

Structure of braid graphs for reduced words in simply-laced Coxeter systems

Seminario di Algebra e Geometria

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Coxeter systems

Definition

A **Coxeter system** (W, S) consists of a **Coxeter group** W generated by a set of involutions S together with a function $m : S \times S \rightarrow \mathbb{N} \cup \{\infty\}$ such that for $s \neq t \dots$

$$m(s, t) = 2 \quad \Longleftrightarrow \quad st = ts \quad \left. \vphantom{m(s, t) = 2} \right\} \text{commutation relation}$$

$$\left. \begin{array}{lcl} m(s, t) = 3 & \Longleftrightarrow & sts = tst \\ m(s, t) = 4 & \Longleftrightarrow & stst = tsts \\ & \vdots & \end{array} \right\} \text{braid relations}$$

Definition

We can encode (W, S) with a unique Coxeter graph Γ having:

- Vertex set = S
- $\{s, t\}$ edge labeled with $m(s, t)$ whenever $m(s, t) \geq 3$

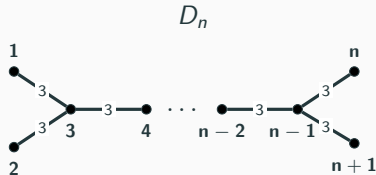
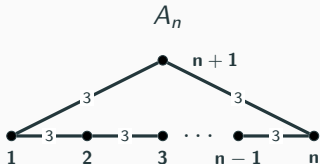
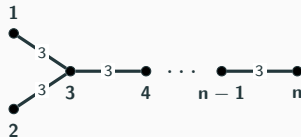
Comments

- Edges correspond to non-commuting pairs of generators.
- If all $m(s, t) \leq 3$, then Γ and W are called simply laced.
- If Γ has no 3-cycles, then Γ and W are called triangle free.
- If both simply laced and triangle free, then Γ and W are of type Λ .

Coxeter graphs (continued)

Example

Here are Coxeter graphs for four common simply-laced Coxeter systems. With the exception of $\tilde{A}_2$, the rest are of type Λ . We write i in place of s_i .



$\tilde{A}_n$

$\tilde{D}_n$

The top two Coxeter graphs yield finite groups while the bottom two yield infinite groups.

Reduced expressions

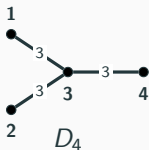
Definition

A word $\alpha = s_{x_1} s_{x_2} \cdots s_{x_\ell} \in S^*$ is called an **expression** for $w \in W$ if it is equal to w when considered as a group element. If ℓ is minimal among all expressions for w , α is called a **reduced expression**.

$\mathcal{R}(w)$ = set of reduced expressions for w

Example

Consider the expression $\alpha = 2132343$ in the Coxeter system of type D_4 .



$$21\textcolor{violet}{3}\textcolor{violet}{2}43 = \textcolor{blue}{2}123243 = \textcolor{violet}{1}\textcolor{green}{2}\textcolor{green}{2}3243 = 13243$$

It follows that α is not reduced. However, 13243 is reduced.

Matsumoto graphs

Matsumoto's Theorem

Any two reduced expressions for $w \in W$ differ by a sequence of commutation & braid moves.

Definition

For $w \in W$, define the **Matsumoto graph** $\mathcal{M}(w)$ via:

- Vertex set = $\mathcal{R}(w)$
- $\{\alpha, \beta\}$ iff α and β are related via a **commutation** or **braid** move

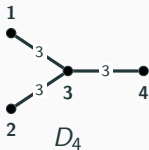
Comments

- Matsumoto's Theorem implies that $\mathcal{M}(w)$ is connected.
- Every cycle in a Matsumoto graph has even length (Bergeron, Ceballos, Labbé / Grinberg, Postnikov).
- Every Matsumoto graph is bipartite.

