}]$ is of the form $\hat{\boldsymbol{x}}$ where $\boldsymbol{x} \in X_{\boldsymbol{\sigma}}$. As a consequence of Corollary 3.8, $\mathcal{B}(\hat{\boldsymbol{\sigma}})$ is isomorphic to $\mathcal{B}(\boldsymbol{\alpha})[X_{\boldsymbol{\sigma}}]$. By induction, $\mathcal{B}(\hat{\boldsymbol{\sigma}})$ is median and as a result, $\mathcal{B}(\boldsymbol{\alpha})[X_{\boldsymbol{\sigma}}]$ is also median. Hence by Proposition 1.20, $\mathcal{B}(\boldsymbol{\alpha})[X_{\boldsymbol{\sigma}}]$ can be obtained from a series of convex expansions starting from a single vertex. Now, we must show that $\mathcal{B}(\boldsymbol{\alpha})$ can be obtained by doing a single convex expansion to $\mathcal{B}(\boldsymbol{\alpha})[X_{\boldsymbol{\alpha}}]$.

Define $C = \{\boldsymbol{\beta} \in X_{\boldsymbol{\sigma}} \mid \boldsymbol{\beta}_{[\![2r-1,2r+1]\!]} = tst\} \subseteq X_{\boldsymbol{\sigma}}$. Certainly, if $\boldsymbol{\beta} \in C$, then $[\![2r-1, 2r+1]\!] \in \mathcal{S}(\boldsymbol{\beta})$. We argue that C is convex. Let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in C$. By Corollary 3.20, $X_{\boldsymbol{\sigma}}$ is convex, and since $C \subseteq X_{\boldsymbol{\sigma}}$, it must be the case that every link on a geodesic between $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ must lie in $X_{\boldsymbol{\sigma}}$. From the definition of C and the convexity of $X_{\boldsymbol{\sigma}}$ $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ end in tst, and every link on a geodesic between $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ must end in st. Toward a contradiction, suppose that there exists a geodesic between $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ that includes a vertex $\boldsymbol{\kappa} \in X_{\boldsymbol{\sigma}} \setminus C$. Then it must be the case that $\boldsymbol{\kappa}_{[\![2r-2,2r+1]\!]} = tust$, where $m(t,u) = 3$ and $m(s,u) = 2$. Consider this geodesic between $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ passing through $\boldsymbol{\kappa}$. Since $\boldsymbol{\gamma}_1$ ends in tst, at least the braid move b^{r-1} must be performed to obtain $\boldsymbol{\kappa}$ from $\boldsymbol{\gamma}_1$. Likewise, b^{r-1} must be performed to obtain $\boldsymbol{\gamma}_2$ from $\boldsymbol{\kappa}$ since $\boldsymbol{\gamma}_2$ ends in tst while $\boldsymbol{\kappa}$ does not. Thus, there is a geodesic from $\boldsymbol{\gamma}_1$ to $\boldsymbol{\gamma}_2$ passing through $\boldsymbol{\kappa}$ that includes two edges labeled with $r-1$. This is prohibited by Proposition 3.10, a contradiction. Therefore, C is convex.

Now, consider $Y_{\boldsymbol{\sigma}} = \{\boldsymbol{\beta} \in [\boldsymbol{\alpha}] \mid \text{sig}_r(\boldsymbol{\beta}) \neq \text{sig}_r(\boldsymbol{\sigma})\}$. Recall that, for all $\boldsymbol{\beta} \in Y_{\boldsymbol{\sigma}}$, $[\![2r-1, 2r+1]\!] \in \mathcal{S}(\boldsymbol{\beta})$ and $\boldsymbol{\beta}_{[\![2r-1,2r+1]\!]} = sts$. Note that no link in $Y_{\boldsymbol{\sigma}}$ has $[\![2r-3, 2r-1]\!]$ as a braid shadow. Moreover, a braid move must be performed in the last braid shadow to pass from $X_{\boldsymbol{\sigma}}$ to $Y_{\boldsymbol{\sigma}}$ according to Proposition 2.18.

Since C is a finite set, we can let $C = \{\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_n\}$. If we perform a braid move in the last braid shadow of these elements, we obtain $b^r(C) = \{b^r(\boldsymbol{\gamma}_1), \dots, b^r(\boldsymbol{\gamma}_n)\}$. Define $\boldsymbol{\gamma}'_i := b^r(\boldsymbol{\gamma}_i)$ for each i. Since a braid move was performed in the last position, it must be the case that $\text{supp}_{[\![2r]\!]}(\boldsymbol{\gamma}'_i) \neq \text{supp}_{[\![2r]\!]}(\boldsymbol{\gamma}_i)$. By Proposition 2.18 and the definition of $Y_{\boldsymbol{\sigma}}$, $\{\boldsymbol{\gamma}'_1, \boldsymbol{\gamma}'_2, \cdots, \boldsymbol{\gamma}'_n\} = Y_{\boldsymbol{\sigma}}$. It follows that $\{\boldsymbol{\gamma}_i, \boldsymbol{\gamma}'_i\}$ is an edge in $\mathcal{B}(\boldsymbol{\alpha})$ labeled by r. Moreover, by Proposition 3.14, these are the only edges labeled with r in $\mathcal{B}(\boldsymbol{\alpha})$. Certainly, since those reduced expressions in $X_{\boldsymbol{\sigma}}$ that have an incident edge labeled with r are in C, the edges labeled r connect C with $Y_{\boldsymbol{\sigma}}$.

