Linear Approximation

1) Local linearization: $\ell_a(x) = f(a) + f'(a)(x-a)$

a) let $a = \pi/6$ $\quad f(a) = \tan(\pi/6) = 1/\sqrt{3}$

$f'(x) = \sec^2(x)$ $\quad f'(a) = \sec^2(\pi/6) = 4/3$

$\ell_{\pi/6}(x) = 1/\sqrt{3} + 4/3\,(x - \pi/6)$

b) let $a = 0$ $\quad f(a) = \ln(e^0 \overset{2}{+} e^{2(0)}) = \ln(2)$

$f'(x) = \frac{1}{e^x + e^{2x}}(e^x + 2e^{2x})$ $\quad f'(0) = \frac{1}{e^0 + e^{2(0)}}(e^0 \overset{3}{+} 2e^{2(0)}) = \frac{3}{2}$

$\ell_0(x) = \ln(2) + \frac{3}{2}(x - 0)$

2) $\ell_2(x) = 3x - 9 \Rightarrow a = 2,\ g'(2) = 3$ (slope of line)

$h'(2) = f'(g(2)) \cdot g'(2)$ $\quad f'(x) = 4e^{4x}$

$= 4e^{4(-3)} \cdot 3$ $\quad g(2) = 3(2) - 9 = -3$

$= 12e^{-12}$

3) Approximations using local linearization

a) let $a = 1$ (since we know $\ln(1)$)

$f(x) = \ln(x)$

$f'(a) = \frac{1}{1} = 1$

$\Rightarrow \ell_1(x) = f(1) + f'(1)(x-1)$

$= \underset{0}{\cancel{\ln(1)}} + 1(x-1) = x - 1$, so $\ln(0.9) \approx 0.9 - 1 = -0.1$

b) let $a = 100$ (since we know $\sqrt{100}$)

$f(x) = \sqrt{x}$ $\quad f'(a) = \frac{1}{2}(100)^{-1/2} = \frac{1}{2\sqrt{100}} = \frac{1}{20}$ $\quad f(a) = \sqrt{100} = 10$

$\Rightarrow \ell_{100}(x) = f(100) + f'(100)(x - 100)$

$= 10 + \frac{1}{20}(x - 100)$

So $\sqrt{101} \approx 10 + \frac{1}{20}(101 - 100) = 10 + \frac{1}{20}$ or 10.05

4. Provide an example of each of the following.

(a) An *equation* of a function f such that f has a critical number at $x = 0$, but f does not have a local maximum or local minimum at $x = 0$.

$$f(x) = x^3$$

$$Since\ f'(x) = 3x^2\ has\ a\ zero\ at\ x = 0\ (and\ thus\ a\ critical\ point)\ and\ also,$$

$$the\ derivitive\ is\ postive\ (and\ the\ function\ increasing)\ both\ before\ and\ after\ x = 0$$

$$indicating\ the\ function\ is\ not\ at\ a\ local\ minimum\ or\ maximum\ at\ x = 0.$$

(b) An *equation* of a function g such that $g''(0) = 0$, but g does not have an inflection point at $x = 0$.

$$f(x) = x^4$$

$$Since\ f''(x) = 12x^2\ has\ a\ zero\ at\ x = 0\ \ and\ also\ since$$

$$the\ second\ derivitive\ is\ postive\ (and\ the\ function\ concave\ up)\ both\ before\ and\ after\ x = 0,$$

$$it\ indicates\ the\ function\ does\ not\ have\ an\ inflection\ point\ at\ x = 0.$$

5. Find the critical numbers for each of the following functions.

(a) $f(t) = 2t^3 + 3t^2 + 6t + 4$

$$f'(t) = 6t^2 + 6t + 6$$

$$0 = 6t^2 + 6t + 6 \rightarrow 0 = t^2 + t + 1 \rightarrow No\ real\ solutions: No\ critical\ numbers.$$