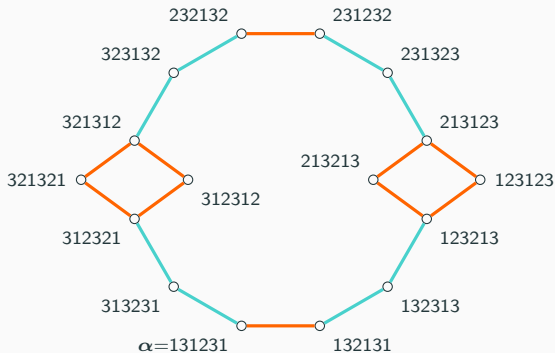
Matsumoto graphs (continued)

Example

Consider the reduced expression $\alpha = 131231$ in the Coxeter system of type D_4 .



Coxeter graph



Matsumoto graph

Braid graphs

Definition

Reduced expressions α and β are **braid equivalent** iff they are related by a sequence of braid moves. The corresponding equivalence classes are called **braid classes**, denoted $[\alpha]$.

Definition

We can encode a braid class $[\alpha]$ in a **braid graph**, denoted $\mathcal{B}(\alpha)$:

- Vertex set = $[\alpha]$
- $\{\gamma, \beta\}$ is an edge iff γ and β are related via a single **braid move**

Braid graphs are the maximal **blue** connected components in the Matsumoto graph.

Note

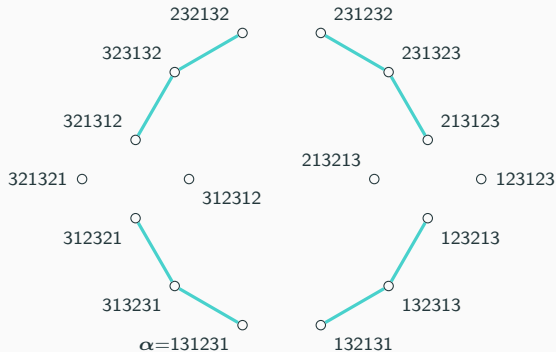
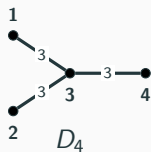
If we replace each $m(s, t) = 2$ with $m(s, t) = \infty$, then

braid graph = Matsumoto graph.

Braid graphs (continued)

Example

Consider the reduced expression $\alpha = 131231$ in type D_4 from earlier.



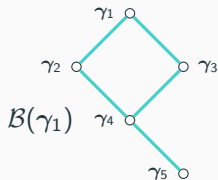
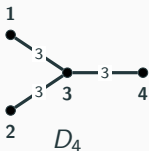
Eight braid graphs

Braid graphs (continued)

Example

In the Coxeter system of type D_4 , the expression $\gamma_1 = 2321434$ is reduced and its braid class consists of the following reduced expressions:

$$\gamma_1 = \underline{2321434}, \gamma_2 = \underline{3231434}, \gamma_3 = \underline{2321343}, \gamma_4 = \underline{3231343}, \gamma_5 = 3213143.$$



Example of Fibonacci cube

Braid shadows

For $i \leq j$, we define the interval

$$[i, j] := \{i, i+1, \dots, j-1, j\}.$$

Important!

We now assume all Coxeter systems are simply laced, often of type A .

Definition

Let α be a reduced expression.

- $[i, i+2]$ is **braid shadow for α** if $\alpha_{[i, i+2]} = sts$ with $m(s, t) = 3$.
- $\mathcal{S}(\alpha)$ = set of braid shadows for α
- Collection of braid shadows for braid class $[\alpha]$ is given by

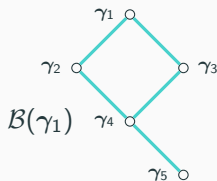
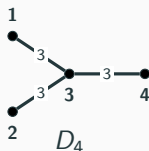
$$\mathcal{S}([\alpha]) := \bigcup_{\beta \in [\alpha]} \mathcal{S}(\beta).$$

