EXPLORATION OF THE TYPE $\widetilde{C}$ TEMPERLEY–LIEB ALGEBRA

By Kevin Salmon

A Thesis

Submitted in Partial Fulfillment

of the Requirements for the Degree of

Master of Science

in Mathematics

Northern Arizona University

May 2016

Approved:

Dana C. Ernst, Ph.D., Chair

Michael Falk, Ph.D.

Nándor Sieben, Ph.D.

ABSTRACT

EXPLORATION OF THE TYPE $\widetilde{C}$ TEMPERLEY–LIEB ALGEBRA

Kevin Salmon

Given the Hecke algebra corresponding to an arbitrary Coxeter system of type Γ, there is a basis of particular interest, called the canonical basis, that has some remarkable properties but is computationally difficult to work with. The change of basis matrix between the defining basis of the Hecke algebra and the canonical basis is determined by a set of polynomials, called the Kazhdan–Lusztig polynomials. One crux to computing these polynomials is determining the so-called μ-values, which are the coefficients of the highest possible degree terms of the polynomials. In this thesis, we study a quotient of the Hecke algebra of type affine C, a type of generalized Temperley–Lieb algebra, which provides a combinatorially tractable model for Kazhdan–Lusztig theory. In particular, we obtained several original results concerning the computation of μ-values and products of canonical basis elements involving fully commutative elements of Coxeter groups of type affine C. Moreover, we construct a diagram algebra that mirrors these results and which we believe is a faithful representation of the corresponding Temperley–Lieb algebra.

Contents

List of Figures

Chapter 1

Preliminaries

1.1 Introduction

The (type A) Temperley–Lieb algebra $\mathrm{TL}(A)$, invented by Temperley and Lieb in 1971, is a finite dimensional associative algebra that arose in statistical mechanics [26]. Kauffman and Penrose showed that this algebra can be realized as a particular diagram algebra, which is a type of associative algebra with a basis determined by certain diagrams, where multiplication is given by applying local combinatorial rules to the diagrams [17, 21]. In 1987, V. Jones showed that $\mathrm{TL}(A)$ occurs naturally as a quotient of the type A Hecke algebra, $\mathcal{H}(A)$, whose underlying group is the symmetric group [16]. Jones introduced a Markov trace on $\mathcal{H}(A)$ that is degenerate (the trace is the matrix trace of a transfer matrix algebra), but its radical is an ideal of $\mathcal{H}(A)$, and so we obtain a generically nondegenerate trace on the quotient algebra. This quotient algebra is isomorphic to $\mathrm{TL}(A)$.

Eventually, this realization of the Temperley–Lieb algebra as a Hecke algebra quotient was generalized by Graham to the case of an arbitrary Coxeter graph Γ, which we denote by $\mathrm{TL}(\Gamma)$ [6]. Each $\mathrm{TL}(\Gamma)$ has several bases indexed by the so-called fully commutative elements (in the sense of Stembridge [25]). The algebra $\mathrm{TL}(\Gamma)$ provides a combinatorially tractable model for Kazhdan–Lusztig theory.

In a series of papers, Green constructed faithful diagrammatic representations of $\mathrm{TL}(\Gamma)$, where Γ is of type B, D, or H [7]. Martin and Saleur introduced a diagram calculus for the generalized Temperley–Lieb of type affine A [20], but faithfulness was later proved by Green and Fan [4]. T. tom Dieck described a diagrammatic representation of the generalized Temperley–Lieb algebra of type E [27], which was proved to be faithful in a recent paper by Green [10]. Ernst [3] constructed an associative diagram algebra and proved that it is a faithful representation of $\mathrm{TL}(\widetilde{C})$. Since Coxeter groups of type $\widetilde{C}$ have an infinite number of fully commutative elements, $\mathrm{TL}(\widetilde{C})$ is infinite dimensional. This is the first faithful representation of an infinite dimensional non-simply-laced generalized Temperley–

Lieb algebra (in the sense of Graham [6]).

For W a Coxeter group of type Γ, the Hecke algebra $\mathcal{H}(\Gamma)$ is an algebra with a basis given by $\{T_w \mid w \in W\}$ and relations that deform the relations of W by a parameter q. If we set q to 1, we recover the group algebra of W. In their 1979 paper, Kazhdan and Lusztig [18] defined a remarkable basis $\{C'_w \mid w \in W\}$ for $\mathcal{H}(\Gamma)$ in terms of the natural basis.

The entries in the change of basis matrix between the T-basis and the C'-basis give rise to the Kazhdan–Lusztig polynomials $\{P_{x,w} \mid x, w \in W\}$. If $x < w$ (in the Bruhat order), then $P_{x,w}$ is a polynomial in q of degree at most $(\ell(w) - \ell(x) - 1)/2$. We let $\mu(x, w)$ denote the (integer) coefficient of $q^{(\ell(w)-\ell(x)-1)/2}$ in $P_{x,w}$. Note that $\mu(x, w)$ can only be nonzero if $x < w$ and $\ell(w) - \ell(x)$ is odd. The μ-values also appear in multiplication formulas for the Kazhdan–Lusztig basis elements $\{C'_w\}$.

The Kazhdan–Lusztig polynomials are of great importance in algebra and geometry. They have applications to the representation theory of semisimple algebraic groups, Verma modules, algebraic geometry, topology of Schubert varieties, canonical bases, immanant inequalities, etc. Unfortunately, computing the polynomials $P_{x,w}$ efficiently is quite difficult, even in finite groups of moderate size. The only obvious way to compute $P_{x,w}$ is by means of a recurrence formula:

$$P_{x,w} = q^{1-c} P_{sx,v} + q^c P_{x,v} - \sum_{sz<z} \mu(z, v) q_z^{-1/2} q_w^{1/2} P_{x,z},$$

where we define $c = 0$ if $x < sx$ and $c = 1$ otherwise. Note that the μ-values play a major role in the recursive structure of the Kazhdan–Lusztig polynomials. Computing each μ-value is not known to be any easier than computing the entire polynomial $P_{x,w}$. However, one can see from the recurrence above that the computation of $P_{x,w}$ would be simplified if one could quickly compute the μ-values.

In 1999, Green and Losonczy [12] showed that $\mathrm{TL}(\Gamma)$ admits a canonical basis, that is, $\{c_w \mid w \text{ fully commutative}\}$. This basis is analogous to the Kazhdan–Lusztig basis, or C'-basis, for $\mathcal{H}(\Gamma)$ [12]. In addition, under some circumstances, c_w is the image of the Kazhdan–Lusztig basis element C'_w in the quotient when w is fully commutative. In particular, this is true for $\mathrm{TL}(\widetilde{C}_n)$.

Using the corresponding diagram algebra of $\mathrm{TL}(\Gamma)$ when Γ is of types A, B, D, or E, Green constructed a trace on $\mathcal{H}(\Gamma)$ similar to the Jones trace in the case that Γ is of type A [4]. This trace satisfies the Markov condition, which arises in the context of knot theory. The coefficient $\mu(x, w)$ appears as the coefficient of $q^{-1/2}$ in the trace of $C'_x C'_w$. Remarkably, this trace is easy to compute in the known examples if x and w are both fully commutative, even though the problem of computing the product $C'_x C'_w$ is difficult in general. Unfortunately, the diagrammatic representation of $\mathrm{TL}(\widetilde{C}_n)$ established by Ernst is described in terms of the so-called monomial basis, which does not coincide with the canonical basis.

In this thesis, we obtained several original results concerning the computation of μ-values and products of canonical basis elements involving fully commutative elements of Coxeter groups of type $\widetilde{C}$. Moreover, we constructed a diagram algebra that mirrors these results, which we believe is a faithful representation of the canonical basis corresponding to $\mathrm{TL}(\widetilde{C})$. Ultimately, we intend to use the results of this thesis to relate the two diagram algebras in order to construct a trace on $\mathcal{H}(\widetilde{C})$ and then use this trace to non-recursively compute $\mu(x,w)$ for x and w both fully commutative.

We begin by introducing Coxeter systems and their necessary properties in Chapter 1. These properties are further developed for Coxeter systems of certain types in Chapter 2, where we also introduce visualizations of group elements of Coxeter groups (called heaps) in order to clarify many of the arguments presented in subsequent chapters. We introduce the Hecke algebra corresponding to a Coxeter system of arbitrary type in Section 3.1, and subsequently develop the Temperley–Lieb algebra in Section 3.2. In Chapter 4, we provide computations of several products of interest that are conjectured to correspond to the diagram algebra encountered in Chapter 5.

1.2 Coxeter systems

A *Coxeter system* is a pair (W,S) where S is a finite set of involutions generating a group W, called a *Coxeter group*, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e\rangle,$$

where e is the identity element in W, $m(s,t) = m(t,s) < \infty$, and $m(s,t) = 1$ if and only if $s = t$. By [15], $m(s,t)$ is exactly the order of the group element $st \in W$. We call $m(s,t)$ the *bond strength* of s and t. It turns out that the elements of S are distinct as group elements. Coxeter groups may be thought of as generalized reflection groups, where each $s \in S$ is a reflection and st is a rotation, where $s,t \in S$ with $s \neq t$ and $m(s,t)$ is the order of the rotation.

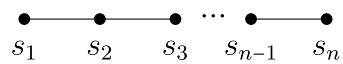
Given a Coxeter system (W,S), a word $s_{x_1}s_{x_2}\cdots s_{x_r}$ spelled in the alphabet S is called an *expression* for $w \in W$ if it is equal to w when considered as a group element. If r is minimal among all expressions for w, the corresponding word is called a *reduced expression* for w, in which case, the *length* of w is defined to be $\ell(w) := r$. Any $w \in W$ may have many reduced expressions representing the element. In the event we opt for a fixed expression for $w \in W$, possibly reduced, we write $\mathsf{w} = s_{x_1}\cdots s_{x_r}$ (written in sans serif font). For $u,v \in W$, we say uv is a *reduced product* if $\ell(uv) = \ell(u) + \ell(v)$.

For a given Coxeter system (W,S), we have an associated *Coxeter graph* Γ having

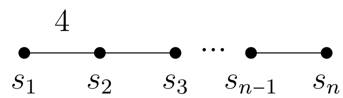
(a) vertex set S and

(b) edges $\{s_i, s_j\}$ for each $m(s_i,s_j) \geq 3$.

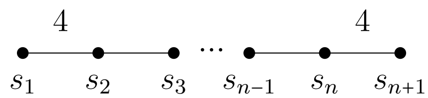
Each edge $\{s_i, s_j\}$ shall be labeled with the corresponding bond strength, although it is standard to omit the label when $m(s_i, s_j) = 3$. When provided a Coxeter graph Γ, by [15, §2.2], we may uniquely reconstruct the associated Coxeter system (W, S), in which case, the associated Coxeter system is said to be of type Γ and we let $W(\Gamma)$ and $S(\Gamma)$ denote the corresponding Coxeter group and generating set, respectively.



(a) A_n



(b) B_n



(c) $\widetilde{C}_n$

Figure 1.1: Coxeter graphs of type A_n, B_n, and $\widetilde{C}_n$.

Suppose (W, S) is of type A_n, whose graph is provided in Figure 1.1(a). It is known that $W(A_n) \cong \mathrm{Sym}_{n+1}$, the symmetric group on $n+1$ objects, under the map $s_i \longmapsto (i, i+1)$. This thesis will focus primarily on Coxeter systems of types B_n and $\widetilde{C}_n$, which are provided in Figures 1.1(b) and 1.1(c), respectively. The group $W(B_n)$ is isomorphic to the group $S_n \wr \mathbb{Z}_2$, which is of order $n!2^n$. Interestingly, the group $W(\widetilde{C}_n)$ has infinite order. One should notice that the Coxeter graph of type $\widetilde{C}_n$ has two subgraphs isomorphic to the graph of type B_n. Note that $W(\widetilde{C}_n)$ has defining relations

(a) $s_i s_i = e$ for all i;

(b) $s_i s_j = s_j s_i$ for $|i - j| > 1$;

(c) $s_i s_j s_i = s_j s_i s_j$ for $|i - j| = 1$ and $1 < i, j < n + 1$;

(d) $s_i s_j s_i s_j = s_j s_i s_j s_i$ for $\{i, j\} = \{1, 2\}$ or $\{n, n + 1\}$.

Note that we can obtain $W(B_n)$ from $W(\widetilde{C}_n)$ by removing the generator s_{n+1} and the corresponding relations [15]. Similarly, we may obtain a Coxeter group of type B_n by removing the generator s_1 and the corresponding relations. To distinguish the two cases, we let $W(B_n)$

denote the subgroup of $W(\widetilde{C}_n)$ generated by $\{s_1, \ldots, s_n\}$ and let the subgroup of $W(\widetilde{C}_n)$ generated by $\{s_2, \ldots, s_{n+1}\}$ be denoted $W(B'_n)$. Clearly $W(B'_n) \cong W(B_n)$.

Since elements of $S(\Gamma)$ have order two, the relation $(st)^{m(s,t)} = e$ can be written

$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)}$$

having $m(s,t) \geq 2$ factors. If $m(s,t) = 2$, then $st = ts$ is called a *commutation relation* as s and t are commuting elements. If $m(s,t) \geq 3$, the relation above is called a *braid relation.* The act of replacing

$$\underbrace{sts\cdots}_{m(s,t)} \longmapsto \underbrace{tst\cdots}_{m(s,t)}$$

is called a *commutation move* if $m(s,t) = 2$ and a *braid move* if $m(s,t) \geq 3$.

Each element $w \in W(\Gamma)$ can have several reduced expressions that represent it. The following theorem, which is known as Matsumoto's Theorem, addresses the relationship between the reduced expressions for a group element.

Theorem 1.2.1. [5] In a Coxeter system (W, S) of type Γ, any two reduced expressions for the same group element in $W(\Gamma)$ differ by a sequence of commutation and braid moves.

As a result of Matsumoto's Theorem, we find that all reduced expressions for $w \in W(\Gamma)$ have the same generators appearing in every reduced expression, possibly with different multiplicities. We define the *support* of an element $w \in W(\Gamma)$, denoted $\text{supp}(w)$, to be the set of all generators appearing in any reduced expression for w. If $\text{supp}(w) = S(\Gamma)$, we say w has *full support.*

Let (W, S) be of type Γ and let $w \in W(\Gamma)$. We define the *left descent of w* via

$$\mathcal{L}(w) := \{s \in S \mid \ell(sw) < \ell(w)\},$$

and the *right descent set of w* via

$$\mathcal{R}(w) := \{s \in S \mid \ell(ws) < \ell(w)\}.$$

It turns out that $s \in \mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) if and only if w has a reduced expression beginning (respectively, ending) with s [15].

Given a reduced expression w for $w \in W(\Gamma)$, we define any expression obtained by deleting some subsequence of generators appearing in w as a *subexpression* for w. Any consecutive subexpression of w shall be deemed a *subword.* If there exists some $z \in W(\Gamma)$ such that z is represented by a subexpression of w, where w is a reduced expression for w, we write $z \leq w$.

This relation is a well-defined partial ordering on $W(\Gamma)$ and is called the *Bruhat order* [15, § 5.9].

In the remainder of this thesis, when confusion is not likely to arise, we abbreviate the expression $s_{x_1}s_{x_2}\cdots s_{x_r}$ as $x_1x_2\cdots x_r$.

Example 1.2.2. The Hasse diagram for the Bruhat order on $W(A_2)$ is given in Figure 1.2. This example provides a visualization of the elements of $W(A_2)$ and their respective subwords.

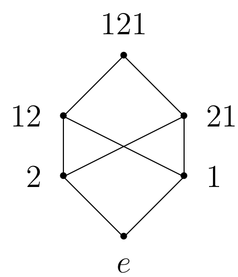


Figure 1.2: Hasse diagram for the Bruhat order of $W(A_2)$.

1.3 Fully commutative elements

Let (W, S) be a Coxeter system of type Γ and let $w \in W(\Gamma)$. As in [25], define the relation $\sim$ on the set of reduced expressions for w as follows. Let w and w' be two reduced expressions for w. We define $\mathsf{w} \sim \mathsf{w}'$ if there is a single commutation relation that carries w to w' via some commutation $ij \longmapsto ji$ where $m(i,j) = 2$.

Now, define $\approx$ to be the reflexive transitive closure of the relation $\sim$. The relation $\approx$ is an equivalence relation on the set of reduced expressions for w. Each equivalence class under $\approx$ is called a *commutation class*. If the set of reduced expressions for w has a single commutation class, we say w is *fully commutative*. Define $\mathrm{FC}(\Gamma)$ to be the set of all fully commutative elements of $W(\Gamma)$.

Example 1.3.1.

(a) Consider $w \in W(\widetilde{C}_3)$ with fixed expression $\mathsf{w} = 323121231$. Then using braid moves and commutations we see that

$$\begin{aligned}(323)(1212)31 &= (232)(2121)31\\ &= 23121(31)\\ &= 23121(13)\\ &= 2(31)23\\ &= 21(323)\\ &= 21(232).\end{aligned}$$

It turns out that 23123, 21323, and 21232 are all reduced expressions for w and they are the only reduced expressions for w. Then, because 23123 and 21323 are related through commutation, and 21232 is not related to either of the other two through any commutation, the commutation classes of w are $\{23123, 21323\}$ and $\{21232\}$. So $w \notin \mathrm{FC}(\widetilde{C}_3)$. Moreover, $\mathrm{supp}(w) = \{1,2,3\}$, which is not equal to S, so w does not have full support. Furthermore, $\mathcal{L}(w) = \{2\}$ and $\mathcal{R}(w) = \{2,3\}$.

(b) Now consider $w \in W(B_3)$ with fixed reduced expression $\mathsf{w} = 323121231$. Since the relations on B_3 are the relations on $\widetilde{C}_3$ after removing the generator 4 and its corresponding relations, we have the same support, commutation classes, and descent sets as in Part (a). The only difference is that w now has full support.

(c) Consider $w \in W(\widetilde{C}_3)$ such that $\mathsf{w} = 132413$ is a reduced expression for w. The unique commutation class of w is

$$\{132413, 134213, 134231, 132431, 312413, 314213, 312431, 314231\}.$$

Notice that we cannot perform any braid moves on any reduced expression for w. Thus $w \in \mathrm{FC}(\widetilde{C}_3)$. Also, $\mathrm{supp}(w) = \{1,2,3,4\}$, so w has full support while $\mathcal{L}(w) = \{1,3\} = \mathcal{R}(w)$. Furthermore, $(1324)w$ is a reduced product representing a fully commutative element in $\widetilde{C}_3$, as well. If we pursue this pattern for any arbitrary length, we find that each $(1324)^k w$ for $k > 0$, is a reduced product, and that the corresponding group element is fully commutative. Thus, $\mathrm{FC}(\widetilde{C}_3)$ has infinitely many elements. We will frequently encounter this type of element in $\widetilde{C}_n$, for arbitrary n, throughout this thesis.

The following theorem, proven by Stembridge, presents criteria sufficient to determine whether or not an element is fully commutative.

Theorem 1.3.2. [25] Let (W,S) be a Coxeter system of type Γ. An element $w \in W(\Gamma)$ is fully commutative if and only if no reduced expression for w contains a subword of the form $\underbrace{sts\cdots}_{m(s,t)}$ for $m(s,t) \geq 3$.

Loosely speaking, an element is fully commutative if and only if no reduced expression provides an opportunity to perform a braid move.

Remark 1.3.3. As a result of Theorem 1.3.2, $\mathrm{FC}(\widetilde{C}_n)$ consists strictly of elements whose reduced expressions avoid the following two types of consecutive subwords;

(a) iji for $|i-j| > 1$ and $1 < i, j < n+1$, and

(b) $ijij$ for $\{i, j\} = \{1, 2\}$ or $\{n, n+1\}$.

Note that all elements of $\mathrm{FC}(B_n)$ and $\mathrm{FC}(B_n')$ avoid the relevant consecutive subwords, as well.

Of course, if $W(\Gamma)$ is a finite group, $\mathrm{FC}(\Gamma)$ must be finite as well, such as in type B_n. However, both $W(\widetilde{C}_n)$ and $\mathrm{FC}(\widetilde{C}_n)$ are infinite. Interestingly, there do exist examples of infinite Coxeter systems that have finitely many fully commutative elements, such as Coxeter systems of type E_n for $n \geq 9$, as seen in [25].

Chapter 2

Combinatorics of Coxeter groups of types B and $\widetilde{C}$

2.1 Heaps

Associated with every reduced expression in a Coxeter group is a labeled partially ordered set, called a heap. Heaps render a visual representation for any reduced expression while preserving the relations among generators. Our development of heaps follows from [1] and [25] with our focus primarily on Coxeter groups of types B_n and $\widetilde{C}_n$.

Let (W, S) be a Coxeter system of type Γ and let $\mathsf{w} = s_{x_1} \cdots s_{x_r}$ be a reduced expression for $w \in W(\Gamma)$. We define a partial ordering on the indices $\{1, \ldots, r\}$ by the transitive closure of the relation $j \lessdot i$ if $i < j$ and $m(s_{x_i}, s_{x_j}) > 2$. Moreover, since w is reduced, $j \lessdot i$ if $i < j$ and $s_{x_i} = s_{x_j}$ by transitivity. This partial order is called the *heap* of w, where i is labeled as s_{x_i}.