Let $\{\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_j\}$ be an edge in $\mathcal{B}(\boldsymbol{\alpha})[C]$ labeled with $k \neq r$. Note that the edges $\{\boldsymbol{\gamma}_i, \boldsymbol{\gamma}'_i\}$ and $\{\boldsymbol{\gamma}_j, \boldsymbol{\gamma}'_j\}$ are labeled with r. Certainly $[\![2k-1, 2k+1]\!]$ is disjoint from $[\![2r-1, 2r+1]\!]$ since $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are elements of C. It follows that $[\![2k-1, 2k+1]\!] \in \mathcal{S}(\boldsymbol{\gamma}'_i) \cap \mathcal{S}(\boldsymbol{\gamma}'_j)$. Hence $\{\boldsymbol{\gamma}'_i, \boldsymbol{\gamma}'_j\}$ is an edge labeled with k in $\mathcal{B}(\boldsymbol{\alpha})[Y_{\boldsymbol{\sigma}}]$. Similarly, if $\{\boldsymbol{\gamma}'_i, \boldsymbol{\gamma}'_j\}$ is an edge in $\mathcal{B}(\boldsymbol{\alpha})[Y_{\boldsymbol{\sigma}}]$, then $\{\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_j\}$ is an edge in $\mathcal{B}(\boldsymbol{\alpha})[C]$. Therefore, $\mathcal{B}(\boldsymbol{\alpha})[C] \cong \mathcal{B}(\boldsymbol{\alpha})[Y_{\boldsymbol{\sigma}}]$ via the bijection between C and $Y_{\boldsymbol{\sigma}}$ explicitly given by $b^r(\boldsymbol{\gamma}_i) \overset{b^r}{\mapsto} \boldsymbol{\gamma}'_i$.

It follows that $\mathcal{B}(\boldsymbol{\alpha})$ is obtained by a convex expansion from the median graph $\mathcal{B}(\boldsymbol{\alpha})[X_{\boldsymbol{\sigma}}]$, and hence $\mathcal{B}(\boldsymbol{\alpha})$ is median. □

The above theorem combined with Proposition 1.18 and 3.5 yields the following.

Corollary 4.9. If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a reduced expression, then $\mathcal{B}(\boldsymbol{\alpha})$ is median.

It turns out that the converse of the above corollary is not true. That is, not every median graph can be realized as a braid graph in a Coxeter system of type Λ.

Example 4.10. Consider the graph G in Figure 4.3. It turns out that G is a median graph and hence a partial cube by Proposition 1.16. In particular $\dim_I(G) = 4$. For sake of contradiction, suppose that $\boldsymbol{\alpha}$ is a link with braid graph G. Consistent with Theorem 4.1, we have colored opposite edges of each 4-cycle with the same color to indicate that they correspond to the same braid shadow. It follows that $\text{rank}(\boldsymbol{\alpha}) = 4$, and hence $\ell(\boldsymbol{\alpha}) = 9$. Then there exists $\boldsymbol{\varphi} \in [\boldsymbol{\alpha}]$ such that all 4 braid shadows occur. According to [2], $\boldsymbol{\varphi}$ is a so-called Fibonacci link. Moreover, in [2], the authors prove that the braid graph for a Fibonacci link must be a Fibonacci cube. It turns out that all Fibonacci cubes have a Fibonacci number of vertices, but G does not—a contradiction. Therefore, G is not the braid graph for a link in a type Λ Coxeter system.

Corollary 4.9 states that the braid graph of any reduced expression in a Coxeter system of type Λ is median. But perhaps braid graphs in all simply-laced Coxeter systems, even those that are not triangle free, are median as the next example suggests.

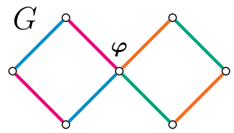


Figure 4.3: A graph that is median but does not arise as a braid graph in a Coxeter system of type Λ as described in Example 4.10.

Example 4.11. Recall that the Coxeter system of type $\widetilde{A}_2$ is not triangle free. Consider the reduced expression $\boldsymbol{\delta} = \underline{12\overline{1}}\overline{3\underline{1}}\underline{21}$ in type $\widetilde{A}_2$. The braid graph $\mathcal{B}(\boldsymbol{\delta})$ is shown in Figure 4.4. One may check that this graph is indeed median.

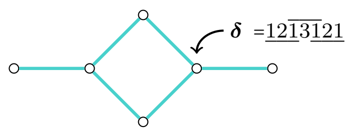


Figure 4.4: The braid graph of $\boldsymbol{\delta} = 1213121$ in the Coxeter system of type $\widetilde{A}_2$ as described in Example 4.11.

The next section of this chapter describes how we would find the median of three vertices in a braid graph. We need a few intermediate results and definitions to proceed. Before stating the next proposition, we must establish necessary notation.