- $\text{rank}(\alpha) = \text{card}(\mathcal{S}([\alpha]))$

Example

Recall the reduced expression $\gamma_1 = 2321434$ in the Coxeter system of type D_4 with braid class:

$$\gamma_1 = \underline{2321434}, \gamma_2 = \underline{3231434}, \gamma_3 = \underline{2321343}, \gamma_4 = \underline{3231343}, \gamma_5 = 3213143.$$



We see that

$$\mathcal{S}(\gamma_1) = \{[1, 3], [5, 7]\}, \mathcal{S}([\gamma_1]) = \{[1, 3], [3, 5], [5, 7]\}, \text{rank}(\gamma_1) = 3.$$

Links (continued)

Theorem

If α is a reduced expression and I and J are distinct braid shadows of $[\alpha]$ then $\text{card}(I \cap J) \leq 1$. That is, braid shadows are either disjoint or overlap by a single position.

Definition

If α is a reduced expression of length m , then α is a **link** provided it either $m = 1$ or $m \geq 3$ is odd and

$$\mathcal{S}([\alpha]) = \{\llbracket 1, 3 \rrbracket, \llbracket 3, 5 \rrbracket, \dots, \llbracket m-4, m-2 \rrbracket, \llbracket m-2, m \rrbracket\}.$$

Theorem

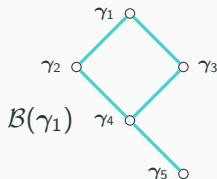
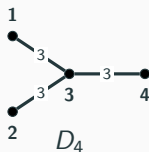
Suppose α and β are braid equivalent links in a **type A** Coxeter system. If $I \in \mathcal{S}(\alpha) \cap \mathcal{S}(\beta)$ such that $\alpha_I = sts$, then $\beta_I \in \{sts, tst\}$.

Links (continued)

Example

Recall the reduced expression $\gamma_1 = 2321434$ in the Coxeter system of type D_4 with braid class:

$$\gamma_1 = \underline{2321434}, \gamma_2 = \underline{3231434}, \gamma_3 = \underline{2321343}, \gamma_4 = \underline{3231343}, \gamma_5 = 32\underline{13143}.$$



Recall

$$\mathcal{S}([\gamma_1]) = \{[1, 3], [3, 5], [5, 7]\}.$$

So, γ_1 is a link of rank 3.

Link factorizations

Definition

If α is a reduced expression, then β is a **link factor** of α if:

- β is factor of α ,
- β is a link, and
- β is not a proper factor of a link that is a factor of α .

Theorem

Every reduced expression for a nonidentity group element can be written uniquely as a product of link factors.

For emphasis, we write the **link factorization** as:

$$\alpha = \beta_1 \mid \beta_2 \mid \cdots \mid \beta_m.$$

Braid graphs for link factorizations

Theorem

If α is reduced expression with link factorization

$$\alpha = \beta_1 \mid \beta_2 \mid \cdots \mid \beta_m,$$

then:

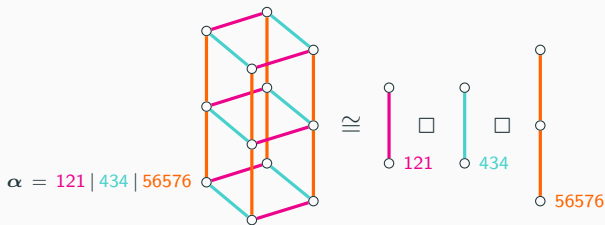
- $[\alpha] = \{\beta_1 \mid \beta_2 \mid \cdots \mid \beta_k : \beta_i \in [\alpha_i] \text{ for } 1 \leq i \leq k\}$
- $\text{card}([\alpha]) = \prod_{i=1}^k \text{card}([\alpha_i])$
- $\text{rank}(\alpha) = \sum_{i=1}^k \text{rank}(\alpha_i)$
- $\mathcal{B}(\alpha) \cong \mathcal{B}(\beta_1) \square \cdots \square \mathcal{B}(\beta_m).$

Braid graphs for link factorizations (continued)