Following from [25, Proposition 2.2], we find that heaps are well-defined up to commutation class. In particular, there is a one-to-one correspondence between commutation classes and heap representations for any $w \in W(\Gamma)$. This means that if w and w' are two reduced expressions for $w \in W(\Gamma)$ that are in the same commutation class, then the heaps of w and w' are equal. If $w \in \mathrm{FC}(\Gamma)$, then there is a unique heap corresponding to w since a fully commutative element has a single commutation class.

Example 2.1.1. If we let $\mathsf{w} = s_3 s_2 s_1 s_2 s_5 s_4 s_6 s_5$ be a reduced expression for $w \in W(\widetilde{C}_5)$, we find that w has indexing set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. As an example, $3 \lessdot 2$ since $2 < 3$ and the generators s_2 and s_1 (which are the second and third generators present in w) do not commute. The visualization of w is provided in the Hasse diagram for the heap poset found in Figure 2.1. Note that $w \in \mathrm{FC}(\widetilde{C}_n)$.

Let w be a fixed reduced expression for $w \in W(\widetilde{C}_n)$. As encountered in [1], a heap for w

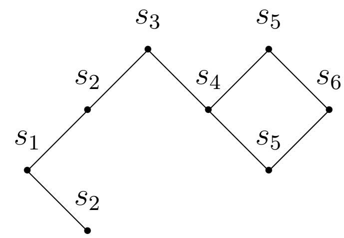


Figure 2.1: Labeled Hasse diagram for the heap of a fully commutative element.

may be represented as a collection of lattice points embedded in $\{1,2,\ldots,n+1\}\times\mathbb{N}$. In order to accomplish this, the (not unique) coordinates $(x,y)\in\{1,2,\ldots,n+1\}\times\mathbb{N}$ are assigned to each entry of the labeled Hasse diagram representing the heap of w as follows:

(a) The entry with coordinates (x,y) will be labeled s_i in the heap if and only if $x=i$;

(b) The entry with coordinates (x,y) is greater than the entry with coordinates (x',y') in the heap if and only if $y>y'$.

In the case of type $\widetilde{C}_n$ (and any other Coxeter system with straight line Coxeter graph), we find the defining relations imply (x,y) covers (x',y') in the heap if and only if $x=x'\pm 1$, $y>y'$, and no entries (x'',y'') exist such that $x''\in\{x,x'\}$ and $y'<y''<y$. Subsequently, we may completely reconstruct the edges of the Hasse diagram and the corresponding heap poset from a lattice point representation. The lattice point representation of a heap enables us to visualize and simplify arguments that have potential to become unwieldy. Note that entries on top of a heap correspond to generators occurring to the left in the associated reduced expression(s).

Let w be a reduced expression for $w\in W(\widetilde{C}_n)$, and let $H(\mathsf{w})$ denote the lattice representation of the heap poset in $\{1,2,\ldots,n+1\}\times\mathbb{N}$. If $w\in \mathrm{FC}(\widetilde{C}_n)$, then the heap corresponding to every reduced expression for w is identical, that is, the choice of reduced expression for w is irrelevant. In this case, we will often write $H(w)$ and we will refer to $H(w)$ as the heap of w. Furthermore, if $w\in \mathrm{FC}(\widetilde{C}_n)$ and the entry i occurs on the top (respectively, bottom) of $H(w)$, then $s_i\in\mathcal{L}(w)$ (respectively, $s_i\in\mathcal{R}(w)$).

Given a heap, every generator with coordinates (x,y) will have fixed x-coordinate, but may have multiple y-coordinates, meaning two entries labeled by the same generator must possess the same x-coordinate but may differ by the amount of vertical space separating them.

Let $\mathsf{w}=s_{x_1}\cdots s_{x_r}$ be a reduced expression for $w\in \mathrm{FC}(\widetilde{C}_n)$. If s_{x_i} and s_{x_j} are adjacent generators in the associated Coxeter graph with $i<j$, then we place the point labeled by s_{x_i} at a level that is *above* the level of the point labeled by s_{x_j}. Because generators that are

not adjacent to one another commute, points whose x-coordinates differ by more than one are allowed to vertically slide past each other or land at the same level. In order to place emphasis on the covering relations occurring in the lattice representation, we will encase each entry of the heap in a rectangle (with rounded corners) in a such a manner that if one entry covers another, the rectangles overlap halfway.

Example 2.1.2. Recall Example 2.1.1, where $\mathsf{w} = 32125465$ is a reduced expression for $w \in W(\widetilde{C}_5)$. Since $w \in \mathrm{FC}(\widetilde{C}_n)$, we may consider $H(w)$. Then two equivalent lattice point representations for $H(w)$ appear in Figure 2.2.

(a) (b)

Figure 2.2: Two equivalent lattice point representations for the heap of a fully commutative element.

Example 2.1.3. Let w be as in Example 1.3.1(a), where the expressions $\mathsf{w}_1 = 23123$, $\mathsf{w}_2 = 21323$, and $\mathsf{w}_3 = 21232$ are the reduced expressions representing $w \in W(\widetilde{C}_3)$. Recall that w has commutation classes $\{\mathsf{w}_1, \mathsf{w}_2\}$ and $\{\mathsf{w}_3\}$. Visualizations of $H(\mathsf{w}_1)$, $H(\mathsf{w}_2)$, and $H(\mathsf{w}_3)$ are presented in Figure 2.3.

(a) $H(\mathsf{w}_1)$ and $H(\mathsf{w}_2)$ (b) $H(\mathsf{w}_3)$

Figure 2.3: Two heaps for a single non-fully commutative element.

There may be many different ways to represent a heap, each differing by some combination of vertical placement of blocks. For example, we may choose to place blocks as high as possible, as low as possible, or anywhere in between. In this thesis, we will choose the representation we find to be the most efficient and enlightening in each example.

Let $\mathsf{w} = x_1 \cdots x_r$ be a fixed reduced expression for $w \in W(\widetilde{C}_n)$ and let $\mathsf{w}' = y_1 \cdots y_k$ be a reduced subexpression (not necessarily a subword) of w. A heap H' is defined to be a *subheap* of $H(w)$ if $H' = H(\mathsf{w}')$.

A subposet Q of a poset P is called *convex* if $y \in Q$ whenever $x < y < z$ in P and $x, z \in Q$. We refer to a subheap as a *convex subheap* if the corresponding subposet is convex.

Example 2.1.4. As in Example 2.1.1, let $\mathsf{w} = 32125465$ be a reduced expression for $w \in W(\widetilde{C}_5)$. Consider the subexpression $\mathsf{w}' = 5465$ resulting from the deletion of the first through fourth generators in w and the subexpression $\mathsf{w}'' = 545$ resulting in the deletion of all but the fifth, sixth, and eighth generators in w. We may represent $H(\mathsf{w}')$ in Figure 2.4(a) and $H(\mathsf{w}'')$ in Figure 2.4(b). Then $H(\mathsf{w}'')$ is not a convex subheap as $5 < 6 < 5$, yet 6 is not present in $H(\mathsf{w}'')$. However, the inclusion of the generator 6 does render a convex subheap.

(a) $H(\mathsf{w}')$ (b) $H(\mathsf{w}'')$

Figure 2.4: Subheaps of the heap given in Figure 2.2.

The following is implicit using the above definitions. In particular, one may refer to [25, §3.3].

Proposition 2.1.5. Let $w \in \mathrm{FC}(\Gamma)$. Then H' is a convex subheap of $H(w)$ if and only if H' is the heap for some subword for some reduced expression for w.

The following result is a consequence of Theorem 1.3.2 and Remark 1.3.3 and enables us to quickly recognize when a heap corresponds to an element of $\mathrm{FC}(\widetilde{C}_n)$.

Proposition 2.1.6. Let $w \in \mathrm{FC}(\widetilde{C}_n)$. Then $H(w)$ never shall contain any of the convex subheaps found in Figure 2.5, where $1 < k < n+1$ and we use the blank rectangle to explicitly state the absence of any element occupying the corresponding position.

2.2 Star operations

In this section, we introduce the notion of non-cancellable elements in arbitrary Coxeter systems, which shall be refined for type B_n and type $\widetilde{C}_n$. Let (W, S) be a Coxeter system of arbitrary type Γ and let $I = \{s, t\}$ be a pair of non-commuting generators. Then there are

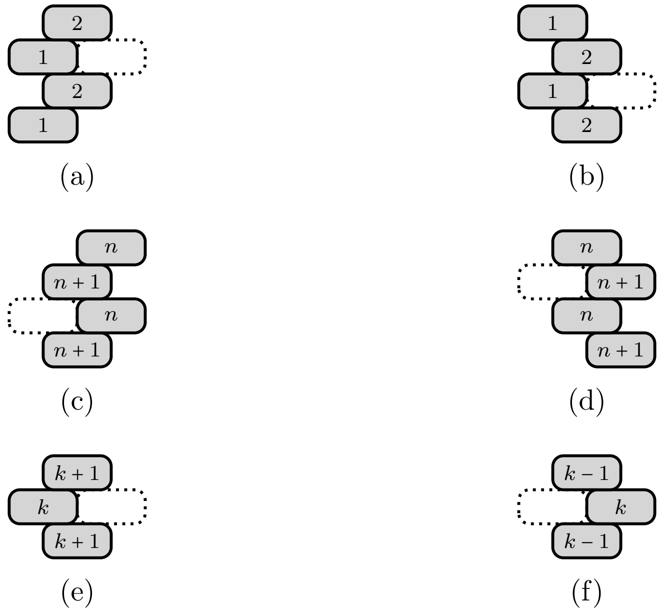


Figure 2.5: Impermissible convex subheaps for elements in $\mathrm{FC}(\widetilde{C}_n)$.

four partially defined maps from W to itself induced by I, known as star operations. When the operation is defined, it will respect the partition $W = \mathrm{FC}(\Gamma)\dot{\cup}(W\setminus\mathrm{FC}(\Gamma))$ and affects the length of w by increasing or decreasing $\ell(w)$ by 1.

Let $w \in W(\Gamma)$. We say w is *left star reducible by s with respect to t* to sw if

(a) $s \in \mathcal{L}(w)$ and

(b) $t \in \mathcal{L}(sw)$ with $m(s,t) \geq 3$.

We define *right star reducible by s with respect to t* analogously. Note that if $m(s,t) \geq 3$, then w is left (respectively, right) star reducible by s with respect to t if and only if $w = stz$ (respectively, $w = zts$), for $z \in W(\Gamma)$, where the product is reduced, meaning no reduced expression for z may start (respectively, end) with s. We will say w is *star reducible* if w is either left or right star reducible by some $s \in S(\Gamma)$. We say w is *left star expandable by t with respect to s* to tw if

(a) $s \in \mathcal{L}(w)$ and

(b) $\ell(tw) > \ell(w)$ with $m(s,t) \geq 3$.

We define *right star expandable by t with respect to s* analogously.

Suppose $I = \{s,t\}$ is a non-commuting pair. A left star reduction will be denoted $\lfloor\underline{st}(w)$, and a left star expansion will be denoted $\lceil\overline{st}(w)$. The respective right-handed notation is $\overline{st}\rceil(w)$ and $\underline{st}\rfloor(w)$. We denote left star operations that are either an expansion or reduction as $[\overline{\underline{st}}(w)$. Similarly, we denote right star operations that are either an expansion or reduction as $\overline{\underline{st}}](w)$. Note the appearance of st in the notation; the comma is omitted for cleanliness in the notation, and should not cause any confusion throughout subsequent chapters. When convenient, we may replace st with ts or simply I in the notation for each operation, where st, ts, and I all represent the unordered set $\{s,t\}$. If the reduction or expansion does not exist with respect to I, the respective symbol is undefined.

A similar concept is that of weak star reductions. Let $w \in \mathrm{FC}(\Gamma)$. Then w is *left weak star reducible by s with respect to t* to sw if the following are satisfied:

(a) $s \in \mathcal{L}(w)$;

(b) $t \in \mathcal{L}(sw)$ with $m(s,t) \geq 3$;

(c) $tw \notin \mathrm{FC}(\Gamma)$.

One should first note that we are restricting to the case where w is fully commutative. Then notice that $\ell(sw) = \ell(w) - 1$ by Condition (b) and $\ell(tw) = \ell(w) + 1$ by Condition (c). We define w to be *right weak star reducible by s with respect to t* to ws analogously. When w is left (respectively, right) weak star reducible by s with respect to t, we refer to the mapping $w \longmapsto sw$ (respectively, $w \longmapsto ws$) as a *left* (respectively, *right*) *weak star reduction.* As before, if w is weak star reducible on either the left or right, we say w is *weak star reducible.* Otherwise, meaning w is not weak star reducible, then $w \in \mathrm{FC}(\Gamma)$ is said to be *non-cancellable.*

Example 2.2.1. Let $w, w' \in \mathrm{FC}(\widetilde{C}_2)$ where $\mathsf{w} = 121$ is a reduced expression for w and $\mathsf{w}' = 12$ is a reduced expression for w'. Note that $1 \in \mathcal{L}(w)$, $2 \in \mathcal{L}(1w)$, and $m(s_1,s_2) = 4$, but $2121 = 2w \notin \mathrm{FC}(\widetilde{C}_2)$. Thus w is left weak star reducible by 1 with respect to 2 to 21, meaning w is not non-cancellable. However, notice that w' is left star reducible by 1 with respect to 2 to the group element $1(12) = 2$. Since $212 = 2w' \in \mathrm{FC}(\widetilde{C}_2)$, we find w' in not left weak star reducible. Similarly, we find w' is right star reducible but not right weak star reducible. Therefore w' is non-cancellable.

Here are a few observations regarding weak star operations:

(a) If $w \in \mathrm{FC}(\Gamma)$ and $s \in \mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$), then $sw \in \mathrm{FC}(\Gamma)$. Consequently, if some fully commutative w is weak star reducible to v, then $v \in \mathrm{FC}(\Gamma)$, as well.

(b) By definition, if w is weak star reducible to v, then w is also star reducible to v. But the converse is not always true, as seen in Example 2.2.1.

(c) If $w \in \mathrm{FC}(\widetilde{C}_n)$, $\mathrm{FC}(B_n)$, or $\mathrm{FC}(B'_n)$, then w is weak star reducible by s with respect to t if and only if

$$w = \begin{cases} stsz, & \text{for } m(s,t) = 4 \\ stz, & \text{for } m(s,t) = 3, \end{cases}$$

where $stsz$ and stz are reduced as products.

2.3 Type I and type II elements

Following [1], we define a family of elements, called type I elements, whose corresponding heap representations are likened to zigzagging shapes. They are defined as follows.

(a) If $i < j$, let

$$\mathsf{z}_{i,j} = i(i+1)\cdots(j-1)j$$

and

$$\mathsf{z}_{j,i} = j(j-1)\cdots(i-1)i.$$

We also let $\mathsf{z}_{i.i} = i$.

(b) If $1 < i \leq n+1$ and $1 < j \leq n+1$, let

$$\mathsf{z}_{i,j}^{L,2k} = \mathsf{z}_{i,2}(\mathsf{z}_{1,n}\mathsf{z}_{n+1,2})^{k-1}\mathsf{z}_{1,n}\mathsf{z}_{n+1,j}.$$

(c) If $1 < i \leq n+1$ and $1 \leq j < n+1$, let

$$\mathsf{z}_{i,j}^{L,2k+1} = \mathsf{z}_{i,2}(\mathsf{z}_{1,n}\mathsf{z}_{n+1,2})^{k}\mathsf{z}_{1,j}.$$

(d) If $1 \leq i < n+1$ and $1 \leq j < n+1$, let

$$\mathsf{z}_{i,j}^{R,2k} = \mathsf{z}_{i,n}(\mathsf{z}_{n+1,2}\mathsf{z}_{1,n})^{k-1}\mathsf{z}_{n+1,2}\mathsf{z}_{1,j}.$$

(e) If $1 \leq i < n+1$ and $1 < j \leq n+1$, let

$$\mathsf{z}_{i,j}^{R,2k+1} = \mathsf{z}_{i,n}(\mathsf{z}_{n+1,2}\mathsf{z}_{1,n})^{k}\mathsf{z}_{n+1,j}.$$

If w is equal to one of the above, we say w is of type I.

The notation may appear cumbersome, so let's make a few comments. The indices i and j determine the point at which we start and stop, respectively. The L or R superscript indicate the zigzag begins moving toward the left or right, respectively (when drawing the corresponding heap from top to bottom). Lastly, the $2k+1$ or $2k$ indicate the number of times we encounter either of the end generators, with the end generators being s_1 and s_{n+1}. If s_i is an end generator, it is not counted, but if s_j is an end generator, it is counted. In other words, if we start our zigzagging on an end generator, it is not counted, but if we finish our zigzag on an end generator, it is counted. Note that each type I element is fully commutative, so $H(w)$ is well-defined if w is of type I.

Example 2.3.1. Consider the type I element $w = \mathsf{z}_{i,j}^{R,2k}$ in $\mathrm{FC}(\widetilde{C}_n)$, where $1 < i, j \leq n+1$. The heap representation for w is provided in Figure 2.6.

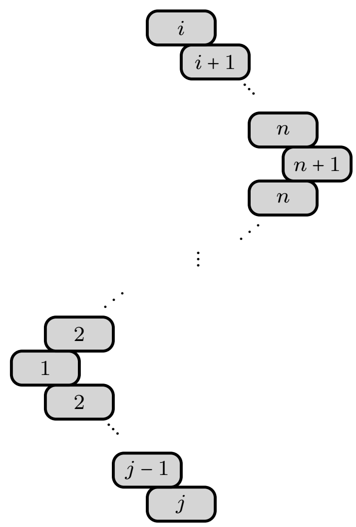


Figure 2.6: Heap of a type I element.

Now define

$$X_{k,k+2m} = k(k+2)(k+4)\cdots(k+2m-2)(k+2m) \in W(\widetilde{C}_n)$$

for $m \in \mathbb{N}$. For example, if n is even we have

$$X_{1,n+1} = 13\cdots(n-1)(n+1)$$

and

$$X_{2,n} = 24\cdots(n-2)n,$$

where all even indices appear exactly once in $X_{2,n}$ and all odd indices appear exactly once in $X_{1,n+1}$. If $w \in W(\widetilde{C}_n)$ is equal to an alternating product of $X_{1,n+1}$ and $X_{2,n}$, for n even, or an alternating product of $X_{1,n}$ and $X_{2,n+1}$, for n odd, we say w is of *type II*.

Example 2.3.2. Let $w \in W(\widetilde{C}_n)$ with n even, such that w has a fixed expression $\mathsf{w} = X_{2,n}(X_{1,n+1}X_{2,n})^k$. One should notice that each of these elements is reduced. Then $H(w)$ is the heap found in Figure 2.7(a). When encountering type II elements, we will primarily be focusing on elements of the form found in Figure 2.7(b) where the element is 'book-ended' by all of the odd generators. Also note that each type II element is fully commutative as they avoid the relations seen in Figure 2.5.

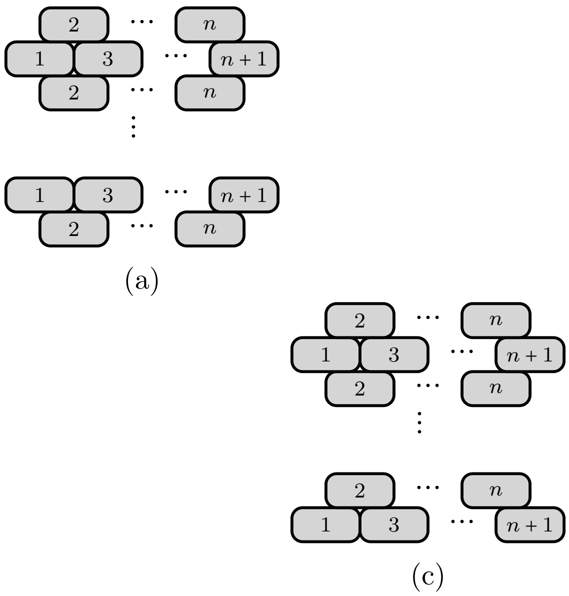


Figure 2.7: Heaps of type II elements for n even.

2.4 Classification of non-cancellable elements in types B_n and $\widetilde{C}_n$

The following theorem classifies all non-cancellable elements in Coxeter systems of type B_n.

Theorem 2.4.1. [1] Let (W, S) be a Coxeter system of type B_n and let $w \in \text{FC}(B_n)$. Then w is non-cancellable if and only if w is equal to one of the following:

(a) a product of commuting generators,

(b) $12u$,

(c) $21u$,

where u is a product of commuting generators and $1, 2, 3 \notin \text{supp}(u)$. There is an analogous statement for $\text{FC}(B'_n)$, where 1 and 2 are replaced with $n+1$ and n, respectively.

The next theorem classifies all non-cancellable elements in Coxeter systems of type $\widetilde{C}_n$.

Theorem 2.4.2. [1] Let $w \in \text{FC}(\widetilde{C}_n)$. Then w is non-cancellable if and only if w is equal to one of the following elements:

(a) uv, where u is a type B non-cancellable element and v is a type B' non-cancellable element with $\text{supp}(u) \cap \text{supp}(v) = \varnothing$;

(b) $\mathsf{z}_{1,1}^{R,2k}, \mathsf{z}_{n+1,n+1}^{L,2k}, \mathsf{z}_{n+1,1}^{L,2k+1}, \mathsf{z}_{1,n+1}^{R,2k+1}$;

(c) any type II element.