The following definition and subsequent proposition are from [3]. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are braid equivalent links, we define

$$\overline{\mathrm{sig}}(\boldsymbol{\alpha},\boldsymbol{\beta}) := \{\boldsymbol{x} \in [\boldsymbol{\alpha}] \mid \mathrm{sig}_i(\boldsymbol{x}) = \mathrm{sig}_i(\boldsymbol{\alpha}) \text{ whenever } \mathrm{sig}_i(\boldsymbol{\alpha}) = \mathrm{sig}_i(\boldsymbol{\beta})\}.$$

That is, $\overline{\mathrm{sig}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is the set of reduced expressions whose signature agrees with the common signatures of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

The next proposition states that the reduced expressions on any geodesic between a pair of braid equivalent links share the common signature values of the pair.

Proposition 4.12. If (W,S) is of type Λ and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are braid equivalent links, then $I(\boldsymbol{\alpha},\boldsymbol{\beta}) = \overline{\mathrm{sig}}(\boldsymbol{\alpha},\boldsymbol{\beta})$.

Let $\boldsymbol{\alpha},\boldsymbol{\beta}$, and $\boldsymbol{\sigma}$ be braid equivalent links of rank $r \geq 1$. As in [3], we define the ith *majority* of $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma}$ via

$$\mathrm{maj}_i(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma}) := \begin{cases} \mathrm{sig}_i(\boldsymbol{\alpha}), & \text{if } \mathrm{sig}_i(\boldsymbol{\alpha}) = \mathrm{sig}_i(\boldsymbol{\beta}) \text{ or } \mathrm{sig}_i(\boldsymbol{\alpha}) = \mathrm{sig}_i(\boldsymbol{\sigma}) \\ \mathrm{sig}_i(\boldsymbol{\beta}), & \text{otherwise.} \end{cases}$$

That is, when at least two of the generators in the ith position of a triple of braid equivalent reduced expressions agree, we record that generator. For braid equivalent reduced expressions $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}$ of rank $r \geq 1$, we define the *majority* of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}$ via

$$\operatorname{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}) \coloneqq (\operatorname{maj}_1(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}), \ldots, \operatorname{maj}_r(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})).$$

The majority of three reduced expressions results in an ordered list of generators. The next result from [3] states that the intersection of the intervals between the three pairs among three braid equivalent links is the set of reduced expressions in their braid class whose ith signature is the ith majority.

Proposition 4.13. If (W, S) is type Λ and $\boldsymbol{\alpha}, \boldsymbol{\beta}$, and $\boldsymbol{\sigma}$ are braid equivalent links, then

$$I(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cap I(\boldsymbol{\beta}, \boldsymbol{\sigma}) \cap I(\boldsymbol{\alpha}, \boldsymbol{\sigma}) = \{\boldsymbol{x} \in [\boldsymbol{\alpha}] \mid \operatorname{sig}(\boldsymbol{x}) = \operatorname{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})\}.$$

Proposition 2.15 implies that the set in Proposition 4.13 is either empty or consists of a single element. But Corollary 4.9 tells us that the intersection must have cardinality one. The following result confirms a conjecture from [3], which connects the concepts of median and majority.

Proposition 4.14. If (W, S) is type Λ and $\boldsymbol{\alpha}, \boldsymbol{\beta}$, and $\boldsymbol{\sigma}$ are braid equivalent links, then $\operatorname{med}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})$ is the unique $\boldsymbol{x}$ satisfying $\operatorname{sig}(\boldsymbol{x}) = \operatorname{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})$.

Proof. It follows from Proposition 4.12 that the unique $\boldsymbol{x}$ satisfying $\operatorname{sig}(\boldsymbol{x}) = \operatorname{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})$ is in $I(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cap I(\boldsymbol{\alpha}, \boldsymbol{\sigma}) \cap I(\boldsymbol{\beta}, \boldsymbol{\sigma})$. By Proposition 2.15 and Theorem 4.8, there is a unique link in this intersection and hence $\boldsymbol{x}$ must be equal to $\operatorname{med}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})$. □

Example 4.15. Consider the braid equivalent links $\boldsymbol{\alpha} = 34\underline{131}\mathbf{2}\underline{3\mathbf{4}3}\mathbf{5}4$, $\boldsymbol{\beta} = \overline{4\mathbf{34}}\mathbf{1}\overline{232}\mathbf{4}3\mathbf{5}4$, and $\boldsymbol{\sigma} = \overline{4\mathbf{34}}\mathbf{1}3\mathbf{2}43\overline{545}$ in the Coxeter system of type D_5. We have highlighted the signatures found in $\operatorname{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})$ in orange. We see that $\operatorname{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}) = (3, 1, 2, 4, 5)$, which corresponds to the signature of $\boldsymbol{x} = \overline{4\mathbf{34}}\mathbf{1}\underline{3\mathbf{2}3}\overline{\mathbf{4}3}\mathbf{5}4$ in $[\boldsymbol{\alpha}]$. The corresponding braid graph, $\mathcal{B}(\boldsymbol{\alpha})$ with the reduced expression $\boldsymbol{x}$ satisfying $\operatorname{sig}(\boldsymbol{x}) = \operatorname{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})$ is depicted in Figure 4.5.