Example

Consider the reduced expression $\alpha = 12143456576$ in type A_7 with link factorization:

$$121 \mid 434 \mid 56576.$$



A_7

Braid graphs for link factorizations (continued)

Theorem

If α is reduced expression for nonidentity element in type A_n with link factorization $\beta_1 \mid \beta_2 \mid \cdots \mid \beta_m$ such that each β_i has $2l_i - 1$ letters, then

$$B(\alpha) \cong \left\{ \begin{array}{c} \circ \\ \text{---} \\ \circ \\ \vdots \\ \circ \\ \text{---} \\ \circ \end{array} \right\} l_1 \quad \square \quad \left\{ \begin{array}{c} \circ \\ \text{---} \\ \circ \\ \vdots \\ \circ \\ \text{---} \\ \circ \end{array} \right\} l_2 \quad \square \cdots \square \quad \left\{ \begin{array}{c} \circ \\ \text{---} \\ \circ \\ \vdots \\ \circ \\ \text{---} \\ \circ \end{array} \right\} l_m ,$$

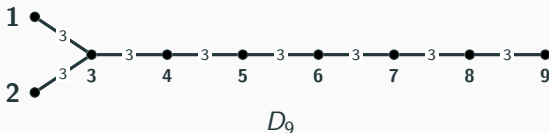
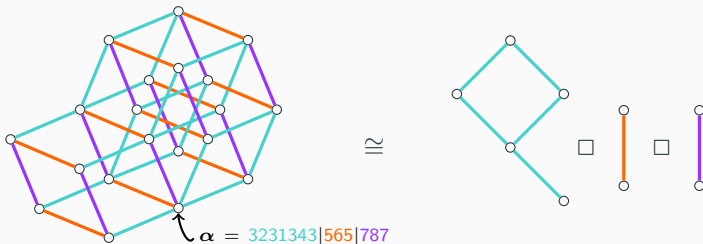
where i th link factor in the decomposition is a path graph with l_i vertices.

Braid graphs for link factorizations (continued)

Example

Consider the reduced expression $\alpha = 3231343565787$ in type D_9 with link factorization:

$$3231343 \mid 565 \mid 787.$$



Definition

If α is a link or rank $r \geq 1$ and $\llbracket 2i - 1, 2i + 1 \rrbracket$ is the i th braid shadow of $[\alpha]$, then the i th signature of α is given by

$$\text{sig}_i(\alpha) = \alpha_{2i}$$

and the signature of α is

$$\text{sig}(\alpha) = (\text{sig}_1(\alpha), \dots, \text{sig}_r(\alpha)).$$

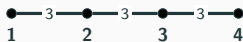
That is, for a link in the simply-laced case, we just recording the entries in the even positions.

Signature (continued)

Example

Consider the link $\beta_1 = 1213243$ in type A_4 .

$$\begin{array}{cccc}\beta_1 = \underline{121}\underline{3}243, & \beta_2 = \underline{212}\underline{3}243, & \beta_3 = 213\underline{23}\underline{4}3, & \beta_4 = 2132\underline{43}\underline{4} \\ \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} \\ (2, 3, 4) & (1, 3, 4) & (1, 2, 4) & (1, 2, 3)\end{array}$$



A_4

Theorem

Suppose α and β are braid equivalent links with $\llbracket 2i - 1, 2i + 1 \rrbracket \in \mathcal{S}(\alpha)$ in a type Λ Coxeter system. Then:

- If $\alpha_{\llbracket 2i-1, 2i+1 \rrbracket} = sts$, then $\text{sig}_i(\beta) \in \{s, t\}$.
- $\text{card}([\alpha]) \leq 2^{\text{rank}(\alpha)}$
- $\alpha = \beta$ iff $\text{sig}(\alpha) = \text{sig}(\beta)$

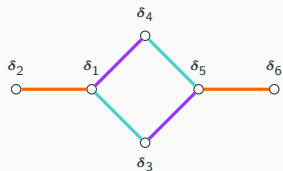
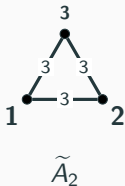
Why type Λ ?