In Theorem 2.4.2, Part (a) includes all possible products of commuting generators, and Part (b) includes all type I elements which have left and right descent sets equal to one of the end generators.

The non-cancellable elements play an important role in the development of a diagram algebras used to simplify concepts appearing in the quotient of the Hecke algebra presented in Chapter 3.

Chapter 3

Hecke algebras and Temperley–Lieb algebras

3.1 Hecke algebras

Let (W, S) be the Coxeter system of type Γ. We define the *Hecke algebra* $\mathcal{H}_q(\Gamma)$ corresponding to $W(\Gamma)$ to be the $\mathbb{Z}[q, q^{-1}]$-algebra with abstract basis $\{T_w \mid w \in W(\Gamma)\}$ satisfying

$$T_s T_w := \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w) \end{cases}$$

for any $s \in S(\Gamma)$ and $w \in W(\Gamma)$. This enables us to compute $T_z T_w$ for any $z, w \in W(\Gamma)$. Each T_s is invertible with

$$T_s^{-1} = q^{-1}T_s - (1 - q^{-1})T_e,$$

meaning each T_w is recursively invertible, although the computation of each inverse becomes progressively more complicated as $\ell(w)$ increases [15]. Note that if $\mathsf{w} = s_{x_1}s_{x_2}\cdots s_{x_k}$ is a reduced expression for $w \in W(\Gamma)$, then $T_w = T_{s_{x_1}s_{x_2}\cdots s_{x_k}} = T_{s_{x_1}}T_{s_{x_2}}\cdots T_{s_{x_k}}$. As before, in an effort to ease notation, we abbreviate $T_{s_{x_1}s_{x_2}\cdots s_{x_k}}$ as $T_{x_1x_2\cdots x_k}$, and in particular, $T_{s_i} = T_i$ for $s_i \in S(\Gamma)$. Note that if we set $q = 1$, we recover the group algebra $\mathbb{Z}[W(\Gamma)]$. It is often convenient to extend the scalars of $\mathcal{H}_q(\Gamma)$ to produce an $\mathcal{A}$-algebra $\mathcal{H}(\Gamma)$, where $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and $v^2 = q$ [15, §7.9]. In particular, $\mathcal{H}(\Gamma) = \mathcal{A} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{H}_q(\Gamma)$. We find the polynomial $v + v^{-1} \in \mathcal{A}$ occurs frequently enough that we opt to denote it as δ.

Since $W(\widetilde{C}_n)$ is an infinite group, $\mathcal{H}(\widetilde{C}_n)$ must be of infinite rank. Similarly, $\mathcal{H}(B_n)$ and $\mathcal{H}(B'_n)$ have finite rank since both $W(B_n)$ and $W(B'_n)$ are finite groups.

Example 3.1.1. Consider $\mathsf{w}_1 = 323$ and $\mathsf{w}_2 = 12123$ where both w_1 and w_2 are reduced expressions for $w_1, w_2 \in W(\widetilde{C}_3)$, respectively. We will calculate $T_{w_1}T_{w_2}$. First notice

$$\begin{aligned}(323)(12123) &= (232)(2121)3 \\ &= 231213\end{aligned}$$

where 231213 is reduced. In order to compute $T_{w_1}T_{w_2}$, observe that 312123 and 2312123 are reduced expressions while $(323)(1212)3 = (232)(2121)3$ is not since $(323)(1212)3 = (232)(2121)3 = 231213$. Then we see that

$$\begin{aligned}T_{w_1}T_{w_2} &= T_{323}T_{12123} \\ &= T_3T_2T_3T_{12123} \\ &= T_3T_2T_{312123} \\ &= T_3T_{2312123} \\ &= T_{231213} + (q-1)T_{2312123}.\end{aligned}$$

While $\{T_w \mid w \in W(\Gamma)\}$ is the natural basis for $\mathcal{H}(\Gamma)$, Kazhdan and Lusztig, in [18], defined the so-called *canonical basis* $\{C'_w \mid w \in W(\Gamma)\}$ for $\mathcal{H}(\Gamma)$ as follows. Let $\iota : \mathbb{Z}[v, v^{-1}] \longmapsto \mathbb{Z}[v, v^{-1}]$ be the $\mathbb{Z}$-linear ring homomorphism that exchanges v and v^{-1} and extend ι to be the ring automorphism of $\mathcal{H}(\Gamma)$ satisfying

$$\iota\left(\sum_{w \in W(\Gamma)} a_w v^{-\ell(w)} T_w\right) = \sum_{w \in W(\Gamma)} \iota\left(a_w\right)\left(v^{-\ell(w)} T_{w^{-1}}\right)^{-1},$$

where the a_w are elements of $\mathbb{Z}[v, v^{-1}]$. In [18], Kazhdan and Lusztig proved the following theorem.

Theorem 3.1.2. There is a unique element $C'_w \in \mathcal{H}(\Gamma)$ such that $\iota(C'_w) = C'_w$ and

$$C'_w = \sum_{z \leq w} P_{z,w} v^{-\ell(w)} T_z,$$

where $\leq$ is the Bruhat ordering on the Coxeter group $W(\Gamma)$, $P_{z,w} \in \mathbb{Z}[v]$ if $z \leq w$, and $P_{w,w} = 1$. The set $\{C'_w \mid w \in W(\Gamma)\}$ forms a $\mathbb{Z}[v, v^{-1}]$-basis for $\mathcal{H}(\Gamma)$ and is called the C'-basis.

The polynomials, $P_{z,w}$, appearing in the previous theorem are called the *Kazhdan–Lusztig polynomials* and determine the change of basis matrix between the T-basis and the C'-basis. This remarkable family of polynomials appears in Lusztig's conjecture about the characters of irreducible modules of reductive algebraic groups in characteristic $p > 0$ and is intimately related to the geometry of Schubert varieties, making this family of key importance in algebra and geometry. The introductory facts regarding the Kazhdan–Lusztig polynomials include $P_{w,w} = 1$ and $P_{z,w} = 0$ unless $z \le w$. If we have $z \le w$, then $P_{z,w}$ is a polynomial (in $\mathbb{Z}[v]$) of degree at most $\ell(w) - \ell(z) - 1$. We define $\mu(z,w)$ to be the integer coefficient attached to the term $v^{\ell(w)-\ell(z)-1}$ in $P_{z,w}$. This implies $\mu(z,w)$ is nonzero only if the maximum bound is attained, which may only occur if both (a) $z \le w$ and (b) $\ell(z)$ and $\ell(w)$ have opposite parities. Using these properties, one may define the Kazhdan–Lusztig polynomials recursively using the formula

$$P_{z,w} = v^{2(1-c)} P_{sz,sw} + v^{2c} P_{z,sw} - \sum_{sx<x<sw} \mu(x,sw) v^{\ell(w)-\ell(x)} P_{z,x},$$

where we define $c = 0$, if $z < sz$, and $c = 1$, otherwise [18]. One should note that knowledge of μ-values is a necessity for computing these polynomials and that the computation of the μ-values is not known to be any easier than computing the entire polynomial. However, the computation of each polynomial would be simplified if we had a method of quickly computing μ-values. Here is the beginning of a collection of well-known results regarding μ-values.

Lemma 3.1.3. Let (W,S) be a Coxeter system of type Γ and let $z, w \in W(\Gamma)$.

(a) If $\ell(z) = \ell(w) \bmod 2$ or $z \nleq w$, then $\mu(z,w) = 0$.

(b) If $z \le w$ and $\ell(z) = \ell(w) - 1$, then $P_{z,w} = 1$, and in particular, $\mu(z,w) = 1$.

(c) If there exists $s \in \mathcal{L}(w) \backslash \mathcal{L}(z)$, then either (i) $\mu(z,w) = 0$ or (ii) both $z = sw$ and $\mu(z,w) = 1$.

(d) If there exists $s \in \mathcal{R}(w) \backslash \mathcal{R}(z)$, then either (i) $\mu(z,w) = 0$ or (ii) both $z = ws$ and $\mu(z,w) = 1$.

Proof. All four parts appear in [18] and [19]. If Γ is simply-laced, i.e., $m(s,t) \le 3$ for all $s,t \in S(\Gamma)$, then Parts (a), (c), and (d) come from Proposition 2.1 of [11]. Also, Part (b) is an exercise in §7.10 of [15]. □

One should note that Parts (c) and (d) of the previous proposition highlight that $\mu(z,w) = 0$ whenever $\mathcal{L}(w) \notin \mathcal{L}(z)$ or $\mathcal{R}(w) \notin \mathcal{R}(z)$, with the only exceptions occurring when the conditions of (c) and (d) are satisfied. The following lemma is another result relating the lengths of two group elements and their corresponding μ-value.

Lemma 3.1.4. Let (W,S) be a Coxeter system of type Γ and suppose $I = \{s,t\} \subseteq S(\Gamma)$ where $m(s,t) \ge 3$. If $z \le w \in W(\Gamma)$ and

(a) $\mathcal{L}(w) \subseteq \mathcal{L}(z)$,

(b) $\ell(w) - \ell(z) \ge 3$,

(c) at least one of ${}^{\ulcorner}st(w)$ and ${}^{\llcorner}st(w)$ exists, and

(d) $|\mathcal{L}(w) \cap I| = |\mathcal{L}(z) \cap I| = 1$,

then $\mu(z,w) = \mu\left(\boxed{st}(z), \boxed{st}(w)\right)$.

Proof. This comes from [19]. □

Here the star operations in $\mu\left(\boxed{st}(z), \boxed{st}(w)\right)$ are independent of each other, that is, $\mu\left({}^{\ulcorner}st(z), st^{\lrcorner}(w)\right)$ is one of four options. We will use this fact several times in the proof of Lemma 4.2.1. There is a right-handed analog of Lemma 3.1.4 stating that, under symmetric conditions, $\mu(z,w) = \mu\left(st^{\lrcorner}(z), st^{\urcorner}(w)\right)$.

Lemma 3.1.5. Let (W,S) be a Coxeter system of type Γ and suppose $I = \{s,t\} \subseteq S(\Gamma)$ where $m(s,t) \geq 3$. If

(a) $\mathcal{L}(z) \cap I \neq \mathcal{L}(w) \cap I$ and

(b) $|\mathcal{L}(z) \cap I| = |\mathcal{L}(w) \cap I| = 1$,

then $\mu\left(z, {}^{\ulcorner}st(w)\right) = \mu\left({}^{\ulcorner}st(z), w\right) + \mu\left({}^{\llcorner}st(z), w\right) - \mu\left(z, {}^{\llcorner}st(w)\right)$.

Proof. This originated in [18] and is stated explicitly as Proposition 5.9 of [9]. □

There is a right-handed analog to Lemma 3.1.5 in which we would conclude

$$\mu\left(z, st^{\urcorner}(w)\right) = \mu\left(st^{\urcorner}(z), w\right) + \mu\left(st^{\lrcorner}(w), w\right) - \mu\left(z, st^{\lrcorner}(w)\right).$$

We will be revisiting μ-values in depth and adding to our collection of results in Chapter 4 where this list will be extended in the context of Coxeter systems of type $\widetilde{C}_n$. For now, we return to the C'-basis to define multiplication of basis elements in the following theorem.

Theorem 3.1.6. Let (W,S) be a Coxeter system of type Γ, let $w \in W(\Gamma)$, and let $s \in S(\Gamma)$. Then multiplication on the C'-basis is determined recursively by

$$C'_s C'_w := \begin{cases} C'_{sw} + \sum \mu(z,w) C'_z & \text{if } \ell(sw) > \ell(w) \\ \delta C'_{sw} & \text{otherwise,} \end{cases}$$

where we are summing over all $sz < z < w$.

Proof. This was stated explicitly in [14] and implicitly in [18, §2.2]. □

Notice the reappearance of μ-values in the above formula. This should come as no surprise, but adds to the motivation for finding streamlined methods of computing μ-values.

3.2 Generalized Temperley–Lieb algebras

Let (W, S) be a Coxeter system of type Γ. Define $J(\Gamma)$ as the two-sided ideal of $\mathcal{H}(\Gamma)$ generated by the elements

$$\sum_{w \in \langle s,t \rangle} T_w$$

where $\langle s, t\rangle$ is the subgroup generated by pairs of adjacent vertices on the Coxeter graph, that is, $m(s,t) \geq 3$.

Example 3.2.1. In type $\widetilde{C}_2$,

$$\langle s_1, s_2 \rangle = \{e, 1, 2, 12, 21, 121, 212, 1212\} \text{ and,}$$
$$\langle s_2, s_3 \rangle = \{e, 2, 3, 23, 32, 232, 323, 2323\},$$

so $J(\widetilde{C}_2)$ is generated by

$$T_e + T_1 + T_2 + T_{12} + T_{21} + T_{121} + T_{212}T_{1212}, \text{ and}$$
$$T_e + T_2 + T_3 + T_{23} + T_{32} + T_{232} + T_{323} + T_{2323}.$$

We now define the *Temperley–Lieb algebra of type* Γ as the quotient $\mathbb{Z}[v, v^{-1}]$-algebra $\mathcal{H}(\Gamma)/J(\Gamma)$, denoted $\mathrm{TL}(\Gamma)$. Let the image of T_w under the canonical epimorphism $\theta : \mathcal{H}(\Gamma) \to \mathrm{TL}(\Gamma)$ be denoted by t_w.

Theorem 3.2.2. [6, Theorem 6.2] Let (W, S) be a Coxeter system of type Γ. Then the set $\{t_w \mid w \in \mathrm{FC}(\Gamma)\}$ is a $\mathbb{Z}[v, v^{-1}]$-basis for $\mathrm{TL}(\Gamma)$.

This basis is called the t-basis and will be used in the construction of other useful bases for the Temperley–Lieb algebra of type Γ. For $s_i \in S(\Gamma)$, define $b_{s_i} := v^{-1}t_i + v^{-1}t_e$ and abbreviate b_{s_i} as b_i. Let $w \in \mathrm{FC}(\Gamma)$ with reduced expression $\mathsf{w} = s_{x_1} \cdots s_{x_r}$ and define

$$b_{\mathsf{w}} := b_{x_1} \cdots b_{x_r}.$$

Note that if w and w' are two different reduced expressions for $w \in \mathrm{FC}(\Gamma)$, then $b_{\mathsf{w}} = b_{\mathsf{w}'}$ since w and w' are commutation equivalent and $b_i b_j = b_j b_i$ when $m(i,j) = 2$. So we will write b_w when we do not have a particular reduced expression in mind.

Theorem 3.2.3. [6] Let (W, S) be a Coxeter system of type Γ. The set $\{b_w \mid w \in \mathrm{FC}(\Gamma)\}$ is a $\mathbb{Z}[v, v^{-1}]$-basis for $\mathrm{TL}(\Gamma)$.

The basis $\{b_w \mid w \in \mathrm{FC}(\Gamma)\}$ is called the *monomial basis* for the Temperley–Lieb algebra of type Γ.

Theorem 3.2.4. [8] Let (W,S) be a Coxeter system of type $\widetilde{C}_n$. Then $\mathrm{TL}(\widetilde{C}_n)$ is generated as a unital algebra by $\{b_1,\dots,b_{n+1}\}$ subject to:

(a) $b_i^2 = \delta b_i$ for all i;

(b) $b_ib_j = b_jb_i$ if $|i-j|>1$;

(c) $b_ib_jb_i = b_i$ if $|i-j|=1$ and $1<i,j<n+1$;

(d) $b_ib_jb_ib_j = 2b_ib_j$ if $\{i,j\}=\{1,2\}$ or $\{n,n+1\}$.

Proof. Here we check that the relations hold in type $\widetilde{C}_n$, but to find the proof in its entirety, the reader shall be referred to [8, Proposition 2.6]. Recall $b_i = v^{-1}t_i + v^{-1}t_e$ and $\delta = v^{-1}+v$. Note that t_e is the multiplicative identity in $\mathrm{TL}(\widetilde{C}_n)$, meaning $t_et_w = t_w$ for $w \in \mathrm{FC}(\Gamma)$. We should also note that $t_i^2 = t_it_i = v^2t_e + (v^{-1}-1)t_i$, using the formula defined in Section 3.1.

(a) We see that

$$\begin{aligned}
b_i^2 &= (v^{-1}t_i + v^{-1}t_e)(v^{-1}t_i + v^{-1}t_e)\\
&= v^{-2}(t_i^2 + 2t_it_e + t_e^2)\\
&= v^{-2}(v^2t_e + (v^1-1)t_i + 2t_it_e + t_e^2)\\
&= v^{-2}(v^2t_e + (v^2-1)t_i + 2t_i + t_e)\\
&= v^{-2}(v^2t_i + v^2t_e + t_i + t_e)\\
&= v^{-2}t_i + v^{-2}t_e + t_i + t_e\\
&= (v^{-1}+v)(v^{-1}t_i + v^{-1}t_e)\\
&= \delta b_i.
\end{aligned}$$

(b) Assume $|i-j|>1$ so that $m(i,j)=2$. Since $is = ji$ and $\ell(ij) > \ell(j)$, $t_it_j = t_{ij} = t_{ji}$. Then

$$\begin{aligned}
b_ib_j &= (v^{-1}t_i + v^{-1}t_e)(v^{-1}t_j + v^{-1}t_e)\\
&= v^{-2}(t_{ij} + t_it_e + t_jt_e + t_e)\\
&= v^{-2}(t_{ij} + t_i + t_j + t_e)\\
&= v^{-2}(t_{ji} + t_i + t_j + t_e)\\
&= (v^{-1}t_j + v^{-1}t_e)(v^{-1}t_i + t_e)\\
&= b_jb_i.
\end{aligned}$$

(c) Let i,j be such that $|i-j|=1$ and $1<i,j<n+1$. Then $m(i,j)=3$. Note that

$$T_{iji} + T_{ij} + T_{ji} + T_i + T_j + T_e \in J(\widetilde{C}_n).$$

Subsequently

$$t_{iji} + t_{ij} + t_{ji} + t_i + t_j + t_e = 0 \in \mathrm{TL}(\widetilde{C}_n).$$

Then we find

$$\begin{aligned}
b_i b_j b_i &= (v^{-1}t_i + v^{-1}t_e)(v^{-1}t_j + v^{-1}t_e)(v^{-1}t_i + v^{-1}t_e) \\
&= v^{-2}(t_i t_j + t_i t_e + t_j t_e + t_e)(v^{-1}t_i + v^{-1}t_e) \\
&= v^{-2}(t_i j + t_i + t_j + t_e)(v^{-1}t_i + v^{-1}t_e) \\
&= v^{-3}(t_{iji} + t_i^2 + t_{ji} + t_i + t_{ij} + t_j + t_e) \\
&= v^{-3}(t_{iji} + v^2 t_e + (v^2-1)t_i + t_{ji} + t_i + t_{ij} + t_j + t_e) \\
&= v^{-3}(v^2 t_e + (v^2-1)t_i + t_i + (t_{iji} + t_{ji} + t_{ij} + t_i + t_j + t_e)) \\
&= v^{-3}(t_i + (v^2-1)t_i + v^2 t_e) + 0 \\
&= v^{-3}(v^2 t_i + v^2 t_e) \\
&= v^{-1}t_i + v^{-1}t_e \\
&= b_i.
\end{aligned}$$

(d) Lastly, let $\{i,j\}$ be such that the set is equal to either $\{1,2\}$ or $\{n, n+1\}$. Then $m(s_i s_j) = 4$. We proceed by playing the same game as above. Note that

$$T_{ijij} + T_{iji} + T_{jij} + T_{ij} + T_{ji} + T_i + T_j + T_e \in J(\widetilde{C}_n),$$

meaning

$$t_{ijij} + t_{iji} + t_{jij} + t_{ij} + t_{ji} + t_i + t_j + t_e = 0 \in \mathrm{TL}(\widetilde{C}_n).$$

Then

$$\begin{aligned}
b_i b_j b_i b_j &= (v^{-1}t_i + v^{-1}t_e)(v^{-1}t_j + v^{-1}t_e)(v^{-1}t_i + v^{-1}t_e)(v^{-1}t_j + v^{-1}t_e) \\
&= v^{-4}(t_i t_j + t_i t_j + t_i + t_j + t_i^2 + t_j^2 + t_i t_j^2 + t_i^2 t_j \\
&\quad + (t_i t_j t_i t_j + t_i t_j t_i + t_j t_i t_j + t_i t_j + t_j t_i + t_i + t_j + t_e)) \\
&= v^{-4}(t_{ij} + t_{ij} + t_i + t_j + t_i^2 + t_j^2 + t_i t_j^2 + t_i^2 t_j \\
&\quad + (t_{ijij} + t_{iji} + t_{jij} + t_{ij} + t_{ij} + t_i + t_j + t_e)) \\
&= v^{-4}(t_{ij} + t_{ij} + t_i + t_j + v^2 t_e + (v^2-1)t_i + v^2 t_e \\
&\quad + (v^2-1)t_j + t_i(v^2 t_e + (v^2-1)t_j) + (v^2 t_e + (v^2-1)t_j)t_j) + 0 \\
&= v^{-4}(v^2 2t_{ij} + v^2 2t_i + v^2 2t_j + v^2 2t_e) \\
&= 2(v^{-1}t_i + v^{-1}t_e)(v^{-1}t_j + v^{-1}t_e) \\
&= 2b_i b_j.
\end{aligned}$$

□

It is known that we may obtain $\mathrm{TL}(B_n)$ and $\mathrm{TL}(B'_n)$ from $\mathrm{TL}(\widetilde{C}_n)$ by deleting the necessary generators and relations [8].

