Supplemental Problems for Exam 2

General Information

Exam 2 focuses on content in Sections 3.1–3.10, 4.1. In particular, here is a list of topics/concepts that you must have an understanding of or be able to do:

- Have an intuitive understanding of derivatives, based on your knowledge of rate of change, speed, velocity, and slope of tangent lines.
- Understand relationship between differentiability and continuity.
- Efficiently use all the derivative shortcut rules:
  - Constant, Sum, Difference, and Constant Multiple Rules
  - Power Rule
  - Product and Quotient Rules
  - Chain Rule
  - Derivatives of Trigonometric Functions
  - Derivatives of Exponential Functions (with an emphasis on e^x)
  - Derivatives of Inverse Trigonometric Functions
  - Derivatives of Logarithmic Functions (with an emphasis on the natural logarithm)
- Be familiar with the proofs of the derivative shortcuts and be able to use the limit definition of the derivative to develop basic derivative formulas.
- Understand and compute higher derivatives (e.g., second derivative).
- Use implicit differentiation to find the derivative of implicitly defined functions.
- Be able to find derivatives of complicated products, quotients, and functions of the form $f(x)^{g(x)}$ using logarithmic differentiation (or the log trick).
- Be able to find an equation of the tangent line at a point on the graph of a function or the graph determined by implicit functions.
- Be able to solve related rates problems.
- Find the local linearization (i.e., tangent line approximation) and the equation of the tangent line to the graph of a function at a particular point.
- Be able to give examples and counter examples to demonstrate different properties of functions.
- Call upon your own mental faculties to respond in flexible, thoughtful and creative ways to problems that may seem unfamiliar on first glance.

The problems that follow will provide you with an opportunity to review the relevant topics. However:

This is not a practice test!

It is possible that problems on your exam will resemble problems seen below, but you should not expect exam problems to be identical. This document contains an abundance of problems and it is not the intention that every student will complete every problem. You should complete as many problems in each section below as you think are necessary to solidify your understanding.

In addition, you should review examples done in class, as well as your homework exercises, especially the ones on the Weekly Homework assignments.

Words of Advice

Here are few things to keep in mind when taking your exam:

- Show all work! The thought process and your ability to show how and why you arrived at your answer is more important than the answer itself.
- The exam will be designed so that you can complete it without a calculator. If you find yourself yearning for a calculator, you might be doing something wrong.
- Make sure you have answered the question that you were asked. Also, ask yourself if your answer makes sense.
- If you know you made a mistake, but you can't find it, explain why you think you made a mistake and indicate where the mistake might be. This shows that you have a good understanding of the problem.
- If you write down an "=" sign, then you better be sure that the two expressions on either side are equal. Similarly, if two things are equal and it is necessary that they be equal to make your conclusion, then you better use "=."
- Don't forget to write limits where they are needed.

Derivative Practice

Find the derivative of each of the following functions.

1. $f(x) = \pi^2$
2. $f(t) = 3e^{4t}$
3. $g(w) = \dfrac{e^3}{3^w}$
4. $h(s) = e^{2s}\ln(2s)$
5. $f(x) = 5\sqrt{\log_3(x)}$
6. $g(x) = x^2 e^{x^2}$
7. $f(x) = x^e$
8. $f(x) = (\pi e)^2$
9. $m(t) = \tan(3t)$
10. $g(y) = y\cos(\ln(y)))$
11. $h(t) = t\sin(t)$
12. $f(x) = \dfrac{x}{\sin(x)}$
13. $f(x) = \ln\left(\dfrac{x^{3/5}\sqrt{3-x}}{(x^2-4)^4}\right)$
14. $y = x^{\cos(x)}$.
15. $f(x) = e^{x^2}\cos(2x)\sqrt{3x+1}$

Derivatives in Multiple Ways

Find the derivative of each of the following functions in at least two different ways.

16. $f(x) = (x+1)(x^2-3)$
17. $g(x) = \dfrac{3x^2+5x}{\sqrt{x}}$
18. $y = \dfrac{x^2-1}{x}$
19. $f(x) = \dfrac{7x+3x^2}{5\sqrt{x}}$
20. $a(t) = 2\sin^2(t) + 2\cos^2(t)$
21. $m(v) = \arccos(\cos(v))$

Implicit Differentiation

Use implicit differentiation to calculate $\frac{dy}{dx}$ for the following implicitly defined functions.

22. $\sqrt{xy} = 1 + x^2 y$
23. $\cos(x - y) = y\sin(x)$
24. $x^3 + x^2 y + 4y^2 = 6$
25. $x^2 \sin(y) = \ln(xy)$

Logarithmic Differentiation

26. Use logarithmic differentiation to show that $\frac{d}{dx}[x^x] = x^x(\ln(x) + 1)$
27. Use logarithmic differentiation to find the derivative of $f(x) = \frac{(x+2)^2 e^{100+x^3}}{\sin^7(x)}$.