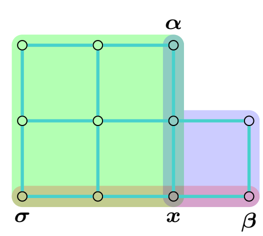


Figure 4.5: Example of median computation for the braid graph discussed in Example 4.15.

Chapter 5

Braid graphs as partially ordered sets

In this penultimate chapter, we introduce the necessary terminology to outline future work in classifying braid graphs. Most of this chapter mimics the development outlined in Chapter 4 of [3].

A *partially ordered set*, which is more commonly known as a *poset*, is a pair $(P, \leq)$, where P is a set and $\leq$ is a relation imposed on this set that is reflexive, antisymmetric, and transitive. For $x, y \in P$, we say x *covers* y, denoted $x \lessdot y$, if $x < y$ and there is no element $z \in P$ such that $x < z < y$. A Hasse diagram is a graphical representation of a poset $(P, \leq)$, where vertices are elements of P, x and y are connected by an edge if $x \lessdot y$, and there is an implied upward orientation, i.e., "smaller" elements are lower in the Hasse diagram.

A poset is said to be *ranked* if there exists a function $\rho : P \to \mathbb{N} \cup \{0\}$ such that $\rho(x) = 0$ if x is a minimal element, and $\rho(y) = \rho(x) + 1$ if $x \lessdot y$. A *lattice* is a special poset in which every pair of elements has a greatest lower bound, referred to as the *meet*, and a least upper bound, called the *join*, in the poset. If P is a lattice, we denote the meet of two elements x and y as $x \wedge y$ and the join as $x \vee y$. A *distributive lattice* is a particular kind of lattice in which the following holds for all $x, y, z \in P$:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \text{ and } x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

To show that P is a distributive lattice, it is sufficient to verify just one of the above identities.

Example 5.1. Consider the Hasse diagrams for the posets given in Figure 5.1. One can check that the poset in Figure 5.1(a) is not a lattice, and thus is not a distributive lattice. In Figure 5.1(b), we see that

$$b \wedge (c \vee d) = b \wedge a = b \neq e = e \vee e = (b \wedge c) \vee (b \wedge d).$$

Hence, the poset in Figure 5.1(b) is not a distributive lattice. On the other hand, one can verify that the poset in Figure 5.1(c) is a distributive lattice.

The next result comes from [6].

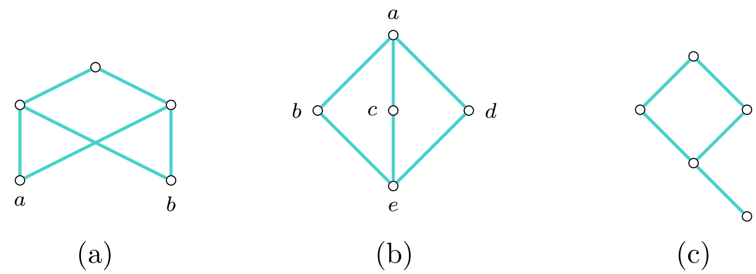


Figure 5.1: Examples of Hasse diagrams for posets.

Proposition 5.2. The underlying graph of the Hasse diagram for a finite distributive lattice is median.

Example 5.3. By Proposition 5.2, the underlying graph for the Hasse diagram of the poset given in Figure 5.1(c) is median, since it is a distributive lattice. On the other hand, the underlying graph for the poset in Figure 5.1(b) is median. However, it is not a distributive lattice, which shows the converse to Proposition 5.2 is false.

An extension of Proposition 5.2 from [1] follows.

Proposition 5.4. A graph G is the underlying graph of the Hasse diagram of a distributive lattice if and only if G is median and there exist two vertices μ and γ such that every vertex in G lies on a geodesic joining μ and γ.

We now mimic the outline in [3] describing how one would construct a poset whose Hasse diagram has $\mathcal{B}(\boldsymbol{\alpha})$ as its underlying graph. Suppose (W, S) is a Coxeter system of type Λ and let $\boldsymbol{\alpha}$ be a link of rank $r \geq 1$. Identify a pair of vertices $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ of $\mathcal{B}(\boldsymbol{\alpha})$ such that $d(\boldsymbol{\mu}, \boldsymbol{\gamma}) = \text{diam}(\mathcal{B}(\boldsymbol{\alpha}))$. That is, $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are diametrical. By Corollary 3.12, $d(\boldsymbol{\mu}, \boldsymbol{\gamma}) = \Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\gamma}))$, so $\text{diam}(\mathcal{B}(\boldsymbol{\alpha})) = \Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\gamma}))$, and by Corollary 3.13, $\text{diam}(\mathcal{B}(\boldsymbol{\alpha})) \leq r$. Choose $\boldsymbol{\mu}$ to be the designated smallest vertex and define $\boldsymbol{\beta} \lessdot \boldsymbol{\sigma}$ if there exists a unique i such that $\text{sig}_i(\boldsymbol{\beta}) \neq \text{sig}_i(\boldsymbol{\sigma})$ and $\Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\beta})) + 1 = \Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\sigma}))$. We then naturally have $([\boldsymbol{\alpha}], \leq)$ as the partial order induced by these covering relations. Note that the poset depends on the choice of $\boldsymbol{\mu}$. A different choice of $\boldsymbol{\mu}$ would yield a different poset. We will refer to both the poset and the Hasse diagram as $\mathcal{P}(\boldsymbol{\mu})$.