Let (W,S) be a Coxeter system of type Γ, let $\mathcal{L}$ be the free $\mathbb{Z}[v^{-1}]$-submodule of $\mathrm{TL}(\Gamma)$ with basis $\{v^{-\ell(w)}t_w \mid w \in \mathrm{FC}(\Gamma)\}$, and let $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$ be the canonical projection. We will again use the t-basis to build another basis for $\mathrm{TL}(\Gamma)$.

Theorem 3.2.5. [12, Theorem 2.3] Recall the involution ι from Theorem 3.1.2. There exists a unique basis $\{c_w \mid w \in \mathrm{FC}(\Gamma)\}$ for $\mathcal{L}$ such that $\iota(c_w) = c_w$ and $\pi(c_w) = \pi(v^{-\ell(w)}t_w)$ for all $w \in \mathrm{FC}(\Gamma)$.

This basis is called the c-basis, or the canonical basis. Returning to the canonical epimorphism $\theta : \mathcal{H}(\Gamma) \to \mathrm{TL}(\Gamma)$ presented in Section 3.1, it is known that $\theta(\{C'_w \mid w \in \mathrm{FC}(\Gamma)\})$ is a basis for $\mathrm{TL}(\Gamma)$ [13, §1.2]. Notice we are indexing only by fully commutative elements here. We say the Coxeter system satisfies the *projection property* if $\theta(C'_w) = c_w$ for $w \in \mathrm{FC}(\Gamma)$. Utilizing Lemma 3.1.3(b), we find that $c_i = b_i = v^{-1}t_s + v^{-1}t_e$ for $s_i \in S(\Gamma)$ and c_{s_i} is abbreviated as c_i. In some Coxeter systems, the monomial and canonical bases are equal, such as in types A_n and D_n [12]. However, not all Coxeter systems exhibit this characteristic, such as in type $\widetilde{C}_n$ [12, Remark 3.7(1)].

Theorem 3.2.6. [9, Theorem 5.13] Let (W,S) be a Coxeter system of type Γ that satisfies the projection property and let $w \in \mathrm{FC}(\Gamma)$ and $s \in S(\Gamma)$. Then multiplication is determined recursively via

$$c_s c_w := \begin{cases} c_{sw} + \sum \mu(z,w) c_z, & \text{if } \ell(sw) > \ell(w) \\ \delta c_{sw}, & \text{otherwise,} \end{cases}$$

where we are summing over $sz < z < w$ and we define $c_w = 0$ for $w \notin \mathrm{FC}(\Gamma)$.

Unlike in the monomial basis, if $\mathsf{w} = x_1x_2\cdots x_r$ is a reduced expression for $w \in W(\Gamma)$, $c_{x_1}c_{x_2}\cdots c_{x_r}$ is not necessarily equal to c_w.

Let (W,S) be a Coxeter system of type Γ. There is no known example of a Coxeter system failing to satisfy the projection property. According to [22, Theorem 1.11], we have

$$\theta(C'_w) = \begin{cases} c_w, & \text{if } w \in \mathrm{FC}(\Gamma) \\ 0, & \text{if } w \notin \mathrm{FC}(\Gamma) \end{cases}$$

if and only if Γ is a non-branching Coxeter graph and $\Gamma \neq \widetilde{F}_4$. This follows immediately from the work of Shi in [23] and [24]. This implies that in Coxeter systems of type $\widetilde{C}_n$, $\theta(C'_w) = c_w$, for $w \in \mathrm{FC}(\widetilde{C}_n)$. Note that the additional condition that $\theta(C'_w) = 0$ for $w \notin \mathrm{FC}(\Gamma)$ is not a requirement of the projection property.

The summation in Theorem 3.2.6 has a few subtleties for nonzero μ-values that we should make explicit. With s and w as in the theorem, the only candidates $z \in W(\widetilde{C}_n)$ for nonzero $\mu(z, w)$ are if

(a) $z < w$,

(b) $sz < z$, and

(c) $z \in \mathrm{FC}(\widetilde{C}_n)$.

To conclude this chapter we provide two results, one of which incorporates star operations and multiplication of elements of $\{c_w \mid w \in W(\widetilde{C}_n)\}$.

Lemma 3.2.7. [9, Proposition 6.3(ii)] Let (W, S) be a Coxeter system of type $\widetilde{C}_n$ and let $w \in \mathrm{FC}(\widetilde{C}_n)$ with $tw < w$. Then

$$c_s c_w = c_{\lceil st \rceil(w)} + c_{\lfloor st \rfloor(w)}.$$

Lemma 3.2.7 applies to all Coxeter systems of type Γ that have the projection property, but we will only be using it in the context of Coxeter systems of type $\widetilde{C}_n$. If any of the terms in the equation are undefined, then they are defined to be zero.

Lemma 3.2.8. Let (W, S) be a Coxeter system of type $\widetilde{C}_n$ and let $w = stx$ be a reduced product where $m(s,t) \geq 3$ and $tx \in \mathrm{FC}(\widetilde{C}_n)$. Then

$$c_s c_{tx} = \begin{cases} c_w, & \text{if } s \notin \mathcal{L}(x) \\ c_w + c_x, & \text{if } s \in \mathcal{L}(x). \end{cases}$$

Proof. First notice that $\ell(stx) > \ell(tx)$ so that we have

$$c_s c_{tx} = c_{stx} + \sum \mu(z, tx) c_z.$$

Assume $\mu(z, tx) \neq 0$ in the sum above for some z. Then $sz < z < tx$ and $z \in \mathrm{FC}(\widetilde{C}_n)$, by Theorem 3.2.6. Since $m(s,t) \geq 3$, $s \in \mathcal{L}(z)$, and $z \in \mathrm{FC}(\widetilde{C}_n)$, we find $t \notin \mathcal{L}(z)$. Since $s \in \mathcal{L}(z) \backslash \mathcal{L}(tx)$, we have $\mathcal{L}(z) \notin \mathcal{L}(tx)$, so the only possible nonzero μ-value occurs in the case that $y \in \mathcal{L}(tx) \backslash \mathcal{L}(z)$ and $z = y(tx)$, by Lemma 3.1.3(c). Since we require $s \in \mathcal{L}(z)$, we find that the only generator satisfying $y \in \mathcal{L}(tx) \backslash \mathcal{L}(z)$ is $y = t$.

If $z = t(tx) = x$, we have $\mu(z, tx) = 1$, and otherwise $\mu(z, tx) = 0$. We now check two cases.

(a) Suppose $s \notin \mathcal{L}(x)$. Since $s \in \mathcal{L}(z)$ and $t(tx) = x$, we find that $z \neq t(tx)$. So there is no z such that $\mu(z, tx) \neq 0$, meaning $c_s c_{tx} = c_{stx} = c_w$.

(b) Suppose $s \in \mathcal{L}(x)$. Then $\mu(z, tx) = 1$ for $z = t(tx) = x$, meaning $c_s c_{tx} = c_w + c_x$.

Note that we do not require w to be fully commutative. In the event that $c_s c_{tx} = c_w + c_x$, meaning $s \in \mathcal{L}(x)$, we find that $c_w = 0$ unless $m(s,t) = 4$. □

Over the course of Chapter 4, we will implement results encountered thus far to compute products of the form $c_{x_1} c_{x_2} \cdots c_{x_r}$ such that $x_1 x_2 \cdots x_r$ is a type I or type II element.

Chapter 4

Computations in $\mathrm{TL}(\widetilde{C})$

Let (W, S) be a Coxeter system of type $\widetilde{C}_n$ for n even. The goal of this chapter is to compute products of elements of the c-basis. In addition, we will be computing μ-values related to these products. We will say $c_{x_1}c_{x_2}\cdots c_{x_r}$ is of type I if $x_1x_2\cdots x_r \in W(\widetilde{C}_n)$ is reduced and a type I element and of type II if $x_1x_2\cdots x_r \in W(\widetilde{C}_n)$ is reduced and a type II element. Note that the product $c_{x_1}c_{x_2}\cdots c_{x_r}$ may be a linear combination of elements from $\{c_w \mid w \in \mathrm{FC}(\widetilde{C}_n)\}$.

Any product that is missing a generator from $\{c_i \mid i \in S(\Gamma)\}$ is handled by known results for Coxeter systems of type A_n and B_n. It turns out that non-cancellable elements with full support provide the foundation for inductive arguments that are used to prove the faithfulness of the desired diagram algebra since computations of μ-values are mostly well-behaved with respect to star operations.

In Section 4.1, we consider a type I product that begins and ends with end generators; however, the tools presented allow for the computation of products that have the zigzagging shape of a type I element, but begin and end with any generator. In Section 4.3, we present type II products in the case where n is even and our product begins and ends with $c_1c_3\cdots c_{n+1}$ since the corresponding type II element in $W(\widetilde{C}_n)$ is not star reducible to a product of commuting generators. Our process will also allow us to compute any type II product that begins with $c_2c_4\cdots c_n$. Furthermore, by using the symmetry of the Coxeter graph, we are also able to compute any type II product that ends with $c_2c_4\cdots c_n$, thus granting the ability to compute all type II products. Furthermore, we only present the case when n is even, but we believe that the case where n is odd is simpler since every type II element of $\mathrm{FC}(\widetilde{C}_n)$ may be made into a product of commuting generators by some sequence of star reductions. Let's get started.

4.1 Computations involving type I elements

Our first task is to compute products involving type I elements in the c-basis. Our first result, Lemma 4.1.2, is a restatement of Lemma 3.2.7 designed to explicitly handle the product $c_s c_w$ where sw is a type I element in $W(\widetilde{C}_n)$. Lemma 4.1.3 is introduced to manage elements that occur in the star reductions caused by Lemma 4.1.2. We restricted n to be even at the beginning of the chapter, however, the results presented in Section 4.1 are independent of the parity of n.

Remark 4.1.1. Observe that if $w \in \mathrm{FC}(\widetilde{C}_n)$ and $s \in S$ such that $s \notin \mathrm{supp}(w)$, then $c_s c_w = c_{sw}$ since $\ell(sw) > \ell(w)$ but $sw \nleq w$ rendering an empty summation in the calculation $c_s c_w = c_{sw} + \sum \mu(z,w) c_z$ by Theorem 3.2.6.

Lemma 4.1.2. Let (W,S) be a Coxeter system of type $\widetilde{C}_n$. Let $z \in \mathrm{FC}(\widetilde{C}_n)$ and $s,t \in S$ such that z, tz, and stz are all of type I. Then

$$c_s c_{tz} = \begin{cases} c_{stz}, & \text{if } s \notin \mathcal{L}(z) \\ c_{stz} + c_z, & \text{if } s \in \mathcal{L}(z). \end{cases}$$

Proof. This is an immediate application of Lemma 3.2.8. □

Notice that certain products cause a 'split' in Lemma 4.1.2 with terms c_{stz} and c_z. If we multiply by another generator $s' \in S(\Gamma)$ for which $\ell(s'z) > \ell(z)$, we find that $c_{s'z}$ may no longer be indexed by a fully commutative element (in which case part of our summation will become 0). Lemma 4.1.3 is designed to explicitly handle this situation.

Lemma 4.1.3. Let (W,S) be a Coxeter system of type $\widetilde{C}_n$ and suppose $z \in \mathrm{FC}(\widetilde{C}_n)$ and $t \in S(\Gamma)$ such that z and tz are of type I, and let $s \in S(\Gamma)$ such that $m(s,t) \geq 3$ and stz is no longer of type I. Then

$$c_s c_{tz} = \begin{cases} c_{stz} + c_z, & \text{if } m(s,t) = 4, \\ c_z, & \text{if } m(s,t) = 3. \end{cases}$$

Proof. Since $m(s,t) \geq 3$ and $t(tz) < z$, we have $c_s c_{tz} = c_{\ulcorner st \urcorner (tz)} + c_{\llcorner st \lrcorner (tz)}$, by Lemma 3.2.7. Note that $\ulcorner st \urcorner (tz) = stz$. But stz is no longer of type I, forcing $stz = stsz'$ for some $z' \in \mathrm{FC}(\widetilde{C}_n)$ to be the case since sts must be a subword of stz. Thus, if $m(s,t) = 3$, then $stz \notin \mathrm{FC}(\widetilde{C}_n)$ and so $c_{stz} = 0$. Otherwise, if $m(s,t) > 3$, we have $c_{stz} \neq 0$. Also note $\llcorner st \lrcorner (tz) = z \in \mathrm{FC}(\widetilde{C}_n)$. Therefore we have the desired result. □

An example of the type of element encountered in Lemma 4.1.3 is given in Figure 4.1. Using Lemmas 4.1.2 and 4.1.3, we may compute products of elements of the c-basis that are of type I.

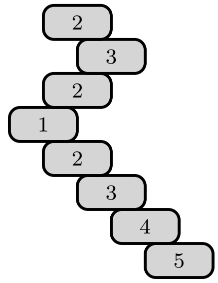


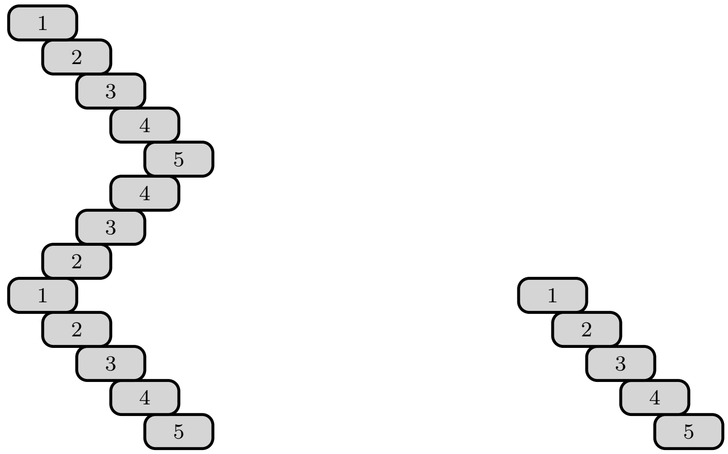
Figure 4.1: Example of a heap involved in the proof of Lemma 4.1.3.

Example 4.1.4. Let (W, S) be a Coxeter system of type $\widetilde{C}_4$. We will compute the product $c_1c_2c_3c_4c_5c_4c_3c_2c_1c_2c_3c_4c_5 \in \mathrm{TL}(\widetilde{C}_4)$ as follows. Note that Lemma 4.1.2 is used in every step of this computation and Lemma 4.1.3 is used on every line after its first noted occurrence. We see that

$$\begin{aligned}
c_1c_2c_3c_4c_5c_4c_3c_2c_1c_2c_3c_4c_5 &= c_1c_2c_3c_4c_5c_4c_3c_2c_{12345} && (4.1.1)\\
&= c_1c_2c_3c_4c_5c_4c_3\left(c_{212345} + c_{2345}\right)\\
&= c_1c_2c_3c_4c_5c_4\left(c_{3212345} + c_{345}\right) && (4.1.3)\\
&= c_1c_2c_3c_4c_5\left(c_{43212345} + c_{45}\right)\\
&= c_1c_2c_3c_4\left(c_{543212345} + c_{545} + c_5\right)\\
&= c_1c_2c_3\left(c_{4543212345} + c_{43212345} + 2c_{45}\right)\\
&= c_1c_2\left(c_{34543212345} + c_{3212345} + 2c_{345}\right)\\
&= c_1\left(c_{234543212345} + c_{212345} + 2c_{2345}\right)\\
&= c_{1234543212345} + c_{12345} + 2c_{12345}\\
&= c_{1234543212345} + 3c_{12345},
\end{aligned}$$

where the heaps of the elements in the final sum of the computation may be found in Figures 4.2(a) and 4.2(b).

Remark 4.1.5. Recall the notation presented in Section 2.3 for type I elements in Coxeter systems of type $\widetilde{C}_n$ where $H(\mathsf{z}_{1,5}^{R,3})$ and $H(\mathsf{z}_{1,5}^{R,1})$ may be found in Figures 4.2(a) and 4.2(b), respectively. Let $Z = c_nc_{n-1}\cdots c_2c_1c_2\cdots c_nc_{n+1}$. Then, using the techniques from Example 4.1.4,



(a) $H(1234543212345)$ (b) $H(12345)$

Figure 4.2: Heaps involved in Example 4.1.4.

we find that

$$
\begin{aligned}
c_1 c_2 \cdots c_{n+1} &= c_{\mathsf{z}_{1,n+1}^{R,1}}, \\
c_1 c_2 \cdots c_{n+1} Z &= c_{\mathsf{z}_{1,n+1}^{R,3}} + 3c_{\mathsf{z}_{1,n+1}^{R,1}}, \\
c_1 c_2 \cdots c_{n+1} Z^2 &= c_{\mathsf{z}_{1,n+1}^{R,5}} + 5c_{\mathsf{z}_{1,n+1}^{R,3}} + 10c_{\mathsf{z}_{1,n+1}^{R,1}}, \\
c_1 c_2 \cdots c_{n+1} Z^3 &= c_{\mathsf{z}_{1,n+1}^{R,7}} + 7c_{\mathsf{z}_{1,n+1}^{R,5}} + 21c_{\mathsf{z}_{1,n+1}^{R,3}} + 35c_{\mathsf{z}_{1,n+1}^{R,1}},
\end{aligned}
$$

and so on. Notice that the lead term in each sum is indexed by $\mathsf{z}_{1,n+1}^{R,2k+1}$, where k corresponds to the exponent of Z^k. Furthermore, there is also a term indexed by $\mathsf{z}_{1,n+1}^{R,2j+1}$ for every $0 \le j < k$. Moreover, we have the following calculations, which appear during the previous computations but have been relocated to draw attention to their similarities:

$$
\begin{aligned}
c_{n+1} Z &= c_{\mathsf{z}_{n+1,n+1}^{L,2}} + c_{(n+1)n(n+1)} + c_{n+1}, \\
c_{n+1} Z^2 &= c_{\mathsf{z}_{n+1,n+1}^{L,4}} + 4c_{\mathsf{z}_{n+1,n+1}^{L,2}} + 3c_{(n+1)n(n+1)} + 3c_{n+1}, \\
c_{n+1} Z^3 &= c_{\mathsf{z}_{n+1,n+1}^{L,6}} + 6c_{\mathsf{z}_{n+1,n+1}^{L,4}} + 15c_{\mathsf{z}_{n+1,n+1}^{L,2}} + 10c_{(n+1)n(n+1)} + 10c_{n+1}.
\end{aligned}
$$

Again we are finding that the lead term of each sum is indexed by $\mathsf{z}_{n+1,n+1}^{L,2k}$, where k corresponds to the exponent of Z^k and that there is also a term indexed by $\mathsf{z}_{n+1,n+1}^{L,2j}$ for every $0 \le j < k$.

These computations are fairly straightforward, requiring minimal tools. However, the computations of type II products require many tools and can become unexpectedly unwieldy. Section 4.2 is dedicated to a single lemma that is crucial in the calculation of type II products.

4.2 A crucial lemma

This section is dedicated to the proof of Lemma 4.2.1 in the context of Coxeter systems of type $\widetilde{C}_n$ where n is restricted to be even. This lemma is necessary in Section 4.3, where we present computations of type II products in $\mathrm{TL}(\widetilde{C}_n)$.

Lemma 4.2.1. Let (W, S) be a Coxeter system of type $\widetilde{C}_n$, where n is even. Consider the reduced products

$$y = X_{1,n+1}$$

and

$$w = X_{3,n+1}X_{2,n}X_{1,n+1}X_{2,n}X_{1,n+1},$$

which are both elements of $\mathrm{FC}(\widetilde{C}_n)$. Then $\mu(y, w) = 1$.

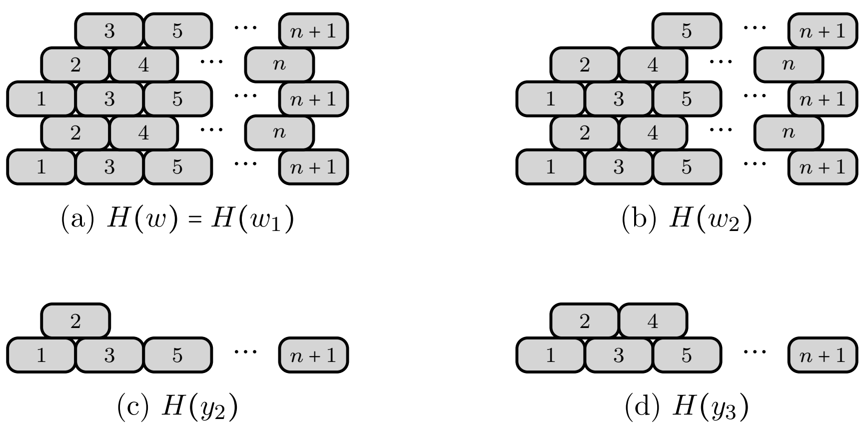


Figure 4.3: Heaps involved in the proof of Lemma 4.2.1.