Example

Consider the link $\delta_1 = 12121312121$ in the Coxeter system given below with braid class:

$$\delta_1 = \underline{12}\overline{13}\underline{12}1, \quad \delta_2 = 1\underline{23}\overline{13}21, \quad \delta_3 = \underline{21}\underline{23}\underline{12}1$$

$$\delta_4 = \underline{12}\underline{13}\underline{21}2, \quad \delta_5 = \underline{21}\underline{23}\underline{21}2, \quad \delta_6 = 2\underline{13}\overline{23}12$$



Median graphs

Definition

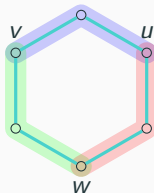
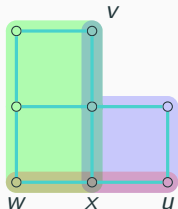
A connected graph is **median** if for any three vertices:

$$|\text{med}(u, v, w) := I(u, v) \cap I(u, w) \cap I(v, w)| = 1.$$

That is, there is a unique vertex, called the **median**, that simultaneously lies on a geodesic between u and v , a geodesic between u and w , and a geodesic between v and w .

Example

The graph on the left is median while the one on the right is not.

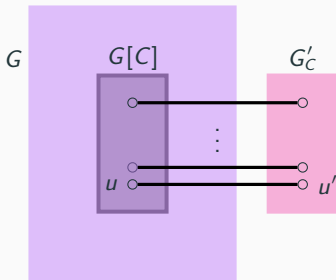


Median graphs (continued)

Definition

Given a graph G and a convex set $C \subseteq V(G)$, we define the **peripheral expansion of G along C** :

- Start with a graph G ;
- Make an isomorphic copy of $G[C]$, denoted G'_C , where each $u \in C$ corresponds to $u' \in C' := V(G'_C)$;
- For each $u \in C$, join u and u' with an edge.



Proposition (Mulder)

A graph is **median** iff it can be obtained from a single vertex by a sequence of peripheral expansions.

Example

○

Proposition (Mulder)

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Example



Median graphs (continued)

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Median graphs (continued)

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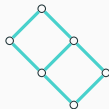


Median graphs (continued)

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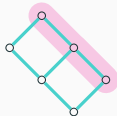


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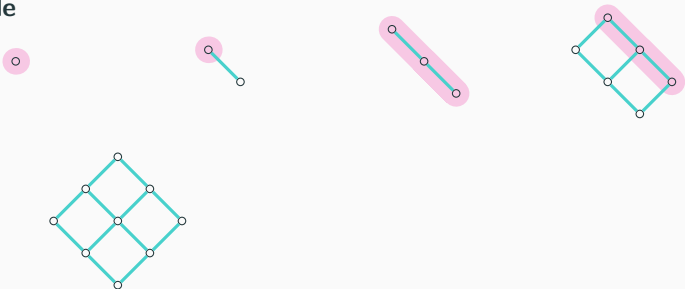


Median graphs (continued)

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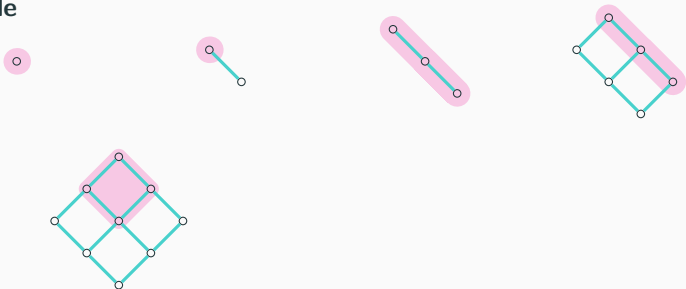


Median graphs (continued)