Proofs

28. Prove the Product Rule using the limit definition of the derivative.
29. Prove that $\frac{d}{dx}[\sin(x)] = \cos(x)$ using the limit definition of the derivative.
30. Prove that $\frac{d}{dx}[\tan(x)] = \sec^2(x)$ and $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$ using well-know trigonometric identities, the quotient rule, and the derivatives of sine and cosine.
31. Prove that $\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1 - x^2}}$ using implicit differentiation.
32. Prove that $\frac{d}{dx}[\arctan(x)] = \frac{1}{x^2 + 1}$ using implicit differentiation.
33. Use logarithmic differentiation to prove that $\frac{d}{dx}[b^x] = b^x \ln(b)$.
34. Prove that $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$ for $x > 0$ using implicit differentiation and the fact that $\frac{d}{dx}[e^x] = e^x$.
35. Prove that $\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$.

Miscellaneous

36. If $f(2) = 4$ and $f'(2) = 7$ determine the derivative of f^{-1} at 4.
37. If $f(x) = \frac{2x - 1}{3x + 4}$, determine $\frac{d}{dx}\left[f^{-1}(x)\right]$ in two different ways.
38. If $g(d) = ab^2 + 3c^3 d + 5b^2 c^2 d^2$, then what is $g''(d)$?
39. If $\frac{dy}{dx} = 5$ and $\frac{dx}{dt} = -2$ then what is $\frac{dy}{dt}$?
40. A ball is thrown into the air and its height h (in meters) after t seconds is given by the function $h(t) = 10 + 20t - 5t^2$. When the ball reaches its maximum height, its velocity will be zero.

    (a) At what time will the ball reach its maximum height?

(b) What is the maximum height of the ball?

41. Find the the values of x such that $f''(x) = 0$, where

$$f(x) = \frac{x^5}{20} - \frac{x^4}{6} + \frac{x^3}{6} + 5x + 1.$$

42. Find an equation of the tangent line to the graph of $y = x^3$ at $x = 2$?

43. Find an equation of the tangent line to the graph of $y = 2e^x$ at $x = 1$?

44. Find an equation of the tangent line to the graph of $x^{2/3} + y^{2/3} = 4$ at the point $(-3\sqrt{3}, 1)$.

45. Suppose that $h(x) = f(x) + g(x)$, where $f(x) = 17 - \sqrt{x}$ and the equation of the tangent line to g at $x = 4$ is $y = 3x + 10$. Find $h'(4)$.

46. If $\ln(x) - y = 0$, find $\dfrac{dx}{dy}$.

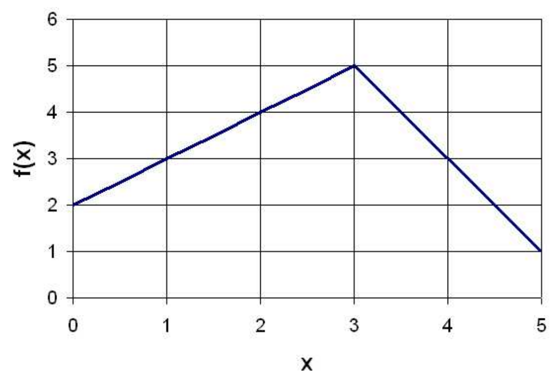
47. Provide an example of a function f such that f is continuous at 0, but $f'(0)$ does not exist.

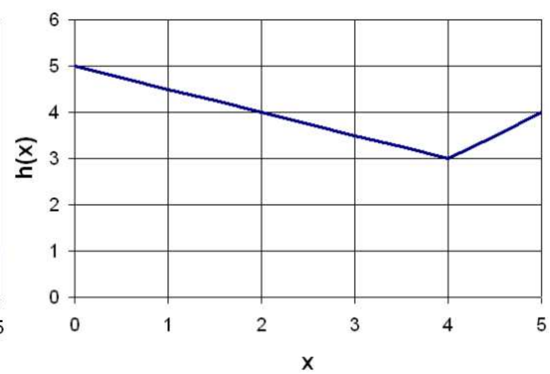
48. Determine the value of the constant a so that

$$f(x) = \begin{cases} a^2x - a, & x \leq 1 \\ ax + 3x^2, & x > 1 \end{cases}$$

is continuous and differentiable.

49. The graphs of the functions f and h are given below.





(a) Define $g = fh$. What is $g'(2)$?

(b) Define $k = f \circ h$. What is $k'(2)$?