Proof. Suppose $n \geq 4$ is even. The proof that follows is easily modified for the case when $n = 2$. We will make repeated use of previous lemmas, especially Lemmas 3.1.4 and 3.1.5, to achieve the desired result. Along the way, a sequence $\{(y_k, w_k, I_k)\}$ will be defined. First, define

$$y_1 := y, \qquad w_1 := w, \qquad I_1 := \{2, 3\}.$$

One may find $H(w)$ in Figure 4.3(a). Then

$$\mathcal{L}(y_1) = \{1, 3, \ldots, n+1\}$$

and

$$\mathcal{L}(w_1) = \{3, 5, \ldots, n+1\},$$

with $\mathcal{L}(w_1) \subseteq \mathcal{L}(y_1)$. Also, note that $\ell(w_1) - \ell(y_1) \geq 3$ and $|\mathcal{L}(w_1) \cap I_1| = |\mathcal{L}(y_1) \cap I_1| = 1$. With respect to I_1, we obtain

$$y_2 := \lceil 23 \rceil (y_1) = 2X_{1,n+1},$$

where $H(y_2)$ may be found in Figure 4.3(c), and

$$w_2 := \lfloor 23 \rfloor (w_1) = X_{5,n+1}X_{2,n}X_{1,n+1}X_{2,n}X_{1,n+1},$$

where $H(w_2)$ is presented in Figure 4.3(b). Since $m(2,3) = 3$ and $\lfloor 23 \rfloor (w_1)$ exists, Lemma 3.1.4 may be applied to determine

$$\begin{aligned} \mu(y, w) &= \mu(y_1, w_1) \\ &= \mu(\lceil 23 \rceil (y_1), \lfloor 23 \rfloor (w_1)) \\ &= \mu(y_2, w_2). \end{aligned}$$

Next, let $I_2 = \{4, 5\}$. Then

$$\mathcal{L}(y_2) = \{2\} \cup \{5, \ldots, n+1\}$$

and

$$\mathcal{L}(w_2) = \{2\} \cup \{5, \ldots, n+1\},$$

with $\mathcal{L}(w_2) \subseteq \mathcal{L}(y_2)$. Again, it is the case that $\ell(w_2) - \ell(y_2) \geq 3$ and $|\mathcal{L}(w_2) \cap I_2| = |\mathcal{L}(y_2) \cap I_2| = 1$. With respect to I_2, we obtain

$$y_3 := \lceil 45 \rceil (y_2) = X_{2,4}X_{1,n+1},$$

where $H(y_3)$ is provided in Figure 4.3(d), and

$$w_3 := \lfloor 45 \rfloor (w_2) = X_{7,n+1}X_{2,n}X_{1,n+1}X_{2,n}X_{1,n+1}.$$

For the remainder of the proof, we will not provide heap representations for the remaining sequence of group elements. However, the idea is to use Lemmas 3.1.4 and 3.1.5 to 'whittle away' at the heap of w and build upon the heap of y, so that the two will eventually differ by a single generator from $S(\Gamma)$, rendering a μ-value of 1 by Lemma 3.1.3(b). As $m(4,5)=3$ and $\lfloor_{45}(w_2)$ exists, Lemma 3.1.4 may be applied to see

$$\begin{aligned}\mu(y,w) &= \mu(y_1,w_1)\\ &= \mu(y_2,w_2) && \text{(by above)}\\ &= \mu\left(\lceil^{45}(y_2),\lfloor_{45}(w_2)\right)\\ &= \mu(y_3,w_3).\end{aligned}$$

Proceeding in this fashion, define

$$I_3 := \{6,7\}, I_4 := \{8,9\}, \ldots, I_{(n-2)/2} := \{n-2,n-1\}$$

so that, for $k=3,4,\ldots,(n-2)/2$, we have

$$y_{k+1} := \lceil^{I_k}(y_k) \text{ and } w_{k+1} := \lfloor_{I_k}(w_k).$$

Note that this sequence of star operations has two effects. First, the sequence increases the length of y by one in each step by multiplying on the left by $2,4,\ldots,n-4$, and $n-2$ (i.e., $X_{2,n-2}$). Second, the length of w is decreased by one in each step by multiplying on the left by $3,5,\ldots,n-3$, and $n-1$ (i.e., $X_{3,n-1}$). This process renders

$$y_{n/2} := X_{2,n-2}X_{1,n+1}$$

and

$$w_{n/2} := (n+1)X_{2,n}X_{1,n+1}X_{2,n}X_{1,n+1}.$$

At each step, the conditions of Lemma 3.1.4 are satisfied. That is, for each k, $\mathcal{L}(w_k)\subseteq\mathcal{L}(y_k)$, $\ell(w_k)-\ell(y_k)\geq 3$, $w_{k+1}=\lfloor_{I_k}(w_k)$ exists, $m(2k,2k+1)=3$, and $|\mathcal{L}(w_k)\cap I_k| = |\mathcal{L}(y_k)\cap I_k| = 1$. Thus, it shall be true that

$$\begin{aligned}\mu(y,w) &= \mu(y_1,w_1)\\ &= \mu(y_2,w_2)\\ &\ \vdots\\ &= \mu\left(y_{(n-2)/2},w_{(n-2)/2}\right)\\ &= \mu\left(y_{n/2},w_{n/2}\right).\end{aligned}$$

At this point, there does not exist any $I=\{s,t\}$ such that $m(s,t)=3$ and $\lfloor_{st}\left(w_{n/2}\right)$ exists. So Lemma 3.1.4 is not applicable at this time. Instead, Lemma 3.1.5 shall be of use in the following manner. Define

$$I := I_{n/2} = \{n,n+1\}$$

and

$$w'_{n/2} \coloneqq X_{2,n} X_{1,n+1} X_{2,n} X_{1,n+1},$$

so that $\ulcorner_I \left(w'_{n/2} \right) = w_{n/2}$. Note that

$$\mathcal{L}\left(y_{n/2}\right) = \{2, 4, \ldots, n-2\} \cup \{n+1\}$$

and

$$\mathcal{L}\left(w'_{n/2}\right) = \{2, 4, \ldots, n\}.$$

Then

$$\mathcal{L}\left(y_{n/2}\right) \cap I_{n/2} \neq \mathcal{L}\left(w'_{n/2}\right) \cap I_{n/2},$$

while

$$|\mathcal{L}\left(y_{n/2}\right) \cap I_{n/2}| = |\mathcal{L}\left(w'_{n/2}\right) \cap I_{n/2}| = 1.$$

With respect to $I = I_{n/2}$, we obtain

$$y_{(n+2)/2} \coloneqq \ulcorner_I \left(y_{n/2}\right) = X_{2,n} X_{1,n+1}$$

and

$$\llcorner_I \left(w'_{n/2}\right) = X_{2,n-2} X_{1,n+1} X_{2,n} X_{1,n+1}.$$

However, notice that $\llcorner_I \left(y_{n/2}\right)$ is not defined for $I = I_{n/2}$. Then $\mu\left(\llcorner_I \left(y_{n/2}\right), w'_{n/2}\right) = 0$. Then Lemma 3.1.5 indicates that

$$\begin{aligned}
\mu\left(y_{n/2}, \ulcorner_I \left(w'_{n/2}\right)\right) &= \mu\left(\ulcorner_I \left(y_{n/2}\right), w'_{n/2}\right) + \mu\left(\llcorner_I \left(y_{n/2}\right), w'_{n/2}\right) - \mu\left(y_{n/2}, \llcorner_I \left(w'_{n/2}\right)\right) \\
&= \mu\left(\ulcorner_I \left(y_{n/2}\right), w'_{n/2}\right) + 0 - \mu\left(y_{n/2}, \llcorner_I \left(w'_{n/2}\right)\right) \\
&= \mu\left(\ulcorner_I \left(y_{n/2}\right), w'_{n/2}\right) - \mu\left(y_{n/2}, \llcorner_I \left(w'_{n/2}\right)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\mu\left(y, w\right) &= \mu\left(y_{n/2}, w_{n/2}\right) \\
&= \mu\left(y_{n/2}, \ulcorner_I \left(w'_{n/2}\right)\right) \\
&= \mu\left(\ulcorner_I \left(y_{n/2}\right), w'_{n/2}\right) - \mu\left(y_{n/2}, \llcorner_I \left(w'_{n/2}\right)\right) \\
&= \mu\left(y_{(n+2)/2}, w'_{n/2}\right) - \mu\left(y_{n/2}, \llcorner_I \left(w'_{n/2}\right)\right).
\end{aligned}$$

We utilize Lemma 3.1.5 one more time to determine $\mu\left(y_{(n+2)/2}, w'_{n/2}\right)$. Define

$$I_{(n+2)/2} := \{1, 2\}$$

and

$$w'_{(n+2)/2} := X_{4,n} X_{1,n+1} X_{2,n} X_{1,n+1}$$

so that $w_{(n+2)/2} := \lceil_{12}\left(w'_{(n+2)/2}\right)$ (to maintain the notation of $\lceil_{I}\left(w'_k\right) = w_k$). Also, notice

$$\mathcal{L}\left(y_{(n+2)/2}\right) = \{2, \ldots, n\}$$

and

$$\mathcal{L}\left(w'_{(n+2)/2}\right) = \{1\} \cup \{4, \ldots, n\}.$$

Then

$$\mathcal{L}\left(y_{(n+2)/2}\right) \cap I_{(n+2)/2} \neq \mathcal{L}\left(w'_{(n+2)/2}\right) \cap I_{(n+2)/2}$$

while

$$|\mathcal{L}\left(y_{(n+2)/2}\right) \cap I_{(n+2)/2}| = |\mathcal{L}\left(w'_{(n+2)/2}\right) \cap I_{(n+2)/2}| = 1.$$

With respect to $I_{(n+2)/2}$, we obtain

$$y_{(n+4)/2} := \lceil_{12}\left(y_{(n+2)/2}\right) = 1 X_{2,n} X_{1,n+1}$$

and

$$\lfloor_{12}\left(y_{(n+2)/2}\right) = X_{4,n} X_{1,n+1}.$$

Maintaining consistent labeling, define

$$w_{(n+4)/2} := w'_{(n+2)/2}.$$

Note that $\lfloor_{12}\left(w'_{(n+2)/2}\right)$ is not defined, meaning $\mu\left(y_{(n+2)/2}, \lfloor_{12}\left(w'_{(n+2)/2}\right)\right) = 0$. Then, by Lemma 3.1.5, we get

$$\begin{aligned}
\mu\left(y_{(n+2)/2}, w'_{n/2}\right) &= \mu\left(y_{(n+2)/2}, \lceil_{12}\left(w_{(n+2)/2}\right)\right) \\
&= \mu\left(\lceil_{12}\left(y_{(n+2)/2}\right), w'_{(n+2)/2}\right) + \mu\left(\lfloor_{12}\left(y_{(n+2)/2}\right), w'_{(n+2)/2}\right) \\
&\quad - \mu\left(y_{(n+2)/2}, \lfloor_{12}\left(w'_{(n+2)/2}\right)\right) \\
&= \mu\left(\lceil_{12}\left(y_{(n+2)/2}\right), w'_{(n+2)/2}\right) + \mu\left(\lfloor_{12}\left(y_{(n+2)/2}\right), w'_{(n+2)/2}\right) - 0 \\
&= \mu\left(y_{(n+4)/2}, w_{(n+4)/2}\right) + \mu\left(\lfloor_{12}\left(y_{(n+2)/2}\right), w'_{(n+2)/2}\right).
\end{aligned}$$

Therefore, recalling $I = \{n, n+1\}$,

$$\begin{aligned}\mu(y,w) &= \mu\left(y_{(n+2)/2}, w'_{n/2}\right) - \mu\left(y_{n/2}, \lfloor_{I}\left(w'_{n/2}\right)\right)\\ &= \mu\left(y_{(n+4)/2}, w_{(n+4)/2}\right) + \mu\left(\lfloor_{12}\left(y_{(n+2)/2}\right), w'_{(n+2)/2}\right) - \mu\left(y_{n/2}, \lfloor_{I}\left(w'_{n/2}\right)\right).\end{aligned}$$

Now, we make several applications of Lemma 3.1.4 to evaluate $\mu\left(y_{(n+4)/2}, w_{(n+4)/2}\right)$. Define

$$I_{(n+4)/2} = \{3,4\},\ I_{(n+6)/2} = \{5,6\},\ \ldots,\ I_n = \{n-1, n\}$$

so that, for each $k = (n+4)/2, (n+6)/2, \ldots, n$, there shall exist

$$y_{k+1} := \lceil^{I_k}(y_k) \text{ and } w_{k+1} := \lfloor_{I_k}(w_k).$$

Note that this sequence of star operations has the effect of multiplying $y_{(n+4)/2}$ on the left by $3, 5, \ldots, n-3, n-1$ (i.e, $X_{3,n-1}$). In each step, the length is increased by one. Similarly, $w_{(n+4)/2}$ is multiplied on the left by $4, 6, \ldots, n-2, n$ (i.e., $X_{4,n}$), where the length is decreased by one in each step. Ultimately, there shall be

$$y_{n+1} := \lceil^{I_n}(y_n) = X_{1,n-1}X_{2,n}X_{1,n+1}$$

and

$$w_{n+1} := \lfloor_{I_n}(w_n) = X_{1,n+1}X_{2,n}X_{1,n+1}.$$

At each step of the sequence, the conditions of Lemma 3.1.4 are satisfied. That is, for each $k \le n$, $\mathcal{L}(w_k) \subseteq \mathcal{L}(y_k)$, $\ell(w_k) - \ell(y_k) \ge 3$, $w_{k+1} = \lfloor_{I_k}(w_k)$ exists, $m(k-1,k) = 3$, and $|\mathcal{L}(w_k) \cap I_k| = |\mathcal{L}(y_k) \cap I_k| = 1$. Thus,

$$\begin{aligned}\mu\left(y_{(n+4)/2}, w_{(n+4)/2}\right) &= \mu\left(y_{(n+6)/2}, w_{(n+6)/2}\right)\\ &\vdots\\ &= \mu(y_n, w_n)\\ &= \mu(y_{n+1}, w_{n+1}).\end{aligned}$$

Observe $y_{n+1} \le w_{n+1}$ and $\ell(y_{n+1}) = \ell(w_{n+1}) - 1$. Then Lemma 3.1.3(b) indicates that

$$\mu(y_{n+1}, w_{n+1}) = 1.$$

This implies (with $I = \{n, n+1\}$) that

$$\begin{aligned}\mu(y,w) &= \mu\left(y_{(n+4)/2}, w_{(n+4)/2}\right) + \mu\left(\lfloor_{12}\left(y_{(n+2)/2}\right), w'_{(n+2)/2}\right) - \mu\left(y_{n/2}, \lfloor_{I}\left(w'_{n/2}\right)\right)\\ &= 1 + \mu\left(\lfloor_{12}\left(y_{(n+2)/2}\right), w'_{(n+2)/2}\right) - \mu\left(y_{n/2}, \lfloor_{I}\left(w'_{n/2}\right)\right).\end{aligned}$$

Now, recall that

$$\begin{aligned}\underline{\lfloor 12}\left(y_{(n+2)/2}\right) &= X_{4,n}X_{1,n+1},\\ w'_{(n+2)/2} &= X_{4,n}X_{1,n+1}X_{2,n}X_{1,n+1},\\ y_{n/2} &= X_{2,n-2}X_{1,n+1},\\ \underline{\lfloor I}\left(w'_{n/2}\right) &= X_{2,n-2}X_{1,n+1}X_{2,n}X_{1,n+1},\end{aligned}$$

and let ϕ be the graph automorphism of $\widetilde{C}_n$ that is the reflection about the graph's vertical axis of symmetry. This map swaps 1 with $n+1$, 2 with n, 3 with $n-1$, and so on. Thus $P_{y,w} = P_{\phi(y),\phi(w)}$, and in particular, $\mu(y,w) = \mu(\phi(y),\phi(w))$. Then notice

$$\phi\left(\underline{\lfloor 12}\left(y_{(n+2)/2}\right)\right) = y_{n/2}$$

and

$$\phi\left(w'_{(n+2)/2}\right) = \underline{\lfloor I}\left(w'_{n/2}\right).$$

Therefore, we have

$$\begin{aligned}\mu\left(\underline{\lfloor 12}\left(y_{(n+2)/2}\right), w'_{(n+2)/2}\right) &= \mu\left(\phi\left(\underline{\lfloor 12}\left(y_{(n+2)/2}\right)\right), \phi\left(w'_{(n+2)/2}\right)\right)\\ &= \mu\left(y_{n/2}, \underline{\lfloor I}\left(w'_{n/2}\right)\right).\end{aligned}$$

This implies

$$\begin{aligned}\mu(y,w) &= 1 + \mu\left(\underline{\lfloor 12}\left(y_{(n+2)/2}\right), w'_{(n+2)/2}\right) - \mu\left(y_{n/2}, \underline{\lfloor I}\left(w'_{n/2}\right)\right)\\ &= 1 + \mu\left(y_{n/2}, \underline{\lfloor I}\left(w'_{n/2}\right)\right) - \mu\left(y_{n/2}, \underline{\lfloor I}\left(w'_{n/2}\right)\right)\\ &= 1.\end{aligned}$$

Thus the desired result is achieved, concluding the proof. □

4.3 Computations involving type II elements

Now that we have Lemma 4.2.1, we may quickly develop a few more results that will allow us to pursue computations of type II products in TL($\widetilde{C}_n$), where n is even. The rest of this section will be dedicated to the computation of such products.

Proposition 4.3.1. Let (W,S) be a Coxeter system of type $\widetilde{C}_n$ with n even. Let $y, w \in$ FC($\widetilde{C}_n$) be the reduced products

$$y = X_{1,n+1}Y_{k-j}$$

and

$$w = X_{3,n+1}Y_k,$$

respectively, where Y_i consists of i copies of $X_{2,n}X_{1,n+1}$ and we define $Y_i = e$ for $i \leq 0$. Then

$$\mu(y,w) = \begin{cases} 1, & \text{if } j = 2 \\ 0, & \text{otherwise.} \end{cases}$$

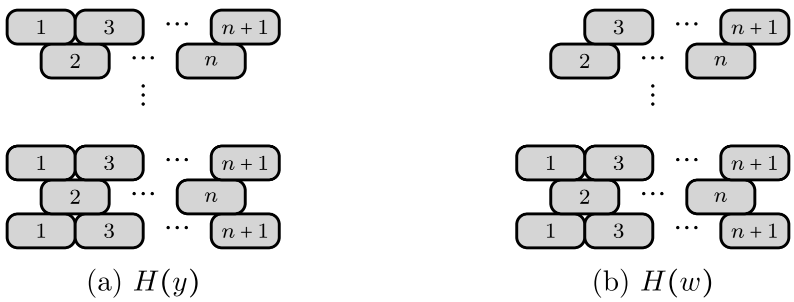


Figure 4.4: Heaps involved in the proof of Lemma 4.3.1.

Proof. As a reference, $H(y)$ and $H(w)$ are provided in Figure 4.4(a), where there is a total of $2(k-j)+1$ layers, and Figure 4.4(b), where there is a total of $2k+1$ layers, respectively. First assume j is odd. Then

$$\ell(y) = \frac{n+2}{2} + (k-j)(n+1)$$

and

$$\ell(w) = \frac{n}{2} + k(n+1).$$

Now compute the difference in length of w and y;

$$\begin{aligned}\ell(w) - \ell(y) &= \left(\frac{n}{2} + k(n+1)\right) - \left(\frac{n+2}{2} + (k-j)(n+1)\right) \\ &= \frac{n}{2} + k(n+1) - \frac{n+2}{2} - k(n+1) + j(n+1) \\ &= j(n+1) - 1\end{aligned}$$

which must be even as both j and $n+1$ are odd. Therefore, y and w must have the same parity. So $\mu(y,w) = 0$ for j odd by Lemma 3.1.3(a).

Next, we consider j even. If $j = 0$, we see $y = X_{1,n+1}Y_k$, and $w = X_{3,n+1}Y_k$, in which case $y \nleq w$, indicating $\mu(y,w) = 0$ by Lemma 3.1.3(a). So assume $k \geq 2$. We chase through a similar set of star operations as seen in Lemma 4.2.1 to see

$$\mu(y,w) = \mu(X_{1,n-1}Y_{k-j+1}, X_{1,n+1}Y_{k-1}).$$

Let

$$y' := X_{1,n-1}Y_{k-j+1}$$

and

$$w' := X_{1,n+1}Y_{k-1}.$$

If $j = 2$, then $\ell(y') = \ell(w') - 1$, and thus

$$\mu(y,w) = \mu(y',w') = \mu(X_{1,n-1}Y_{k-1}, X_{1,n+1}Y_{k-1}) = 1$$

via Lemmas 3.1.3(b) and 3.1.4. However, if $k > 2$, we see that

$$\mathcal{L}(y') = \{1,3,5,\ldots,n-1\}$$

and

$$\mathcal{L}(w') = \{1,3,5,\ldots,n-1,n+1\},$$

where $\mathcal{L}(y') \subseteq \mathcal{L}(w')$ and $\mathcal{L}(w') \backslash \mathcal{L}(y') = \{n+1\}$. Yet $y' \neq (n+1)w'$. Therefore, if $j > 2$, then

$$\mu(y,w) = \mu(y',w') = 0$$

by Lemma 3.1.3(c), thus concluding the proof. □

We have now completed the final necessary μ-value calculation and will begin introducing computations of type II products in the c-basis.