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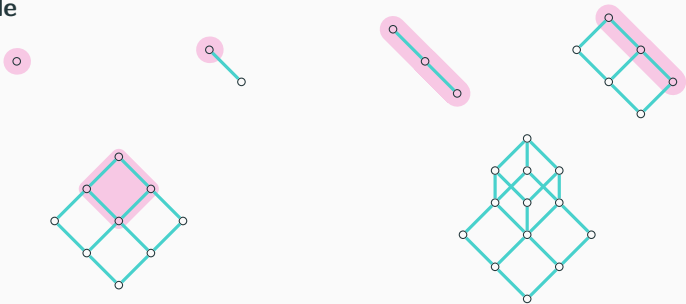


Median graphs (continued)

Proposition (Mulder)

A graph is **median** iff it can be obtained from a single vertex by a sequence of peripheral expansions.

Example



Peripheral expansions



Definition

Suppose α is a link of rank $r \geq 2$ in a type Λ Coxeter system, and let $\sigma \in [\alpha]$ such that the two rightmost braid shadows exist in σ . Define

$$\hat{\sigma} := \text{“chop off last two entries of } \sigma\text{”}.$$

For example: $\sigma = \underline{212}\overline{32} \Rightarrow \hat{\sigma} = 212$.

$$\text{Earth} := \{\beta \in [\alpha] \mid \text{sig}_r(\beta) = \text{sig}_r(\sigma)\}$$

$$\hat{\text{Earth}} := [\hat{\sigma}]$$

$$\text{Moon} := \{\beta \in [\alpha] \mid \text{sig}_r(\beta) \neq \text{sig}_r(\sigma)\}$$

$$\text{Shadow} := \{\beta \in \text{Earth} \mid \text{rightmost braid shadow exists in } \beta\}$$

For simplicity, we refer to the corresponding induced subgraphs using the same names.

Earth, Moon, & Shadow are convex

Theorem

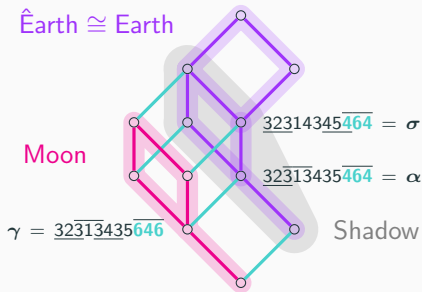
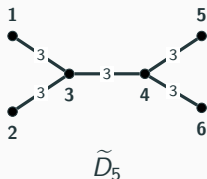
Suppose α is a link of rank at least two in a Coxeter system of type Λ . Choose $\sigma \in [\alpha]$ according to previous definition. Then

- Earth, Moon, and Shadow are convex.
- $\hat{\sigma}$ is a link with rank one less than σ .
- $\beta \in \text{Earth}$ iff $\hat{\beta} \in \hat{\text{Earth}} = [\hat{\sigma}]$.
- $\hat{\text{Earth}} \xrightarrow{\text{isometric}} \mathcal{B}(\alpha)$ with $\hat{\text{Earth}} \cong \text{Earth}$
- $\text{Shadow} \cong \text{Moon}$

Visualizing Earth, Moon, & Shadow

Example

Consider the link $\alpha = 32313435464$ in the Coxeter system of type $\tilde{D}_5$.



Braid graphs for links are median

Theorem

If α is a link in a type Λ Coxeter system, then $\mathcal{B}(\alpha)$ is median.

Outline of Proof

- We induct on rank. Base cases check out.
- Choose $\sigma \in [\alpha]$ with the last two braid shadows locally available.
- By induction, $\text{Earth} \cong \hat{\text{Earth}}$ is median.
- $\mathcal{B}(\alpha)$ is obtained from Earth via a peripheral expansion relative to Shadow .

Braid graphs for reduced expressions are median

Proposition

If graphs G_1 and G_2 are median, then $G_1 \square G_2$ is also median.

Theorem

If α is a reduced expression in a type Λ Coxeter system, then $\mathcal{B}(\alpha)$ is median.