The next result from [3] states that if we designate a smallest vertex $\boldsymbol{\mu} \in [\boldsymbol{\alpha}]$ for a link $\boldsymbol{\alpha}$, then $\mathcal{B}(\boldsymbol{\alpha})$ is the underlying Hasse diagram of $\mathcal{P}(\boldsymbol{\mu})$ and this poset is ranked by the change in signature relative to $\boldsymbol{\mu}$.

Proposition 5.5. Suppose (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link. If $\boldsymbol{\mu} \in [\boldsymbol{\alpha}]$ is the designated smallest vertex, then $\mathcal{P}(\boldsymbol{\mu})$ is ranked by $\Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\beta}))$ for $\boldsymbol{\beta}$ in $[\boldsymbol{\alpha}]$. Moreover, $\mathcal{B}(\boldsymbol{\alpha})$ is the underlying graph for the Hasse diagram of $\mathcal{P}(\boldsymbol{\mu})$.

Example 5.6. Consider the link $\boldsymbol{\alpha} = 343132343$ with $\text{rank}(\boldsymbol{\alpha}) = 4$ in the Coxeter system of type D_4. The braid graph $\mathcal{B}(\boldsymbol{\alpha})$ is depicted in Figure 5.2. One can see that $\text{diam}(\mathcal{B}(\boldsymbol{\alpha})) = 4$

with unique diametrical pair 341312434 and 434123243. If we take $\boldsymbol{\mu} = 341312434$ to be the designated smallest vertex, then the graph given in Figure 5.2 is also the underlying graph of the Hasse diagram for $\mathcal{P}(\boldsymbol{\mu})$. We have highlighted the centers of each reduced expression in orange as they differ from $\boldsymbol{\mu}$ to indicate the rank of each reduced expression in the poset. Notice that $\mathcal{P}(\boldsymbol{\mu})$ is also a distributive lattice according to Proposition 5.4. Note that we may also have chosen $\boldsymbol{\mu}$ to be 434123243, which would have given the upside-down version of the graph in Figure 5.2.

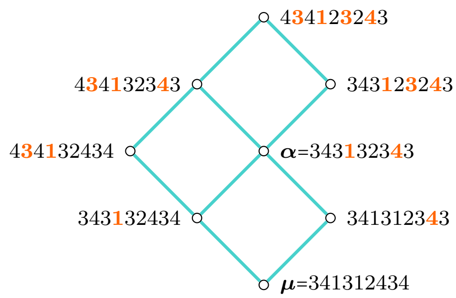


Figure 5.2: The Hasse diagram for $\mathcal{P}(\boldsymbol{\mu})$ in Example 5.6

The following conjecture was given in [3].

Conjecture 5.7. If (W,S) is type Λ and $\boldsymbol{\alpha}$ is a link, then $\mathcal{B}(\boldsymbol{\alpha})$ is the underlying graph for the Hasse diagram of a distributive lattice.

If the previous conjecture holds, then every braid graph in Coxeter systems of type Λ is the underlying graph for the Hasse diagram of a distributive lattice. This is because every braid graph is the box product of the braid graphs of the corresponding link factors of the reduced expression. We provide additional conjectures that may be useful in an attempt to prove Conjecture 5.7. The following conjecture also appears in [3].

Conjecture 5.8. Suppose (W,S) is type Λ and $\boldsymbol{\alpha}$ is a link of rank at least one and choose $\boldsymbol{\mu}$ to be the designated vertex of $\mathcal{P}(\boldsymbol{\mu})$. If $\mathrm{sig}_i(\boldsymbol{\alpha}) = \mathrm{sig}_i(\boldsymbol{\mu})$, then there exists $\boldsymbol{\beta} \in [\boldsymbol{\alpha}]$ such that $\boldsymbol{\alpha} \lessdot \boldsymbol{\beta}$.

The previous conjecture states that we can always "go up" in the poset from an element that has at least one signature entry that agrees with the signature of $\boldsymbol{\mu}$. If this conjecture is true, we would know that if we have a diametrical pair $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ with $\boldsymbol{\mu}$ the designated minimum, then $d(\boldsymbol{\mu},\boldsymbol{\gamma}) = \mathrm{diam}(\mathcal{B}(\boldsymbol{\alpha}))$, and $\boldsymbol{\gamma}$ would be the unique maximum element of $\mathcal{P}(\boldsymbol{\mu})$. In particular, we would have $\mathrm{diam}(\mathcal{B}(\boldsymbol{\alpha})) = \Delta(\mathrm{sig}(\boldsymbol{\mu}), \mathrm{sig}(\boldsymbol{\gamma})) = \mathrm{rank}(\boldsymbol{\alpha})$. This result would settle Conjecture 3.17. In addition, if Conjecture 5.8 is true, then Proposition 5.4 would imply Conjecture 5.7. The next conjecture is more of an interesting observation as opposed to a useful step in verifying Conjecture 5.7.