Corollary 4.3.2. Let (W,S) be a Coxeter system of type $\widetilde{C}_n$, where n is even and suppose we have the reduced product

$$w = X_{3,n+1}Y_k,$$

where $Y_k = e$ for $k < 1$. Then

$$c_1c_w = c_{1w} + c_{X_{1,n+1}Y_{k-2}}.$$

Proof. Recall $H(w)$ may be found in Figure 4.4(b) with a total of $2k+1$ layers. First, recall Theorem 3.2.6, where for any $w \in W(\widetilde{C}_n)$ and any $s \in S$, we have

$$c_sc_w = \begin{cases} c_{sw} + \sum \mu(y,w)\,c_y, & \text{if } \ell(sw) > \ell(w) \\ \delta c_w, & \text{otherwise,} \end{cases}$$

where $c_x = 0$ for $x \notin \mathrm{FC}(\widetilde{C}_n)$ and we are summing over all $y \in W(\widetilde{C}_n)$ satisfying $sy < y < w$ in the Bruhat order. Note that

$$1w = X_{1,n+1}Y_k$$

(see Figure 4.4(a) with a total of $2k+1$ layers) and that

$$\ell(1w) > \ell(w).$$

Then

$$c_1 c_w = c_{1w} + \sum_{1y<y<w} \mu(y,w)\, c_y.$$

By Proposition 4.3.1,

$$\mu(X_{1,n+1}Y_{k-2}, X_{1,n+1}Y_k) = 1,$$

meaning $c_{X_{1,n+1}Y_{k-2}}$ appears in the sum above.

Now we proceed by arguing that no other terms with nonzero coefficient appear in the sum. Let $y \in W(\widetilde{C}_n)$ such that

(a) $y < w$,

(b) $1y < y$, and

(c) $y \in \mathrm{FC}(\widetilde{C}_n)$.

So $1 \in \mathcal{L}(y)$. But $y < w$, and so $\ell(w) - \ell(y) > 1$. By Lemmas 3.1.3(c) and 3.1.3(d), we must have $\mathcal{L}(w) \subseteq \mathcal{L}(y)$ and $\mathcal{R}(w) \subseteq \mathcal{R}(y)$ in order for $\mu(y,w) \neq 0$. Note that

$$\mathcal{L}(w) = \{3, 5, \ldots, n+1\}$$

and

$$\mathcal{R}(w) = \{1, 3, 5, \ldots, n+1\}.$$

Therefore, we must have

$$\mathcal{L}(y) = \{1, 3, 5, \ldots, n+1\} = \mathcal{R}(y).$$

Since $y \in \mathrm{FC}(\widetilde{C}_n)$, it must be the case that y is the reduced product $X_{1,n+1}Y_l$. But by Proposition 4.3.1, $\mu(y,w) = 1$ only when $l = k-2$. So $c_{X_{1,n+1}Y_{k-2}}$ is the only term appearing in the sum. Thus $c_1 c_w = c_{1w} + c_{X_{1,n+1}Y_{k-2}}$. □

We now systematically handle type II products in the c-basis. The first step is to compute $c_2 c_4 \cdots c_n c_1 c_3 \cdots c_{n+1}$.

Lemma 4.3.3. Let (W, S) be a Coxeter system of type $\widetilde{C}_n$, where n is even. Let $w \in \mathrm{FC}(\widetilde{C}_n)$ be the reduced product $X_{2,n}X_{1,n+1}$. Then

$$c_2c_4\cdots c_nc_1c_3\cdots c_{n-1}c_{n+1} = c_{X_{2,n}X_{1,n+1}}.$$

Proof. The calculation of $c_2c_4\cdots c_nc_1c_3\cdots c_{n+1}$ follows from Remark 4.1.1. That is,

$$c_2c_4\cdots c_nc_1c_3\cdots c_{n-1}c_{n+1} = c_{X_{2,n}X_{1,n+1}}.$$

□

Now that we have the foundation of our type II products, we will proceed by building upward and right to left, when considering the heap of our group element (see Figure 2.7(b)). That is, we want to recursively build the type II products that follow the pattern right to left of the reduced product $w = X_{1,n+1}Y_k$ where $k \in \mathbb{N}$.

The next result will help manage computations where we encounter elements that are not fully commutative in $W(\widetilde{C}_n)$. An example of such a situation comes in calculating c_3c_{2135} in the context of $\mathrm{TL}(\widetilde{C}_4)$.

Lemma 4.3.4. Let (W, S) be a Coxeter system of type $\widetilde{C}_n$, where n is even. Consider the reduced product $w = X_{2,i}X_{1,n+1}Y_k \in \mathrm{FC}(\widetilde{C}_n)$ with $k \geq 0$, $2 < i \leq n$, and i even. Then

$$c_{i+1}c_w = \begin{cases} c_{(i+1)w} + c_y, & \text{if } i = n \\ c_y, & \text{otherwise.} \end{cases}$$

where y is the reduced product $X_{2,i-2}X_{1,n+1}Y_k$. In particular,

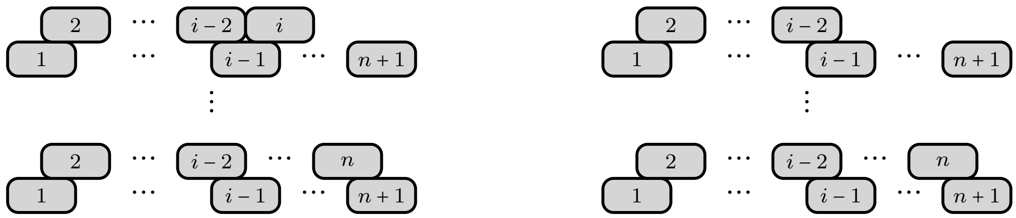
$$c_{i+1}c_{w'} = \begin{cases} c_{(i+1)w'} + c_{y'}, & \text{if } i = n \\ c_{y'}, & \text{otherwise.} \end{cases}$$

where we have the reduced products $w' = X_{2,i}X_{1,n+1}$ and $y' = X_{2,i-2}X_{1,n+1}$.

Proof. For reference, the heaps of w and y may be found in Figures 4.5(a) and 4.5(b), respectively. Note that this result includes the case where $i = n$. First, notice the second statement follows from the first. Next, note the generators i and $i+1$ do not commute and $i + 1 \in \mathcal{L}(y)$. Then by Lemma 3.2.7, we have

$$c_{i+1}c_w = c_{(i+1)w} + c_y.$$

Note that if $i = n$, $(i+1)w \in \mathrm{FC}(\widetilde{C}_n)$, but if $i < n$, then $(i+1)w \notin \mathrm{FC}(\widetilde{C}_n)$, and so $c_{(i+1)w} = 0$ by definition. Hence the desired result. □



(a) $H(w)$ (b) $H(y)$

Figure 4.5: Heaps involved in the proof of Lemma 4.3.4.

(a) $H(13524135)$ (b) $H(135)$

Figure 4.6: Heaps involved in Example 4.3.5.

Now that we have a library of tools, we should see an example.

Example 4.3.5. Consider the Coxeter system (W, S) of type $\widetilde{C}_4$. Then we may compute $c_1c_3c_5c_2c_4c_1c_3c_5$ as follows:

$$\begin{aligned} c_1c_3c_5c_2c_4c_1c_3c_5 &= c_1c_3c_5c_{24135} && (4.3.3) \\ &= c_1c_3\left(c_{524135} + c_{2135}\right) && (4.3.4) \\ &= c_1\left(c_{3524135} + c_{32135} + c_{135}\right) && (3.2.8 \text{ and } 4.3.4) \\ &= c_1\left(c_{3524135} + 0 + c_{135}\right) && (3.2.6) \\ &= c_{13524135} + \delta c_{135} && (4.3.2 \text{ and } 3.2.6). \end{aligned}$$

The heaps of the last two group elements involved in this computation are provided in Figures 4.6(a) and 4.6(b).

Remark 4.3.6. For Coxeter systems of type $\widetilde{C}_n$, where n is even, we follow the same pattern and tools to find

$$
\begin{aligned}
c_1 \cdots c_{n+1} \left(c_2 \cdots c_n c_1 \cdots c_{n+1} \right)^2 &= c_{X_{1,n+1} Y_2} + 2\delta c_{X_{1,n+1} Y_1} + \left(\delta^2 + 1 \right) c_{X_{1,n+1}}, \\
c_1 \cdots c_{n+1} \left(c_2 \cdots c_n c_1 \cdots c_{n+1} \right)^3 &= c_{X_{1,n+1} Y_3} + 3\delta c_{X_{1,n+1} Y_2} \\
&\quad + \left(3\delta^2 + 2 \right) c_{X_{1,n+1} Y_1} + \left(\delta^3 + 3\delta \right) c_{X_{1,n+1}}, \\
c_1 \cdots c_{n+1} \left(c_2 \cdots c_n c_1 \cdots c_{n+1} \right)^4 &= c_{X_{1,n+1} Y_4} + (4\delta)\, c_{X_{1,n+1} Y_3} \\
&\quad + \left(6\delta^2 + 3 \right) c_{X_{1,n+1} Y_2} + \left(4\delta^3 + 8\delta \right) c_{X_{1,n+1} Y_1} \\
&\quad + \left(\delta^4 + 6\delta^2 + 2 \right) c_{X_{1,n+1}},
\end{aligned}
$$

and so on. As one can see, computations can quickly become quite cumbersome, wherein lies much of the motivation for finding a diagrammatic representation that renders computation of large products a manageable task.

Chapter 5

Diagram algebras

The rest of this thesis is dedicated to the presentation of a diagram algebra that we conjecture to be a faithful representation of $\mathrm{TL}(\widetilde{C}_n)$, for n even. We will be providing a correspondence and significant evidence that our diagram algebra is very likely to be an accurate representation. However, we will only painting the big picture and providing a proof of concept at this time.

5.1 Undecorated diagrams

To develop our diagram algebra, we first introduce undecorated diagrams. Let k be a non-negative integer. We define a *standard k-box* to be a rectangle with $2k$ ticks, named *nodes* which will be labeled as in Figure 5.1. The top of the rectangle will be called the *north face*, and the bottom will be called the *south face*. At times, we will find convenience in embedding the standard k-box in the plane so that the lower left corner is at the origin, allowing us to locate each node i (respectively, i') at the point $(i, 1)$ (respectively, $(i, 0)$).

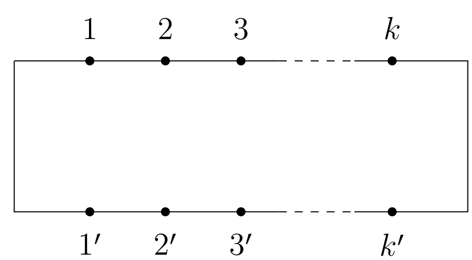
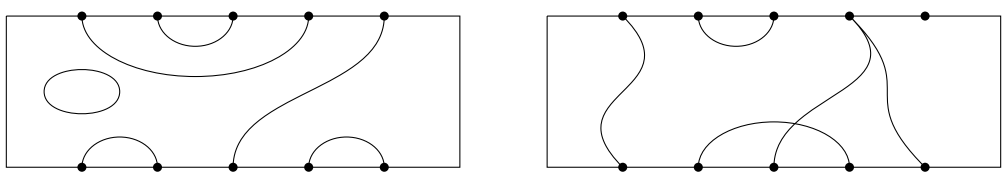


Figure 5.1: Standard k-box.

Now we define a *concrete pseudo k-diagram* consisting of a finite number of disjoint planar curves, called *edges*, embedded in the standard k-box where each node of the box is the endpoint of exactly one edge, meeting the box transversely, and all other edges must be closed (isotopic to circles) and disjoint from the box.

We now define an equivalence relation on the set of concrete pseudo k-diagrams. Two concrete pseudo k-diagrams are *(isotopically) equivalent* if one concrete diagram may be obtained from the other by isotopically deforming the edges such that every intermediate diagram is also a concrete pseudo k-diagram. A *pseudo k-diagram* is defined to be the equivalence class of all equivalent concrete pseudo k-diagrams. We denote the set of pseudo k-diagrams by $T_k(\varnothing)$.

Example 5.1.1. In Figure 5.2, one will find an example of a concrete pseudo 5-diagram, and an example of one that is not.



(a) A concrete pseudo 5-diagram (b) Not a concrete pseudo 5-diagram

Figure 5.2: Examples of diagrams.

When visually representing a pseudo k-diagram, an arbitrary representative will be chosen from the continuum of equivalent concrete k-diagrams.

A closed curve occurring in a pseudo k-diagram is referred to as a *loop edge*, or loop. Most diagram algebras eliminate loops through a process akin to scaling. We intend to preserve the presence of some loops in our diagram algebra in order to obtain infinitely many diagrams, hence our usage of the term pseudo. The appearance of $\varnothing$ when defining $T_k(\varnothing)$ above indicates that the diagrams are undecorated. We will discuss decorated diagrams in subsequent sections.

Let d be a diagram. If there is an edge E in d joining a node i (in the north face) to a node j' (in the south face), then we call E a *propagating edge* from i to j. If E joins i to i', we call E a *vertical propagating edge*. In the case an edge is not a loop or propagating edge, it will be called a *non-propagating edge*.

If there is at least one propagating edge in a diagram d, we say d is *dammed*. Otherwise, d is *undammed* (which can only happen when k is even). Furthermore, the number of non-propagating edges in the north face is equal to the number of non-propagating edges in the south face. Now let $a : T_k(\varnothing) \longmapsto \mathbb{Z}^+ \cup \{0\}$ be defined via

$$a(d) = \text{ number of non-propagating edges in the north face of } d.$$

The diagram d_e, appearing in Figure 5.3, is the only diagram with no loops and a-value 0. Note that $k/2$ is the upper bound of $a(d)$, for any diagram d. If k is even and $a(d) = d/2$, d is undammed, and if k is odd and $a(d) = (k-1)/2$, then d has a single propagating edge.

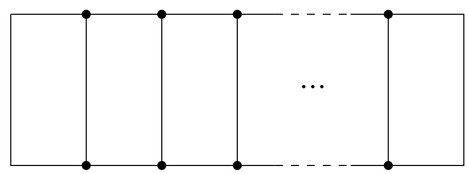
$d_e :=$ 

Figure 5.3: The identity diagram.

Next we define an associative algebra using pseudo k-diagrams as a basis. Let R be a commutative ring with unity. The associative algebra $\mathcal{P}_k(\varnothing)$ over R is the free R-module having $T_k(\varnothing)$ as a basis, with multiplication (referred to as diagram concatenation) defined as follows. Let $d_1, d_2 \in T_k(\varnothing)$. Then the product $d_1 d_2$ is the element of $T_k(\varnothing)$ obtained by stacking d_1 on top of d_2 in the manner satisfying the coincidence of the nodes i' of d_1 and i of d_2, rescaling by a factor of 1/2, and then applying the appropriate translation to recover a standard k-box.

Example 5.1.2. Figure 5.4 depicts the product of three basis diagrams from $\mathcal{P}_5(\varnothing)$.

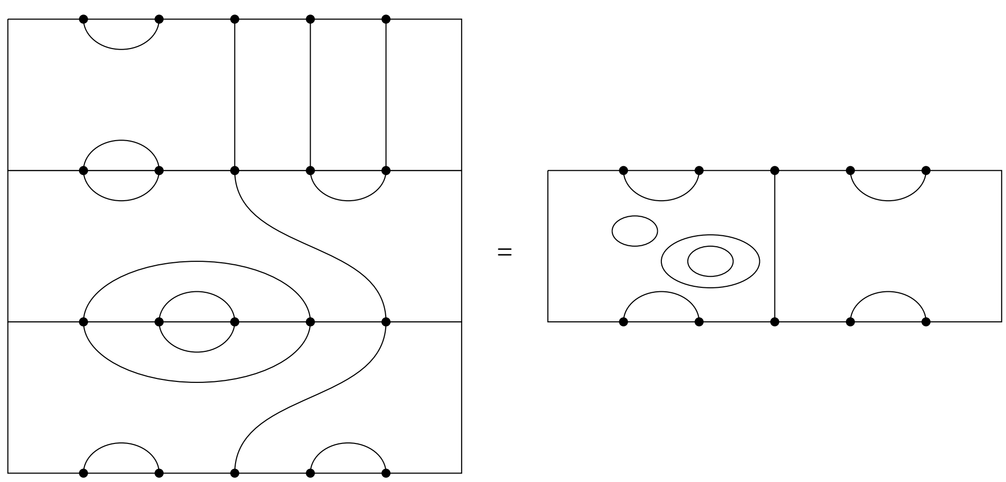


Figure 5.4: Example of multiplication in $\mathcal{P}_5(\varnothing)$.

Now suppose $R = \mathbb{Z}[\delta]$, the ring of polynomials in δ with integer coefficients. Let $\mathbb{D}\mathrm{TL}(A_n)$ be the associative $\mathbb{Z}[\delta]$-algebra equal to the quotient of $\mathcal{P}_{n+1}(\varnothing)$ determined by the relation depicted in Figure 5.5.

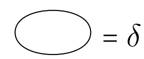


Figure 5.5: Defining relation of $\mathbb{D}\mathrm{TL}(A_n)$.

It is well-known that $\mathbb{D}\mathrm{TL}(A_n)$ is the free $\mathbb{Z}[\delta]$-module with basis given by the elements of $T_{n+1}(\varnothing)$ having no loops [17, 21]. The multiplication is inherited from the multiplication on $\mathcal{P}_{n+1}(\varnothing)$ except we multiply by a factor of δ for each resulting loop and then discard the loop. We will refer to $\mathbb{D}\mathrm{TL}(A_n)$ as the *type A Temperley–Lieb diagram algebra.*

Example 5.1.3. Figure 5.6 depicts the product of three basis diagrams from $\mathbb{D}\mathrm{TL}(A_4)$. Note that this is the same product of diagrams as in Example 5.1.2, however, in this case the three loops are replaced with the coefficient δ^3.

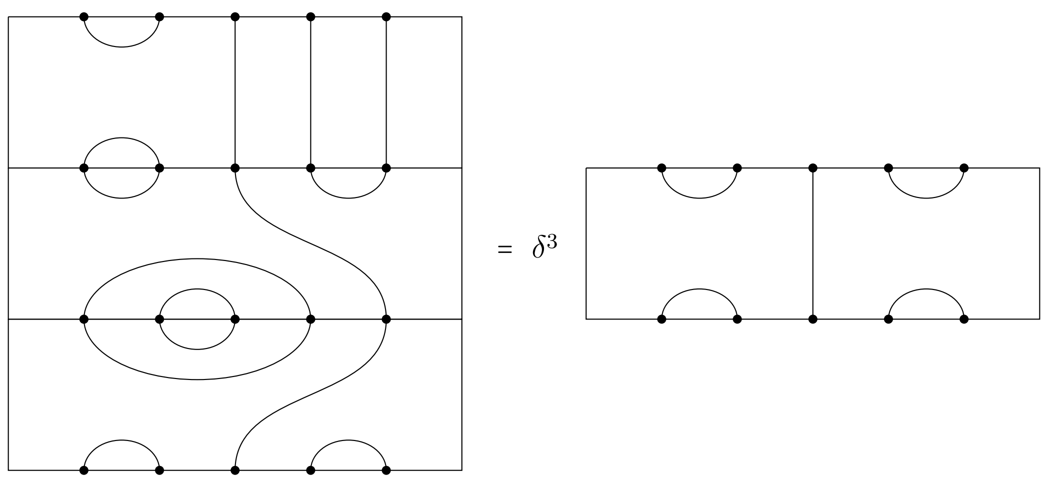


Figure 5.6: Example of multiplication in $\mathbb{D}\mathrm{TL}(A_4)$.

The next theorem describes the connection between $\mathrm{TL}(A_n)$ and $\mathbb{D}\mathrm{TL}(A_n)$ shown in [17] and [21].

Theorem 5.1.4. [17] As $\mathbb{Z}[\delta]$-algebras, the Temperley–Lieb algebra $\mathrm{TL}(A_n)$ is isomorphic to $\mathbb{D}\mathrm{TL}(A_n)$. Moreover, each loop-free diagram from $T_{n+1}(\varnothing)$ corresponds to a unique monomial basis element of $\mathrm{TL}(A_n)$.

Since the monomial and canonical bases are equal in $\mathrm{TL}(A_n)$, the correspondence holds true for the canonical basis in $\mathrm{TL}(A_n)$. Our goal is to establish a diagrammatic correspondence with the canonical basis in $\mathrm{TL}(\widetilde{C}_n)$. However, in order to do this we need to introduce a set of decorations and relations that can handle the bond strength of 4 that is seen in Coxeter systems of type $\widetilde{C}_n$.