(c) Define $m = \dfrac{f}{h}$. What is $m'(2)$?

50. Suppose f and g are differentiable functions such that $f(3) = 2$, $f'(3) = 4$, $g(3) = 1$, $g'(3) = 3$, and $f'(1) = 5$.

(a) If $h(x) = f(x)g(x)$, what is $h'(3)$?

(b) If $k(x) = \dfrac{f(x)}{g(x)}$, what is $k'(3)$?

(c) If $m(x) = f \circ g(x)$, what is $m'(3)$?

51. Find the points on the curve $x^2 + 2y^2 = 6$ where the slope of the tangent line is 1.

52. Find a formula for $\dfrac{d}{dx}\left[f(3x^2)\right]$ assuming f is differentiable.

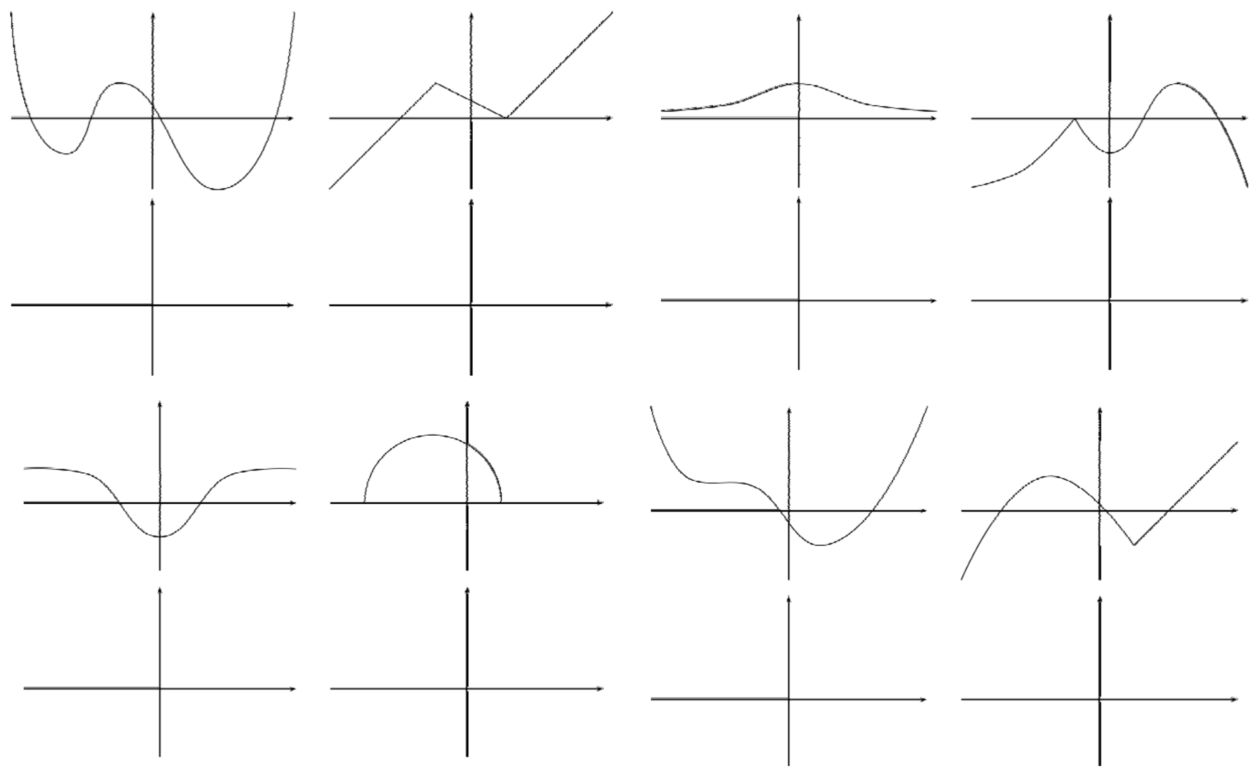
53. What is the 45th derivative of $y = \sin(x)$?

54. Find a formula for $f^{(n)}(x)$ if $f(x) = xe^x$. Recall that $f^{(n)}$ denotes the nth derivative of f.

55. Find a formula for $g^{(n)}(x)$ if $g(x) = \dfrac{1}{x}$. Recall that $g^{(n)}$ denotes the nth derivative of g.

56. Find an equation of the tangent to the graph of $y = f'(x)$ at $x = 1$ where $f(x) = x^5 - 3x^2 + x$.

57. For each of the graphs below, provide a rough sketch of the derivative of the corresponding function.



Related Rates

58. Suppose x and y are differentiable functions of t and are related by $y = x^2 - 1$. Find dy/dt when $x = 2$ given that $dx/dt = 3$.