Conjecture 5.9. If (W,S) is type Λ and $\boldsymbol{\alpha}$ is a link of rank at least one, then there exists a unique pair $\boldsymbol{\mu},\boldsymbol{\gamma} \in [\boldsymbol{\alpha}]$ such that $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are diametrical.

The conjecture above is not true for arbitrary reduced expressions. For example, consider the braid graph given in Figure 3.2(a), which is the braid graph of a reduced expression that is not a link. Observe that there are two choices for pairs of diametrical vertices.

Now we will describe an algorithm, first given in [3], to find the potential meet of two braid equivalent reduced expressions in the poset $\mathcal{P}(\boldsymbol{\mu})$. Let (W,S) be a Coxeter system of type Λ and $\boldsymbol{\alpha}$ be a link of rank $r \geq 1$. Choose a diametrical pair $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ and consider $\mathcal{P}(\boldsymbol{\mu})$. Let $\boldsymbol{\beta},\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$. To find $\boldsymbol{\beta} \wedge \boldsymbol{\sigma}$, first identify all positions i such that $\mathrm{sig}_i(\boldsymbol{\beta}) = \mathrm{sig}_i(\boldsymbol{\sigma})$. These positions will not change. Next identify all positions with $\mathrm{sig}_i(\boldsymbol{\beta}) \neq \mathrm{sig}_i(\boldsymbol{\sigma})$. Then either $\mathrm{sig}_i(\boldsymbol{\beta}) = \mathrm{sig}_i(\boldsymbol{\mu})$ or $\mathrm{sig}_i(\boldsymbol{\sigma}) = \mathrm{sig}_i(\boldsymbol{\mu})$, so choose the generator to match $\mathrm{sig}_i(\boldsymbol{\mu})$. Combining the positions in the signature that were common between $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ with the chosen generators matching $\mathrm{sig}(\boldsymbol{\mu})$, the result will be $\boldsymbol{\beta} \wedge \boldsymbol{\sigma}$. However, we do not know whether the expression given by $\boldsymbol{\beta} \wedge \boldsymbol{\sigma}$ is actually in $[\boldsymbol{\alpha}]$.

Similarly, to find $\boldsymbol{\beta} \vee \boldsymbol{\sigma}$, we start by identifying all positions i such that $\mathrm{sig}_i(\boldsymbol{\beta}) = \mathrm{sig}_i(\boldsymbol{\sigma})$. Again, these positions will not change. Next identify all positions such that $\mathrm{sig}_i(\boldsymbol{\beta}) \neq \mathrm{sig}_i(\boldsymbol{\sigma})$. Then either $\mathrm{sig}_i(\boldsymbol{\beta}) = \mathrm{sig}_i(\boldsymbol{\gamma})$ in that position or $\mathrm{sig}_i(\boldsymbol{\sigma}) = \mathrm{sig}_i(\boldsymbol{\gamma})$, so choose the generator to match $\mathrm{sig}_i(\boldsymbol{\gamma})$. Combining the positions in the signature that were common between $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ with the chosen generators matching $\mathrm{sig}(\boldsymbol{\gamma})$, we obtain $\boldsymbol{\beta} \vee \boldsymbol{\sigma}$. Again, it remains to argue that $\boldsymbol{\beta} \vee \boldsymbol{\sigma}$ corresponds to a reduced expression in $[\boldsymbol{\alpha}]$.

Example 5.10. Recall the poset $\mathcal{P}(\boldsymbol{\mu})$ introduced in Example 5.6. Consider the braid equivalent links $\boldsymbol{\alpha} = 343132343$ and $\boldsymbol{\beta} = 434132434$. We will follow the algorithms described above to find $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \vee \boldsymbol{\beta}$. We have highlighted the signatures of each reduced expression as they differ from $\boldsymbol{\mu}$ in orange in Figure 5.3 to aid the reader.

To find $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$, we see that $\mathrm{sig}_2(\boldsymbol{\alpha}) = \mathrm{sig}_2(\boldsymbol{\beta}) = 1$ and $\mathrm{sig}_3(\boldsymbol{\alpha}) = \mathrm{sig}_3(\boldsymbol{\beta}) = 2$, so these signature positions will stay the same in $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$. Since $\mathrm{sig}_i(\boldsymbol{\alpha}) \neq \mathrm{sig}_i(\boldsymbol{\beta})$ for $i \in \{1,4,5\}$, we select the generators in these positions to match $\mathrm{sig}_i(\boldsymbol{\mu})$. Then $\mathrm{sig}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = (4,1,2,3)$, so we have $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = 343132434$, which does occur in $[\boldsymbol{\alpha}]$ and is labeled in Figure 5.3.

Similarly, we will compute $\boldsymbol{\alpha} \vee \boldsymbol{\beta}$. From above, we know that $\mathrm{sig}_2(\boldsymbol{\alpha}) = \mathrm{sig}_2(\boldsymbol{\beta}) = 1$ and $\mathrm{sig}_3(\boldsymbol{\alpha}) = \mathrm{sig}_3(\boldsymbol{\beta}) = 2$ so we will keep these signature positions the same. Since $\mathrm{sig}_i(\boldsymbol{\alpha}) \neq \mathrm{sig}_i(\boldsymbol{\beta})$ for $i \in \{1,4,5\}$, we choose the the generators in these positions now to match $\mathrm{sig}_i(\boldsymbol{\gamma})$. Then $\mathrm{sig}(\boldsymbol{\alpha} \vee \boldsymbol{\beta}) = (3,1,2,4)$, so we have $\boldsymbol{\alpha} \vee \boldsymbol{\beta} = 434132343$.