5.2 Decorated diagrams

We will follow the development of decorated diagrams that has been presented in [2]. We will now modify our diagrams by allowing the edges to carry symbols referred to as *decorations*. We will use elements of the free monoids on the sets $\mathcal{M} = \{\bullet, \circ, \blacktriangle, \vartriangle\}$ and $\mathcal{C} = \{\bullet, \circ, \blacksquare, \square\}$ as our decorations. The set $\mathcal{M}$ corresponds to the diagrammatic representation of the monomial basis, $\{b_w \mid w \in \mathrm{FC}(\widetilde{C}_n)\}$, and the set $\mathcal{C}$ will correspond to the conjectured diagrammatic representation of the canonical basis, $\{c_w \mid w \in \mathrm{FC}(\widetilde{C}_n)\}$. We will refer to elements of $\{\bullet, \blacktriangle, \blacksquare\}$ as *closed decorations* and elements of $\{\circ, \vartriangle, \square\}$ as *open decorations*. The free monoid on $\mathcal{M}$ (respectively, $\mathcal{C}$) is denoted $\mathcal{M}^*$ (respectively, $\mathcal{C}^*$). Suppose $\mathbf{b} = x_1 \cdots x_r$ is a finite sequence of decorations in either $\mathcal{M}^*$ or $\mathcal{C}^*$. We call $\mathbf{b}$ a *block* of decorations and say $\mathbf{b}$ has *width* r. For example, $\mathbf{b} = \bullet \ \bullet \ \square \ \circ \ \blacksquare \ \square$ is a block of width 6 from $\mathcal{C}^*$.

We now introduce several restrictions on the decoration of the edges in our diagrams. Let d be a fixed concrete pseudo k-diagram and let E be an edge of d.

(D0) If $a(d) = 0$, then E is undecorated.

Note that d_e is the unique diagram with a-value 0 and no loops; also it is undecorated. Subject to some constraints, if $a(d) > 0$, we may adorn E with a finite sequence of blocks of decorations $\mathbf{b}_1, \dots, \mathbf{b}_m$ such that the adjacency of blocks and decorations of each block is preserved as we travel along E.

If E is a non-loop edge, we adopt the convention of placing the decorations in such a manner that we may read off the sequence of decorations as we traverse E from north face to south face if E is propagating, or from left to right if E is non-propagating. If E is a loop, we may choose an arbitrary starting point and direction to read off the decorations.

If $a(d) \neq 0$, then we also require the following:

(D1) All decorated edges can be deformed so as to take closed decorations to the left wall of the diagram and open decorations to the right wall simultaneously without crossing any other edges.

(D2) If E is non-propagating (a loop or otherwise), then we allow adjacent blocks on E to be conjoined to form larger blocks.

(D3) If $a(d) > 1$ and E is propagating, we allow adjacent blocks to be conjoined to form larger blocks, as in (D2).

(D4) If $a(d) = 1$ and E is propagating, then we allow E to be decorated subject to the following restrictions.

(a) All decorations occurring on propagating edges must have vertical position lower (respectively, higher) than the vertical positions of decorations occurring on the (unique) non-propagating edge in the north face (respectively, south face) of d.

(b) If $\mathbf{b}$ is a block of decorations occurring on E, then no other decorations occurring on any other propagating edges may have vertical position in the range of vertical positions that $\mathbf{b}$ occupies.

(c) If $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$ are two adjacent blocks occurring on E, then they may be conjoined to form a larger block only if the previous requirements are not violated.

We define a *concrete LR-decorated pseudo k-diagram* to be any concrete k-diagram decorated by blocks from $\mathcal{M}^*$ or $\mathcal{C}^*$ that satisfy the conditions presented in (D0) through (D4)

Requirement (D4) is nonstandard for diagram algebras and is required to ensure our diagrammatic representation of type I products in $\mathrm{TL}(\widetilde{C}_n)$ is faithful.

Example 5.2.1. Some examples of concrete LR-decorated pseudo k-diagrams may be found in Figure 5.7. Here are a few comments.

(a) The diagram in Figure 5.7(a) is a concrete LR-decorated pseudo 5-diagram with decorations coming from $\mathcal{M}^*$. This diagram has a-value greater than 1, meaning there is no restriction on the vertical placement of the decorations, and the decorations on the single propagating edge may be conjoined to form a block of width 4.

(b) The diagram in Figure 5.7(b) is another concrete LR-decorated pseudo 5-diagram with decorations from $\mathcal{M}^*$. However, this diagram has a-value 1, meaning there are restrictions on the vertical placement of decorations. We use horizontal dotted lines to indicate that the decorations of the left-most propagating edge may not be conjoined into a single block, and are 3 distinct blocks. Similarly, the decorations on the right-most propagating edge form 2 distinct blocks that may not be conjoined to form a single block.

(c) The diagram in Figure 5.7(c) is a concrete LR-decorated pseudo 6-diagram with maximal a-value of 3 and no propagating edges. This diagram is decorated with elements of $\mathcal{M}^*$, as well.

(d) The diagram in Figure 5.7(d) is similar to the diagram in Figure 5.7(c). The only difference is that this diagram is decorated with elements of $\mathcal{C}^*$.

Earlier, we mentioned that concrete pseudo k-diagrams have edges that may be deformed while remaining isotopically equivalent. The only time equivalence is an issue is when $a(d) = 1$. In this case, we wish to preserve the relative vertical position of the blocks. We define two

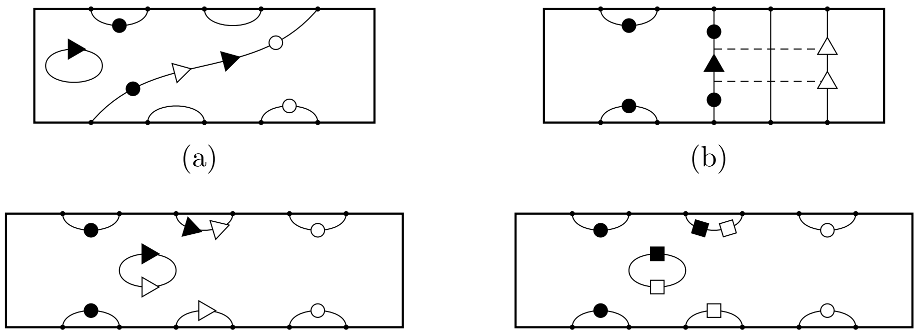


Figure 5.7: Examples of concrete LR-decorated pseudo diagrams.

concrete pseudo LR-decorated k-diagrams to be $\mathcal{M}$*-equivalent* (respectively, $\mathcal{C}$*-equivalent*) if the decoration set is $\mathcal{M}$ (respectively, $\mathcal{C}$) and we can isotopically deform one diagram into the other such that any intermediate diagram is also a concrete LR-decorated pseudo k-diagram. Note that we do allow decorations from the same maximal block to pass each other's vertical position (while maintaining adjacency).

We now define an *LR-decorated pseudo k-diagram* to be the equivalence class of $\mathcal{M}$-equivalent (respectively, $\mathcal{C}$-equivalent) concrete LR-decorated pseudo k-diagrams. We will denote the set of LR-decorated pseudo k-diagrams as $T_k^{LR}(\mathcal{M})$ (respectively, $T_k^{LR}(\mathcal{C})$).

We define $\mathcal{P}_k^{LR}(\mathcal{M})$ (respectively, $P_k^{LR}(\mathcal{C})$) as the free $\mathbb{Z}[\delta]$-module having the LR-decorated pseudo k-diagrams $T_k^{LR}(\mathcal{M})$ (respectively, $T_k^{LR}(\mathcal{C})$) as a basis. Multiplication in $\mathcal{P}_k^{LR}(\mathcal{M})$ (respectively, $\mathcal{P}_k^{LR}(\mathcal{C})$) is defined via basis elements d and d', and then extending bilinearly, that is, to calculate $d'd$, concatenate d' and d and conjoin adjacent blocks while maintaining $\mathcal{M}$-equivalence (respectively, $\mathcal{C}$-equivalence). It turns out that $\mathcal{P}_k^{LR}(\mathcal{M})$ (respectively, $\mathcal{P}_k^{LR}(\mathcal{C})$) is a well-defined infinite dimensional associative $\mathbb{Z}[\delta]$-algebra. The proof in the case of $\mathcal{P}_k^{LR}(\mathcal{M})$ appears in [2], and the proof may easily be adapted to the case of $\mathcal{P}_k^{LR}(\mathcal{C})$.

In the next section, we define a quotient of $\mathcal{P}_k^{LR}(\mathcal{M})$ (respectively, $\mathcal{P}_k^{LR}(\mathcal{C})$) having relations determined by the application of local combinatorial rules on our diagrams in order to create a diagrammatic representation of $\mathrm{TL}(\widetilde{C}_n)$ using the monomial basis. Decorations coming from $\mathcal{C}$ will be abandoned until Section 5.4.

5.3 Monomial relations

Let $R = \mathbb{Z}[\delta]$ and define $\mathcal{V}_{\mathcal{M}}$ to be the quotient of $R\mathcal{M}^*$ by the relations

(a) $\bullet\ \bullet\ =\ \blacktriangle$,

(b) ● ▲ = ▲ ● = 2 ●,

(c) ○ ○ = △, and

(d) ○ △ = △ ○ = 2 ○.

We refer to $\mathcal{V}_{\mathcal{M}}$, which is associative and has a basis consisting of the identity and all finite alternating products of open and closed decorations, as our *decoration algebra on* $\mathcal{M}$.

We now impose some relations on the blocks of decorations coming from $\mathcal{M}^*$ in Figure 5.8. Let $\widehat{\mathcal{P}}_k^{LR}(\mathcal{M})$ be the associative $\mathbb{Z}[\delta]$-algebra equal to the quotient of $\mathcal{P}_k^{LR}(\mathcal{M})$ defined by the given relations. The relations found in Figure 5.9 are a consequence of the relations presented in Figure 5.8.

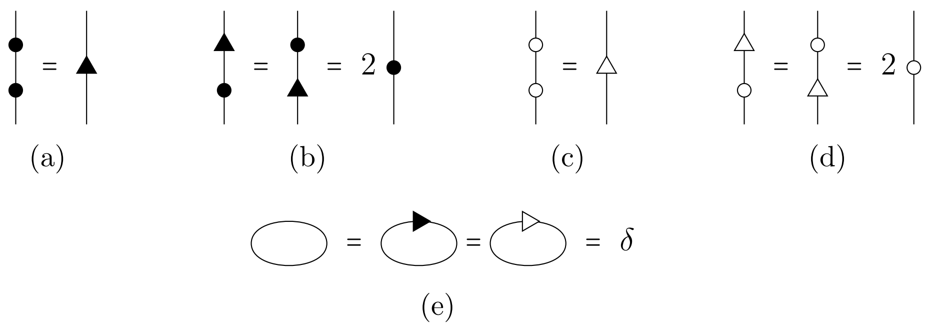


Figure 5.8: Defining relations of $\widehat{\mathcal{P}}_k^{LR}(\mathcal{M})$.

Figure 5.9: Additional relations of $\widehat{\mathcal{P}}_k^{LR}(\mathcal{M})$.

Example 5.3.1. In Figure 5.10, one will find the multiplication of three diagrams in $\widehat{\mathcal{P}}_k^{LR}(\mathcal{M})$, where no diagram involved has a-value of 1. In Figure 5.11, one will find multiplication of three diagrams in $\widehat{\mathcal{P}}_k^{LR}(\mathcal{M})$. However, each diagram involved in Figure 5.11 has a-value of 1, exemplifying the use of the dotted line to preserve vertical placement of distinct blocks.

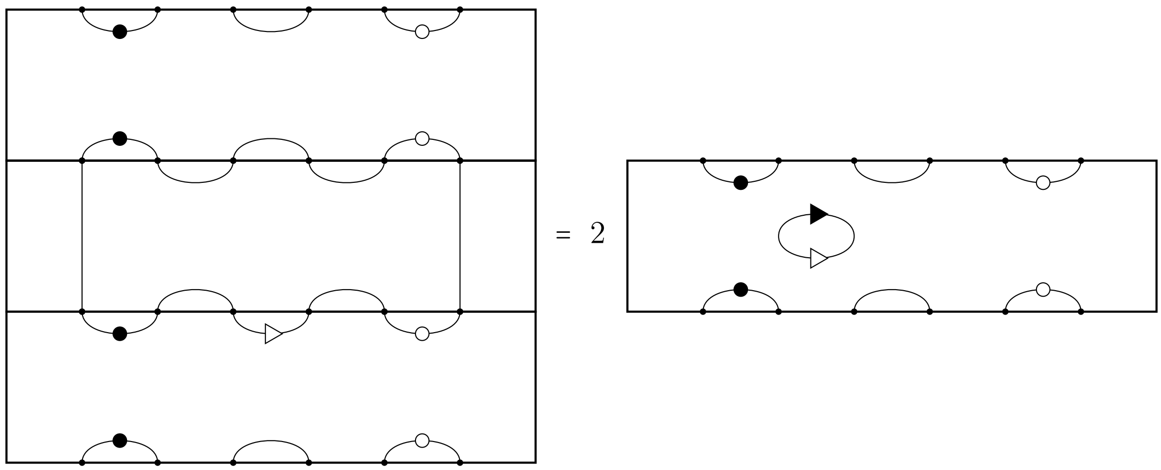


Figure 5.10: Example of multiplication in $\widehat{\mathcal{P}}_6^{LR}(\mathcal{M})$.

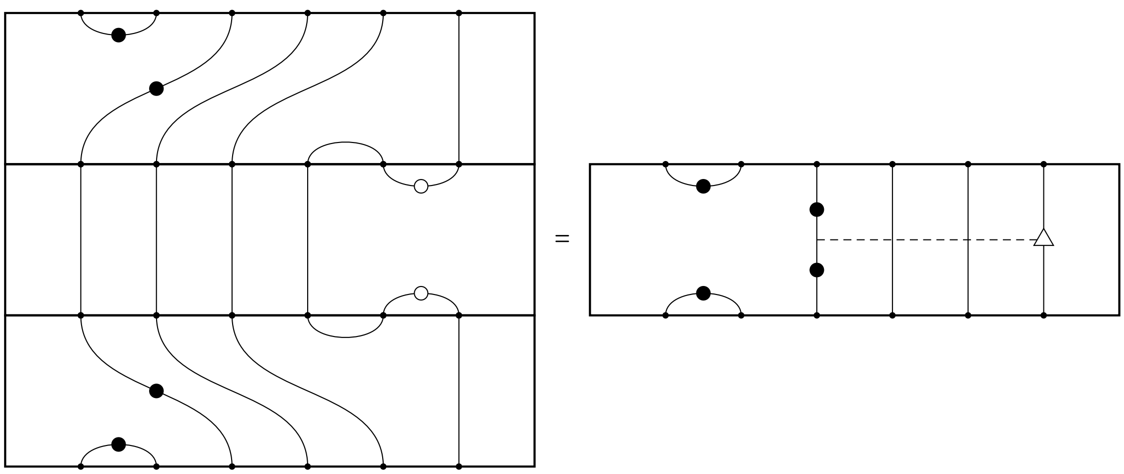


Figure 5.11: Another example of multiplication in $\widehat{\mathcal{P}}_6^{LR}(\mathcal{M})$.

Define the diagrams $d_1, d_2, \ldots, d_{n+1}$ as seen in Figure 5.12, and call them *simple diagrams*. Notice each of $d_1, d_2, \ldots, d_{n+1}$ is a basis element in $\widehat{\mathcal{P}}_{n+2}^{LR}(\mathcal{M})$. Define $\mathbb{D}\mathrm{TL}(\widetilde{C}_n)$ as the $\mathbb{Z}[\delta]$-algebra of $\widehat{\mathcal{P}}_{n+2}^{LR}(\mathcal{M})$ generated by the simple diagrams with multiplication inherited from $\widehat{\mathcal{P}}_{n+2}^{LR}(\mathcal{M})$.

A complete description of admissible diagrams may be found in [2]. That is to say, there exists a set of axioms that describes all diagrams that may be built using the simple diagrams and the relations found in Figure 5.8. In [3], $\mathrm{TL}(\widetilde{C}_n)$ and $\mathbb{D}\mathrm{TL}(\widetilde{C}_n)$ were shown to be isomorphic as $\mathbb{Z}[\delta]$-algebras under the identification $b_i \longmapsto d_i$, where $\{b_w \mid w \in \mathrm{FC}(\widetilde{C}_n)\}$

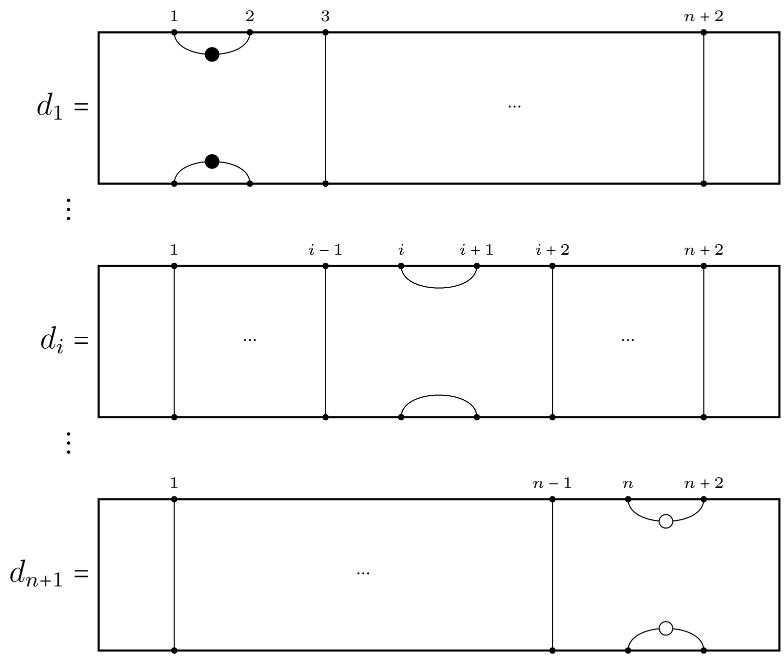


Figure 5.12: Simple diagrams.

is the monomial basis introduced in Section 3.2.

Theorem 5.3.2. Define $\theta : \mathrm{TL}(\widetilde{C}_n) \to \mathbb{D}\mathrm{TL}(\widetilde{C}_n)$ via $\theta(b_i) = d_i$. The map θ is an isomorphism of $\mathrm{TL}(\widetilde{C}_n)$ and $\mathbb{D}\mathrm{TL}(\widetilde{C}_n)$.

Note that if $w = x_1 x_2 \cdots x_r \in W(\widetilde{C}_n)$ is a reduced product, then $d_{x_1} d_{x_2} \cdots d_{x_r} = d_w$. In the following section we build a diagram algebra that we conjecture to be representative of the canonical basis of $\mathrm{TL}(\widetilde{C}_n)$.

5.4 Canonical relations

We now change gears to work with decorations coming from $\mathcal{C}$ instead of $\mathcal{M}$ in order to create a diagram algebra that we conjecture to be a faithful representation of $\mathrm{TL}(\widetilde{C}_n)$ in terms of the canonical basis. The first step in this task is to introduce a set of relations on the blocks of decorations from $\mathcal{C}^*$.

Let $R = \mathbb{Z}[\delta]$ and define $\mathcal{V}_{\mathcal{C}}$ to be the quotient of $R\mathcal{C}^*$ by the relations

(a) $\bullet \; \bullet \; = \; \blacksquare \; + \; e$,

(b) $\bullet\ \blacksquare\ =\ \blacksquare\ \bullet\ =\ \bullet$,

(c) $\circ\ \circ\ =\ \square\ +\ e$, and

(d) $\circ\ \square\ =\ \square\ \circ\ =\ \circ$,

where e is the identity, which translates to a blank edge.

As in Section 5.3, we impose the relations found in Figure 5.13 upon the blocks of decorations in $\mathcal{C}^*$. Let $\widehat{\mathcal{P}}_k^{LR}(\mathcal{C})$ be the associative $\mathbb{Z}[\delta]$-algebra equal to the quotient of $\mathcal{P}_k^{LR}(\mathcal{C})$ defined by the given relations. Consequently, we also have the relations found in Figure 5.14. As in Section 5.3, we define $\mathbb{D}_2\mathrm{TL}(\widetilde{C}_n)$ as the $\mathbb{Z}[\delta]$-algebra of $\widehat{\mathcal{P}}_{n+2}^{LR}(\mathcal{C})$ generated by the simple diagrams seen in Figure 5.12 with multiplication inherited by $\widehat{\mathcal{P}}_{n+2}^{LR}(\mathcal{C})$. Without inspection one may miss the subtle difference between $\mathbb{D}\mathrm{TL}(\widetilde{C}_n)$ and $\mathbb{D}_2\mathrm{TL}(\widetilde{C}_n)$, where the latter is defined by the relations imposed on decorations from $\mathcal{C}$. We conjecture that $\mathbb{D}_2\mathrm{TL}(\widetilde{C}_n) \cong \mathbb{D}\mathrm{TL}(\widetilde{C}_n)$ for n even.

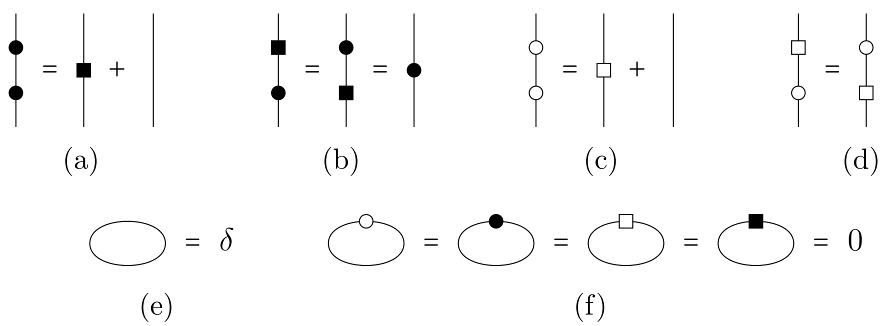


Figure 5.13: Defining relations of $\widehat{\mathcal{P}}_k^{LR}(\mathcal{C})$.