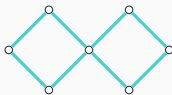
Corollary

If α is a reduced expression in a type Λ Coxeter system, then $\mathcal{B}(\alpha)$ is the one-skeleton of a CAT(0) cube complex.

Not every median graph arises as a braid graph

Example

Not every median graph can be realized as the braid graph for a reduced expression!



Braid graphs are “special” median graphs. What is “special”???

Signature majority determines median

Definition

We define the i th **majority** of braid equivalent links α, β, σ of rank r via

$$\text{maj}_i(\alpha, \beta, \sigma) := \begin{cases} \text{sig}_i(\alpha), & \text{if } \text{sig}_i(\alpha) = \text{sig}_i(\beta) \text{ or } \text{sig}_i(\alpha) = \text{sig}_i(\sigma) \\ \text{sig}_i(\beta), & \text{otherwise,} \end{cases}$$

and their **majority** via

$$\text{maj}(\alpha, \beta, \sigma) := (\text{maj}_1(\alpha, \beta, \sigma), \dots, \text{maj}_r(\alpha, \beta, \sigma)).$$

Corollary

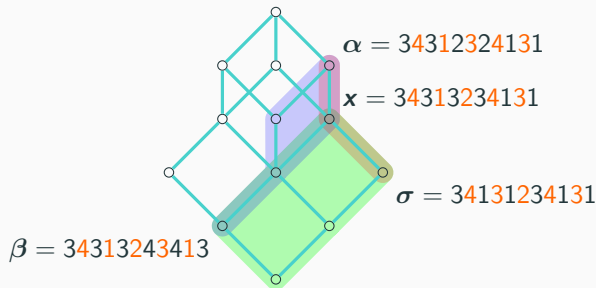
If α, β, σ are braid equivalent links in a Coxeter system of type Λ , then their median is the unique link $\mathbf{x}$ satisfying

$$\text{sig}(\mathbf{x}) = \text{maj}(\alpha, \beta, \sigma).$$

Signature majority determines median (continued)

Example

Consider braid equivalent links $\alpha = 34312324131$, $\beta = 34313243413$, and $\sigma = 34131234131$ in $[\alpha]$ in Coxeter system of type D_4 .



We see that

$$\text{maj}(\alpha, \beta, \sigma) = (4, 1, 2, 4, 3),$$

which corresponds to the signature of $\mathbf{x} = 34313234131$ in $[\alpha]$.

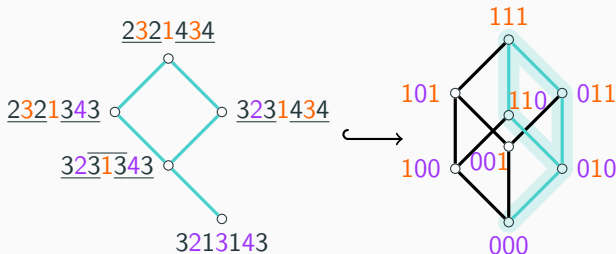
Braid graphs for reduced expressions partial cubes

Corollary

If α is a reduced expression in a type Λ Coxeter system, then $\mathcal{B}(\alpha)$ is a partial cube with isometric dimension equal to $\text{rank}(\alpha)$.

Example

Recall the reduced expression $\gamma_1 = 2321434$ in the Coxeter system of type D_4 from previous examples.



Some additional results

Theorem

Suppose α and β are braid equivalent links of rank at least one in a Coxeter system of type Λ . Then:

- Braid shadows appear once in a geodesic from α to β .
- Any two geodesics from α to β utilize same set of braid shadows.
- $d(\alpha, \beta) = \Delta(\text{sig}(\alpha), \text{sig}(\beta))$.
- $\text{diam}(\mathcal{B}(\alpha)) \leq \text{rank}(\alpha)$.
- If $\mathcal{B}(\alpha)$ is a tree, then it is a path.
- Every isometric cycle in $\mathcal{B}(\alpha)$ is of length 4.
- If α has the property that $\mathcal{S}(\alpha) = \mathcal{S}([\alpha])$, then $\mathcal{B}(\alpha)$ is a Fibonacci cube.