Conjecture 5.11. Suppose (W,S) is type Λ and $\boldsymbol{\alpha}$ is a link of rank $r \geq 1$. If $\boldsymbol{\mu} \in [\boldsymbol{\alpha}]$ is the designated smallest vertex, and $[\![2i-1,2i+1]\!] \in \mathcal{S}(\boldsymbol{\mu})$, then $[\![2i+1,2i+3]\!],[\![2i-3,2i-1]\!] \notin \mathcal{S}(\boldsymbol{\mu})$. That is, we claim that the designated minimum vertex has no overlapping braid shadows.

The next conjecture states that if two reduced expressions in $\mathcal{P}(\boldsymbol{\mu})$ originate from a common reduced expression of rank one or less, then the edges corresponding to the covering relations are labeled with disjoint braid shadows.

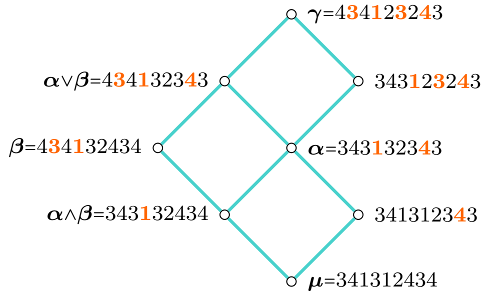


Figure 5.3: The Hasse diagram for $\mathcal{P}(\boldsymbol{\mu})$ in Examples 5.6 and 5.10.

Conjecture 5.12. Suppose (W, S) is type Λ and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are braid equivalent links of rank $r \geq 1$ and let $\boldsymbol{\mu} \in [\boldsymbol{\alpha}]$ be the designated smallest vertex. Further, suppose that $\text{rank}(\boldsymbol{\alpha}) = \text{rank}(\boldsymbol{\beta})$ and there exists $\boldsymbol{x} \in [\boldsymbol{\alpha}]$ such that $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \boldsymbol{x}$ with $\text{rank}(\boldsymbol{x}) = \text{rank}(\boldsymbol{\alpha}) - 1$. Then there exists two braid moves, b^i and b^j, with $b^i(\boldsymbol{\alpha}) = \boldsymbol{x}$ and $b^j(\boldsymbol{\beta}) = \boldsymbol{x}$ such that b^i and b^j commute (i.e., the corresponding braid shadows are disjoint).

Note that Conjecture 5.11 is claiming that as a special case, all pairs of reduced expressions of rank one in $\mathcal{P}(\boldsymbol{\mu})$ satisfy Conjecture 5.12. Figure 5.4 illustrates the location of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in the Hasse diagram for $\mathcal{P}(\boldsymbol{\mu})$ as described in the previous conjecture. Note that this figure only illustrates a portion of $\mathcal{P}(\mu)$.

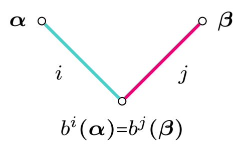


Figure 5.4: An illustration of the scenario described in Conjecture 5.12.

If Conjecture 5.12 holds, then Theorem 4.2 would iteratively imply that $\mathcal{P}(\boldsymbol{\mu})$ has a unique maximum. Then Proposition 5.4 would imply Conjecture 5.7.

Chapter 6

Conclusion

In Chapter 1 we provide an overview of necessary terminology and results regarding simple connected graphs. In particular, we discuss partial cubes and median graphs.

In Chapter 2, using results both from [2] and [3], we introduced Coxeter systems and many concepts related to braid classes. Properties of reduced expressions were discussed along with the notions of braid shadow and link. Using the fact that every reduced expression has a unique factorization in terms of links we focused the remainder of the thesis primarily on links, extending any pertinent results to reduced expressions accordingly. The goal of Chapter 2 is to summarize all of the pertinent information from [2] and [3] regarding the architecture of braid classes in Coxeter systems of type Λ.

Continuing to summarize the results in [2] and [3], we begin Chapter 3 by introducing the notion of a braid graph for a class of braid equivalent reduced expressions. The fact that every reduced expression has a unique factorization in terms of links as described in Chapter 2 implies that the braid graph of any reduced expression is equivalent to the box product of the braid graphs of its link factors. The rest of this chapter focuses on results about links from both [2] and [3], which provide insight into the structure of braid graphs. This allows us to prove the new results about braid graphs in Chapter 4.