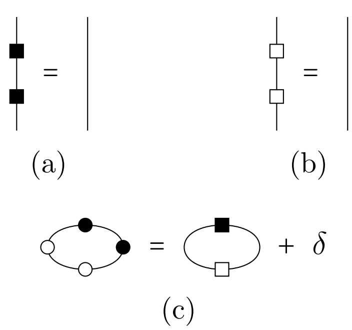


Figure 5.14: Additional relations of $\widehat{\mathcal{P}}_k^{LR}(\mathcal{C})$.

As in Chapter 4, we will define $d_{x_1}d_{x_2}\cdots d_{x_r} \in \mathbb{D}_2\mathrm{TL}(\widetilde{C}_n)$ to be of type I when $x_1x_2\cdots x_r \in W(\widetilde{C}_n)$ is reduced and of type I and of type II when $x_1x_2\cdots x_r \in W(\widetilde{C}_n)$ is reduced and of of type II. Let's have an example.

Example 5.4.1. Consider $\mathbb{D}_2\mathrm{TL}(\widetilde{C}_2)$. Then the type I product $d_1d_2d_3d_2d_1d_2d_3 \in \mathbb{D}_2\mathrm{TL}(\widetilde{C}_n)$ may be found in Figure 5.15(a). Also $d_1d_2d_3\,(d_2d_1d_2d_3)^2$ may be found in Figure 5.15(b), and $d_1d_2d_3\,(d_2d_1d_2d_3)^3$ may be found in Figure 5.15(c).

Furthermore, we have $d_3\,(d_2d_1d_2d_3)^2$ in Figure 5.16(a) and $d_3\,(d_2d_1d_2d_3)^3$ in Figure 5.16(b). These diagrams appear during the calculation of the previous set of diagrams, but was relocated to draw attention to their similarities.

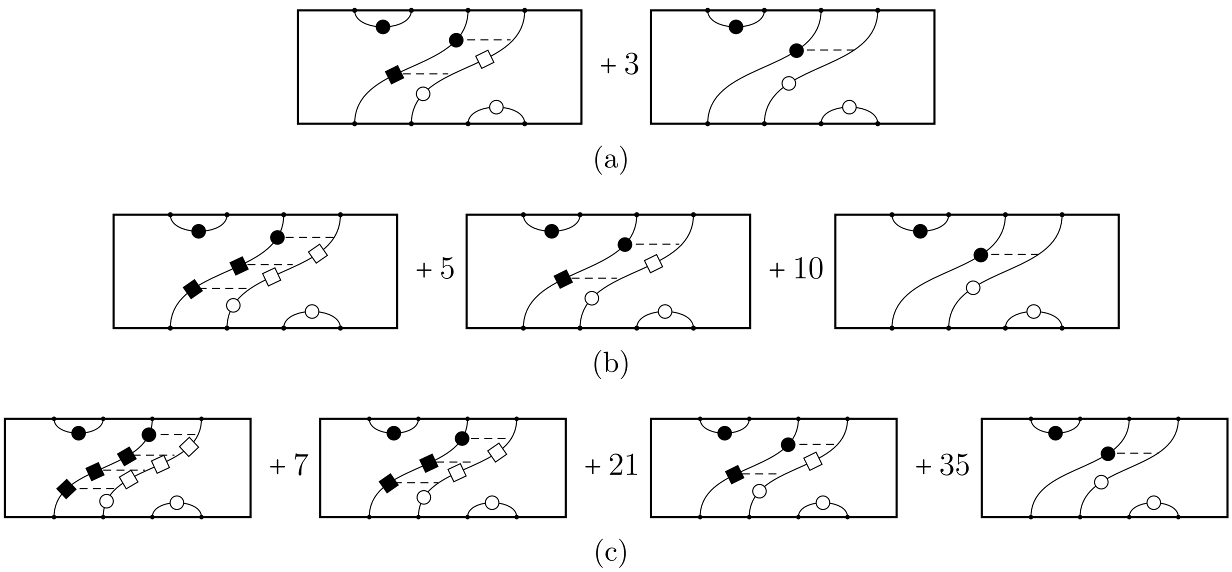
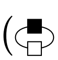


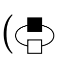
Figure 5.15: Linear combinations of diagrams corresponding to type I products.

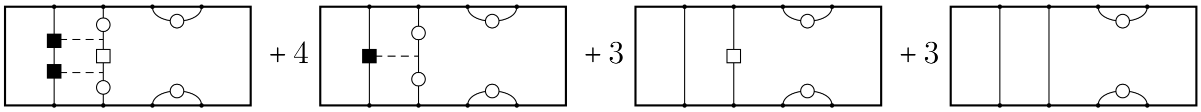
Example 5.4.2. Consider $\mathbb{D}_2\mathrm{TL}(\widetilde{C}_4)$. Let's look at a few type II products.

(a) The computation of $d_1d_3d_5d_2d_4d_1d_3d_5$ is found in Figure 5.17. Since we are using the simple diagrams from Section 5.3, we find that $d_1d_3d_5 = d_{135}$ by Theorem 5.3.2. Then we find

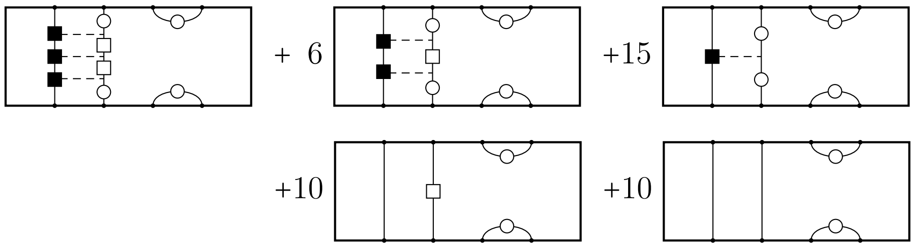
$$d_1d_3d_5d_2d_4d_1d_3d_5 = \Big($$  $$+ \delta\Big)\, d_{135}.$$

(b) In fact, the computation of $d_1d_3d_5\,(d_2d_4d_1d_3d_5)^k$ for $k \in \mathbb{N}$ is found in Figure 5.18. Again, we find

$$d_1d_3d_5\,(d_2d_4d_1d_3d_5)^k = \Big($$  $$+ \delta\Big)^k d_{135}.$$



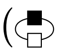
(a)



(b)

Figure 5.16: Additional linear combinations of diagrams corresponding to type I products.

(c) Furthermore, if we consider $\mathbb{D}_2\mathrm{TL}(\widetilde{C}_n)$, where n is even, we find

$d_1 d_3 \cdots d_{n+1} \left(d_2 d_4 \cdots d_n d_1 d_3 \cdots d_{n+1}\right)^k = \Big($  $+ \delta\Big)^k d_{13\cdots(n+1)}$

Note that this does not quite line up with our type II products from Remark 4.3.6, so we are not quite done yet.

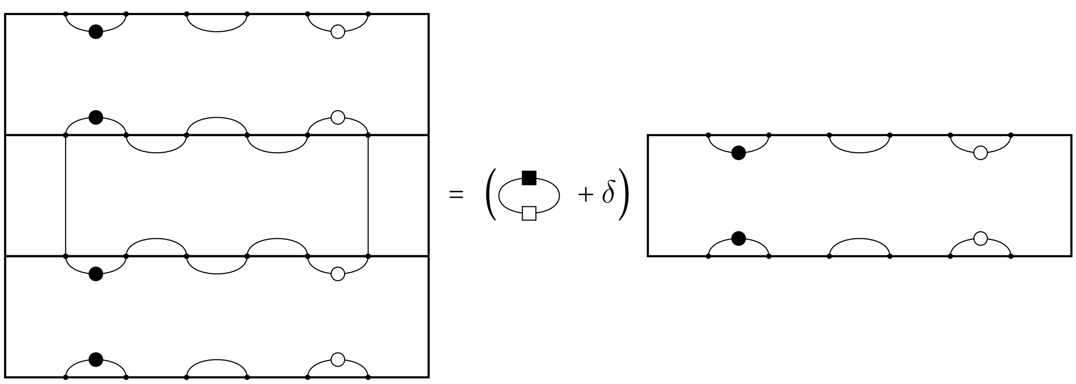


Figure 5.17: Linear combinations of diagrams corresponding to type II products.

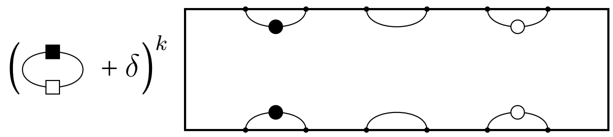


Figure 5.18: Additional linear combinations of diagrams corresponding to type II products.

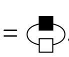
5.5 A diagrammatic correspondence

In this section we present evidence sufficient to indicate there should exist a faithful diagrammatic representation of $\mathrm{TL}(\widetilde{C}_n)$ in terms of the c-basis, and thus an isomorphism between $\mathbb{D}_2\mathrm{TL}(\widetilde{C}_n)$ and $\mathrm{TL}(\widetilde{C}_n)$, where the c-basis corresponds to some set of admissible diagrams from $\mathbb{D}_2\mathrm{TL}(\widetilde{C}_n)$, where n is restricted to be even. Once we have the correspondence between type I and type II elements, we ought to be able to begin the inductive process for showing the correspondence holds across $\mathrm{TL}(\widetilde{C}_n)$. The full proof is left to subsequent studies. As hinted in Chapter 4, we anticipate the case where n is odd to be simpler.

As in Example 5.4.1, there is a clear correspondence between the terms in the type I products presented in Remark 4.1.5 and diagrams with a-value 1. We have depicted the correspondence in Figure 5.19 when $n = 2$. In general, there are $n - 2$ propagating edges inserted between the propagating edge that bears closed decorations and the propagating edge that bears open decorations following the obvious pattern and carrying no decorations. These propagating edges that extend our example to the arbitrary case are omitted for the sake cleanliness in the argument.

A few comments are in order. First notice that the c-basis elements indexed by $\mathsf{z}_{1,3}^{R,2k+1}$ correspond to diagrams that have k instances of the decoration $\blacksquare$, k instances of the decoration $\square$, and are equal when all of the decorations are removed. Similarly, we find the c-basis elements indexed by $\mathsf{z}_{3,3}^{L,2k}$ correspond to diagrams with k instances of $\blacksquare$, $k - 1$ instances of $\square$ (and one instance of the decoration $\circ$ above and below), and are equal when are of the decorations are removed.

Now we address the correspondence in type II products. When convenience demands and there is no chance for confusion, we shall make the following replacement:

$L =$ .

We now introduce Chebyshev polynomials of the second kind, which are defined to be the elements of $\mathbb{Z}[L]$ given by $P_0(L) = 1$, $P_1(L) = L$ and

$$P_n(L) = LP_{n-1}(L) - P_{n-2}(L),$$

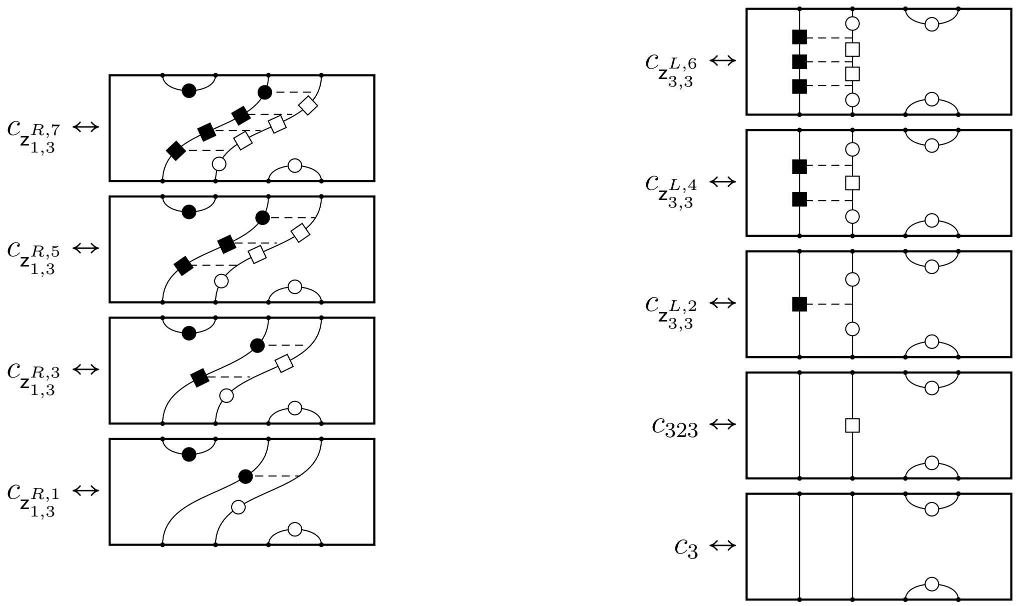


Figure 5.19: Type I identifications for $n = 2$.

where $n \geq 2$. Note that we have the following polynomials:

$$
\begin{aligned}
P_2(L) &= L^2 - 1;\\
P_3(L) &= L^3 - 2L;\\
P_4(L) &= L^4 - 3L^2 + 1;\\
P_5(L) &= L^5 - 4L^3 + 3L;\\
P_6(L) &= L^6 - 5L^4 + 6L^2;\\
&\text{etc.}
\end{aligned}
$$

The purpose of these polynomials will be to create a faithful correspondence between certain diagrams in $\mathbb{D}_2\mathrm{TL}(\widetilde{C}_n)$ and terms arising from the computation of type II products in the c-basis, which we encountered in Section 4.3.

The first step in identifying this correspondence is to collect our thoughts from Example 5.4.2 where we found

$$d_1 d_3 \cdots d_{n+1} d_2 d_4 \cdots d_n d_1 d_3 \cdots d_{n+1} = (L + \delta) d_{X_{1,n+1}}$$

and

$$d_1 d_3 \cdots d_{n+1} \left(d_2 d_4 \cdots d_n d_1 d_3 \cdots d_{n+1}\right)^k = (L + \delta)^k d_{X_{1,n+1}}.$$

We conjecture that there is an isomorphism of algebras where the terms arising from a type II product correspond to the diagram $d_{X_{1,n+1}}$ together with $P_k(L)$ for some k. Specifically, $c_{x_{1,n+1}Y_k} \leftrightarrow P_k(L)d_{X_{1,n+1}}$. We have depicted the correspondence in the case when $n = 4$ in Figure 5.20. In general, there are $n-4$ non-propagating edges are added in each face of the diagram that join nodes i and $i+1$ to extend our example to the case of arbitrary (but even) n, and manipulate addition to form Chebyshev polynomials for our linear combinations of type II diagrams.

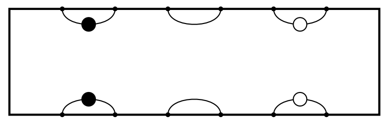
$c_{X_{1,5}Y_k} \leftrightarrow P_k(L)$ 

Figure 5.20: Type II identifications for $n = 4$.

We should have a few examples.

Example 5.5.1. Let n be even. Recalling Remark 4.1.1 and Example 4.3.5, we have the following correspondence between elements of $\mathbb{D}_2\mathrm{TL}(\widetilde{C}_n)$ and $\mathrm{TL}(\widetilde{C}_n)$:

$$\begin{aligned} d_1 d_3 \cdots d_{n+1} &= d_{X_{1,n+1}} \\ &= P_0(L) d_{X_{1,n+1}} \\ &\leftrightarrow c_{X_{1,n+1}} \\ &= c_1 c_3 \cdots c_{n+1}. \end{aligned}$$

Recalling Example 4.3.5, we have:

$$\begin{aligned} d_1 d_3 \cdots d_{n+1} d_2 d_4 \cdots d_n d_1 d_3 \cdots d_{n+1} &= (L + \delta) d_{X_{1,n+1}} \\ &= \left(P_1(L) + P_0(L)\right) d_{X_{1,n+1}} \\ &= P_1(L) d_{X_{1,n+1}} + P_0(L) d_{X_{1,n+1}} \\ &\leftrightarrow c_{X_{1,n+1}Y_1} + c_{X_{1,n+1}} \\ &= c_1 c_3 \cdots c_{n+1} c_2 c_4 \cdots c_n c_1 c_3 \cdots c_{n+1}. \end{aligned}$$

Lastly, recalling Remark 4.3.6, we have:

$$\begin{aligned}
d_1 d_3 \cdots d_{n+1} \left(d_2 d_4 \cdots d_n d_1 d_3 \cdots d_{n+1} \right)^2 &= (L+\delta)^2 d_{X_{1,n+1}} \\
&= \left(L^2 + 2L\delta + \delta^2 \right) d_{X_{1,n+1}} \\
&= \left(L^2 - 1 + 1 + 2\delta L\delta + \delta^2 \right) d_{X_{1,n+1}} \\
&= \left((L^2 - 1) + 2\delta L + (\delta^2 + 1) \right) d_{X_{1,n+1}} \\
&= \left(P_2(L) + 2\delta P_1(L) + (\delta^2+1) P_0(L) \right) d_{X_{1,n+1}} \\
&\leftrightarrow c_{X_{1,n+1}Y_2} + 2\delta c_{X_{1,n+1}Y_1} + (\delta^2+1) c_{X_{1,n+1}} \\
&= c_1 c_3 \cdots c_{n+1} \left(c_2 c_4 \cdots c_n c_1 c_3 \cdots c_{n+1} \right)^2 .
\end{aligned}$$

Bibliography

[1] D.C. Ernst. Non-cancellable elements in type affine C Coxeter groups. *Int. Electron. J. Algebra*, 8, 2010.

[2] D.C. Ernst. Diagram calculus for a type affine C Temperley–Lieb algebra, I. *J. Pure Appl. Alg.*, 216(11):2467–2488, 2012.

[3] D.C. Ernst. Diagram calculus for a type affine C Temperley–Lieb algebra, II. `arXiv:1101.4215`, 2016.

[4] C.K. Fan and R.M. Green. On the affine Temperley–Lieb algebras. *Jour. L.M.S.*, 60:366–380, 1999.

[5] M. Geck and G. Pfeiffer. *Characters of finite Coxeter groups and Iwahori–Hecke algebras.* 2000.

[6] J.J. Graham. *Modular representations of Hecke algebras and related algebras.* PhD thesis, University of Sydney, 1995.

[7] R.M. Green. Decorated tangles and canonical bases. *J. Algebra*, 246:594–628, 2001.

[8] R.M. Green. Star reducible Coxeter groups. *Glasgow Math. J.*, 48:583–609, 2006.

[9] R.M. Green. Generalized Jones traces and Kazhdan–Lusztig bases. *J. Pure Appl. Alg.*, 211:744–772, 2007.

[10] R.M. Green. On the Markov trace for Temperley–Lieb algebras of type E_n. *J. Knot Th. Ramif.*, 18, 2007.

[11] R.M. Green. Leading coefficients of Kazhdan–Lusztig polynomials and fully commutative elements. *J. Algebraic Combin.*, 30(2), 2009.

[12] R.M. Green and J. Losonczy. Canonical bases for Hecke algebra quotients. *Math. Res. Lett.*, 6:213–222, 1999.

[13] R.M. Green and J. Losonczy. A projection property for Kazhdan–Lusztig bases. *Int Math Res Notices*, 1:23–34, 2000.

[14] R.M. Green and J. Losonczy. Fully commutative Kazhdan–Lusztig cells. *Ann. Inst. Fourier Grenoble*, 51:1025–1045, 2001.

[15] J.E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, Cambridge, 1990.

[16] V.F.R. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. of Math. 2*, 126:335–388, 1987.

[17] L.H. Kauffman. State models and the Jones polynomial. *Topology*, 26:395–407, 1987.

[18] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53:165–184, 1979.

[19] G. Lusztig. Cells in affine Weyl groups. In *Algebraic groups and related topics*, Adv. Studies Pure Math 6, pages 255–287. North-Holland and Kinokuniya, Tokyo and Amsterdam, 1985.

[20] P.P. Martin and H. Saleur. The blob algebra and the periodic Temperley–Lieb algebra. *Lett. Math. Phys.*, 30 (3):189–206, 1994.

[21] R. Penrose. Angular momentum: An approach to combinatorial space-time. In *Quantum Theory and Beyond*, pages 151–180. Cambridge University Press, 1971.

[22] A. Pesiri. Kazhdan–Lusztig and R-polynomials of generalized Temperley–Lieb algebras. `arXiv:1401.0611`, 2014.

[23] J.Y. Shi. Fully commutative elements and Kazhdan–Lusztig cells in the finite and affine Coxeter groups. *Proc. Amer. Math. Soc.*, 131:3371–3378, 2003.

[24] J.Y. Shi. Fully commutative elements and Kazhdan–Lusztig cells in the finite and affine Coxeter groups, II. *Proc. Amer. Math. Soc.*, 133:2525–2531, 2005.

[25] J.R. Stembridge. On the fully commutative elements of Coxeter groups. *J. Algebraic Combin.*, 5, 1996.

[26] H.N.V. Temperley and E.H. Lieb. Relations between percolation and colouring problems and other graph theoretical problems associated with regular planar lattices: Some exact results for the percolation problem. *Proc. Roy. Soc. London Ser. A*, 322:251–280, 1971.

[27] T. tom Dieck. Bridges with pillars: a graphical calculus of knot algebra. *Topology Appl.*, 78:21–38, 1997.