Braid graphs as the Hasse diagram of a partial order

Suppose α is a link of rank $r \geq 1$ in Coxeter system of type Λ .

- Identify diametrical pair μ and γ and choose μ to be the designated smallest vertex
- Define $\beta \leq \sigma$ if there exists a unique i such that $\text{sig}_i(\beta) \neq \text{sig}_i(\sigma)$ and $d(\mu, \beta) + 1 = d(\mu, \sigma)$.
- Let $\mathcal{P}(\mu) := ([\alpha], \leq)$ be the partial order induced by these covering relations.

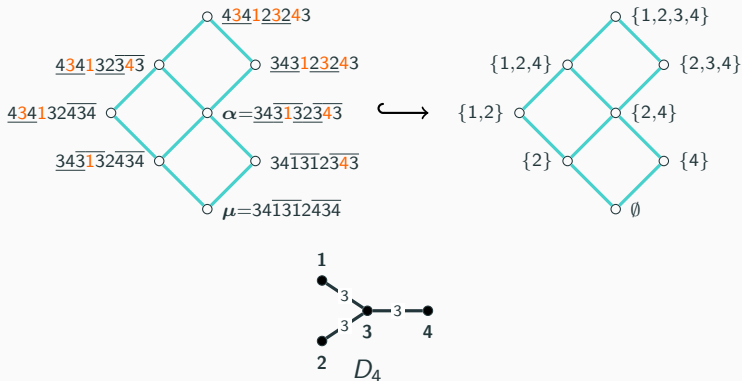
Theorem

If α is a reduced expression in a type Λ Coxeter system, then $\mathcal{B}(\alpha)$ is the underlying graph for the Hasse diagram of $\mathcal{P}(\mu)$.

Braid graphs as the Hasse diagram of a partial order

Example

Consider the link $\alpha = 343132343$ in the Coxeter system of type D_4 .



Braid graphs as the Hasse diagram of a distributive lattice

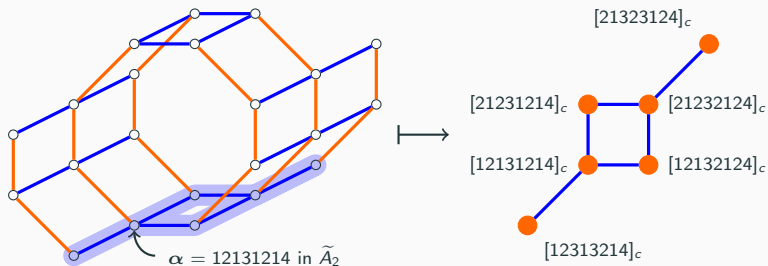
“Theorem”

If α is a reduced expression in a type Λ Coxeter system, then $\mathcal{B}(\alpha)$ is the underlying graph for the Hasse diagram of a distributive lattice.

Future work & open problems

- Not every distributive lattice corresponds to a braid graph in a type A Coxeter system. What additional restrictions do these braid graphs have? Can we completely characterize them?
- Deal with the pesky triangle-free obstruction! We conjecture that in the simply-laced case, every braid graph is median and a distributive lattice.
- In progress: When there is a braid class that is a transversal for the collection of commutation classes, the **quotient commutation graph** is isomorphic to the braid graph for the transversal (and hence it is median!). Fully commutative elements trivially exhibit this transversal property. I think freely braided elements also do. Can we characterize the group elements that have this transversal property?
- In progress: Extend the machinery beyond simply-laced Coxeter systems.

Transversal property magic





F. Awik, J. Breland, Q. Cadman, and D.C. Ernst.

Braid graphs in simply-laced triangle-free Coxeter systems are partial cubes.

European Journal of Combinatorics, 118, 2024.



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Braid graphs in simply-laced triangle-free Coxeter systems are median.

arXiv:2408.16839, 2025.