Chapter 4 contains several new results pertaining to Coxeter systems of type Λ. Collectively, the first several results describe the local cycle-structure in braid graphs. Theorems 4.1 and 4.2 state that the opposite edges in any 4-cycle in a braid graph are labeled with the same braid shadow while the adjacent edges in the 4-cycle correspond to disjoint braid shadows. Corollary 4.4 tells us that any braid graph without a cycle must be a path. The upshot of Theorem 4.7 is that every primitive cycle (i.e., a cycle that is an isometric subgraph) is actually a 4-cycle, and hence has the structure described in Theorems 4.1 and 4.2. Theorem 4.8 constitutes our main result and tells us that every braid graph for a link in a Coxeter system of type Λ is median. Our approach is to apply Mulder's Theorem (Proposition 1.20), which states that every median graph can be obtained from a sequence of convex expansions starting from a single vertex. We utilize induction on rank with the lynchpin being Propositions 2.18 and 3.8. It follows that every braid graph in a Coxeter system of type Λ is median (Corollary 4.9), since every reduced expression has a unique link factorization and the box product of median graphs is median (Proposition 1.18). This is a strengthening of the main result in [2], which states that every braid graph in type Λ Coxeter systems is a partial cube since median graphs are partial cubes (Proposition 1.16). In Example 4.10, we show that not every median graph gives rise to a braid graph in a type Λ Coxeter system. It remains to understand which median graphs correspond to braid graphs. We conclude Chapter 4 with a characterization of the median of any three vertices in a braid graph in terms of signature.

Chapter 5 kicks off by recalling the concepts of posets and distributive lattices. Next, we summarize the construction in [3] of braid graphs as a Hasse diagrams for posets ranked by change in signature relative to the designated minimum vertex. Next, as in [3] we conjecture that every braid graph for a link is a distributive lattice. We conclude Chapter 5 with a few other conjectures, that if proved, would help show that every braid graph for a link is a distributive lattice.

We now summarize a list of open questions concerning braid graphs in Coxeter systems of type Λ that appeared throughout this thesis. For each conjecture below, assume (W, S) is a Coxeter system of type Λ.

- Conjecture 3.17: If $\boldsymbol{\alpha}$ is a link, then $\operatorname{diam}(\mathcal{B}(\boldsymbol{\alpha})) = \operatorname{rank}(\boldsymbol{\alpha})$. If true, it follows that that if $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 \mid \cdots \mid \boldsymbol{\alpha}_k$ is link factorization, then

$$\operatorname{diam}(\mathcal{B}(\boldsymbol{\alpha})) = \sum_{i=1}^{k} \operatorname{rank}(\boldsymbol{\alpha}_i).$$

- Conjecture 5.7: If $\boldsymbol{\alpha}$ is a link, then $\mathcal{B}(\boldsymbol{\alpha})$ is the underlying graph for Hasse diagram for distributive lattice.
- Conjecture 5.8: Let $\boldsymbol{\alpha}$ be a link of rank $r \geq 1$. Choose $\boldsymbol{\mu}$ to be the designated vertex of $\mathcal{P}(\boldsymbol{\mu})$. If $\operatorname{sig}_i(\boldsymbol{\alpha}) = \operatorname{sig}_i(\boldsymbol{\mu})$, then there exists $\boldsymbol{\beta} \in [\boldsymbol{\alpha}]$ such that $\boldsymbol{\alpha} \lessdot \boldsymbol{\beta}$.
- Conjecture 5.9: If $\boldsymbol{\alpha}$ is a link of rank at least one, then there exists a unique diametrical pair $\boldsymbol{\gamma}, \boldsymbol{\mu} \in [\boldsymbol{\alpha}]$.
- Conjecture 5.11: Let $\boldsymbol{\alpha}$ be a link of rank at least one. If $\boldsymbol{\mu} \in [\boldsymbol{\alpha}]$ is the designated smallest vertex, and $[\![2i-1, 2i+1]\!] \in \mathcal{S}(\boldsymbol{\mu})$, then $[\![2i+1, 2i+3]\!], [\![2i-3, 2i-1]\!] \notin \mathcal{S}(\boldsymbol{\mu})$. That is, we claim that the designated minimum vertex has no overlapping braid shadows.
- Conjecture 5.12: Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be braid equivalent links of rank at least one and let $\boldsymbol{\mu} \in [\boldsymbol{\alpha}]$ be the designated smallest vertex. Further, suppose that $\operatorname{rank}(\boldsymbol{\alpha}) = \operatorname{rank}(\boldsymbol{\beta})$ and there exists $\boldsymbol{x} \in [\boldsymbol{\alpha}]$ such that $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \boldsymbol{x}$ with $\operatorname{rank}(\boldsymbol{x}) = \operatorname{rank}(\boldsymbol{\alpha}) - 1$. Then there exists two braid moves, b^i and b^j, with $b^i(\boldsymbol{\alpha}) = \boldsymbol{x}$ and $b^j(\boldsymbol{\beta}) = \boldsymbol{x}$ such that b^i and b^j commute (i.e., the corresponding braid shadows are disjoint).

We conclude with a couple more general open problems.

- As suggested in Example 4.11, perhaps braid graphs in simply-laced yet not triangle-free Coxeter systems are still median. Nearly all of the results in [2] and this thesis fundamentally rely on the Coxeter systems being triangle free. Can we generalize to overcome the 3-cycle obstruction?

- Can we generalize the notion of braid shadow and link to account for arbitrary exponents $m(s, t)$? We conjecture that nearly all of the known results with be straight forward to generalize in the case when all braid moves are of odd length. However, generalizing to even length braid moves will require substantially more retooling.

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