59. A rock is dropped into a calm pond, causing ripples in the shape of concentric circles. The radius of the outer ripple is increasing at a rate of 12 cm/s. When the radius is 30 cm, at what rate is the total area of the outer ripple changing?

60. A spherical balloon is deflated so that its volume decreases at a constant rate of 3 in^3/s. How fast is the balloon's diameter decreasing when the radius is 2 inches?

61. A 10-foot ladder is leaning against a building. If the top of the ladder slides down the wall at a constant rate of 2 feet per second, how fast is the acute angle that the ladder makes with the ground decreasing when the top of the ladder is 5 feet from the ground? (Give your answer in radians per second.)

62. Two cars start moving from the same point. One travels south at 60 miles per hour and the other travels west at 25 miles per hour. At what rate is the distance between the cars increasing two hours later?

Additional Derivative Practice

If you feel that you need additional practice with derivative rules, find the first derivative of each of the following functions.

Power Rule

63. $f(x) = x - x^3$
64. $y = 3x^2 - \sqrt{x} + \frac{4}{x} + \pi^2$
65. $f(x) = \frac{4}{x^2} - \frac{x^2}{4}$
66. $h(x) = \frac{3}{\sqrt{x}}$
67. $f(x) = x^2 - e^2$
68. $g(x) = \sqrt{\sqrt{x}}$

Chain Rule

69. $f(x) = (x^2 - 1)^{10}$
70. $f(x) = \sqrt{1 + \sqrt{1 + 2x}}$
71. $g(x) = (3x^2 + 3x - 6)^{-8}$
72. $f(x) = \sqrt[4]{9 - x}$

Power, Product, Quotient, Chain Rules

73. $h(x) = (\sqrt{x} - 4)^3(\sqrt{x} + 4)^5$
74. $f(x) = x\sqrt{3x^2 - x}$
75. $f(x) = \frac{(5x^2 - 3)(x^2 - 2)}{x^2 + 2}$
76. $g(x) = \frac{x}{x + \frac{17}{x}}$
77. $f(t) = 3t^2 + 2t$
78. $g(w) = \frac{w^3}{(w + 3)^5}$
79. $h(s) = (s^{-2})^3$
80. $f(x) = 5\sqrt{x}$
81. $g(x) = \sqrt[3]{\sqrt[5]{\sqrt{x}}}$
82. $m(t) = \sqrt{t^2 - 5t}$
83. $g(y) = \sqrt{1 + \sqrt{1 + \sqrt{y}}}$
84. $h(s) = (s + 1)^5\sqrt{s - 1}$
85. $f(x) = \frac{2x - 1}{\sqrt{x + 1}}$
86. $f(x) = \frac{(x + 2)^2(3x - 4x^5)^{100}}{(8 - x)^7}$

Exponential Functions

87. $f(t) = e^{3t}$
88. $y = t^2 e^{t^3}$
89. $g(z) = \left(\frac{2}{3}\right)^{3z - z^2}$
90. $h(k) = 7e^{-5} - 7e^{-5k} + k^2\ln(e^4)$
91. $i(r) = 2^{4\sqrt{r}}$
92. $A(t) = Pe^{rt}$ (P, r are constants)

Logarithmic Functions

93. $l(t) = \ln(x^2 - 1)$
94. $y = x \ln(x)$
95. $h(x) = \ln(x^x)$
96. $t(y) = y \ln\left(\frac{1}{y}\right)$
97. $j(x) = \ln\left(\frac{(4x-1)^8(3x^2+14)^7}{\sqrt{x^2-4}}\right)$
98. $k(s) = \log_2((5s^8 - 11)^3)$

Trigonometric, Inverse Trigonometric, and Inverse Functions

99. $b(t) = 4\ln(5\cos(t))$
100. $c(u) = \cos(\sin(u))$
101. $f(w) = \tan(w^2 + 1)$
102. $g(v) = \arcsin(\cos(v)) + \cos(\arcsin(v))$
103. $h(y) = y^2 \arctan(4y)$
104. $i(z) = \sec^7(2z)$

Linear Approximation

Note: The numbering in the solutions is off for these problems, but it should be clear what corresponds to what.

105. Determine the local linearization of each of the following functions at the indicated value.
    (a) $f(x) = \tan(x)$ at $x = \pi/6$.
    (b) $g(x) = \ln(e^x + e^{2x})$ at $x = 0$.
106. Suppose the local linearization of g at 2 is given by $l_2(x) = 3x - 9$. Determine $h'(2)$ if $h = f \circ g$ and $f(x) = e^{4x} - 5$.
107. Use local linearization to approximate each of the following.
    (a) $\ln(0.9)$.
    (b) $\sqrt{101}$.