(b) $g(r) = \dfrac{r}{r^2+1}$

$$g'(r) = \frac{(1)(r^2+1) - (r)(2r)}{(r^2+1)^2} = \frac{1-r^2}{(r^2+1)^2} = \frac{(1-r)(1+r)}{(r^2+1)^2} \quad Note: r^2 + 1 > 0\ for\ all\ r \in (-\infty, \infty)$$

$$0 = \frac{(1-r)(1+r)}{(r^2+1)^2} \rightarrow 0 = (1-r)(1+r) \rightarrow 0 = 1 - r\ \ or\ \ 0 = 1 + r \rightarrow Critical\ numbers:\ r = 1\ or\ r = -1$$

(c) $h(x) = \sqrt{x}(1-x)$

$$h'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1-x) + \sqrt{x}(-1) = \frac{1-x}{2\sqrt{x}} - \sqrt{x} = \frac{(1-x) - 2\sqrt{x}\sqrt{x}}{2\sqrt{x}} = \frac{1-3x}{2\sqrt{x}}$$

$$0 = \frac{1-3x}{2\sqrt{x}} \rightarrow\ 0 = 1 - 3x \rightarrow\ \ x = \frac{1}{3}\ \ is\ a\ critical\ number$$

$$h'(0)\ is\ undefined\ so\ x = 0\ is\ a\ critical\ number.$$

(d) $f(\theta) = \sin^2(2\theta)$

$$f'(\theta) = 2\sin(2\theta)\cos(2\theta)\,(2) = 4\sin(2\theta)\cos(2\theta)$$

$$0 = 4\sin(2\theta)\cos(2\theta) \rightarrow 0 = \sin(2\theta)\cos(2\theta) \rightarrow 0 = \sin(2\theta)\text{ or }\ 0 = \cos(2\theta)$$

Critical Points: $\left\{2\theta \in \mathbb{R} | 2\theta = \frac{n\pi}{2}, n \in \mathbb{Z}\right\} \rightarrow \left\{\theta \in \mathbb{R} | \theta = \frac{n\pi}{4}, n \in \mathbb{Z}\right\}$

6. Let $f(x) = \frac{6}{5}x^{5/3} - \frac{9}{2}x^{2/3}$. Find all critical numbers of f and then classify each critical number as a *local minimum*, *local maximum*, or *neither*. Sufficient work must be shown.

$$f'(x) = \left(\frac{5}{3}\right)\left(\frac{6}{5}\right)\left(x^{\frac{2}{3}}\right) - \left(\frac{2}{3}\right)\left(\frac{9}{2}\right)\left(x^{-\frac{1}{3}}\right) = 2x^{\frac{2}{3}} - 3x^{-\frac{1}{3}} = 2x^{\frac{2}{3}} - \frac{3}{x^{\frac{1}{3}}} = \frac{2x-3}{\sqrt[3]{x}}$$

$$f'(0)\ is\ undefined\ so\ x = 0\ is\ a\ critical\ number$$

$$0 = \frac{2x-3}{\sqrt[3]{x}} \rightarrow 0 = 2x - 3 \rightarrow\ \ x = \frac{3}{2}\ is\ a\ critical\ number$$

x		0		3/2	
f'	+	Undefined	-	0	+
Behavior of f	↗	Local Maximum	↘	Local Minimum	↗

$$f(0) = 0, so\ f\ achieves\ a\ local\ maximum\ at\ the\ point\ (0,0)$$

$$f\left(\frac{3}{2}\right) \cong -3.54\ so\ f\ achieves\ a\ local\ minimum\ at\ approximately\ the\ point\ (\frac{3}{2}, -3.54)$$

7. Let $f(x) = \frac{x^3}{3} + x^2 - 3x$.

(a) Find the critical numbers of f.

$$f'(x) = x^2 + 2x - 3 = (x+3)(x-1)$$

$$0 = (x+3)(x-1) \rightarrow\ \ 0 = x + 3\ or\ 0 = x - 1 \rightarrow critical\ numbers{:}\ x = -3, x = 1$$

(b) List the intervals where f is increasing.

x		-3		1	
$f'(x)$	+	0	-	0	+
$Behavior\ of\ f$	↗		↘		↗

$$f\ is\ increasing\ on\ the\ intervals\ (-\infty, -3) and\ (1, \infty)$$

(c) List the intervals where f is decreasing.

$$f\ is\ decreasing\ on\ the\ interval\ (-3,1)$$

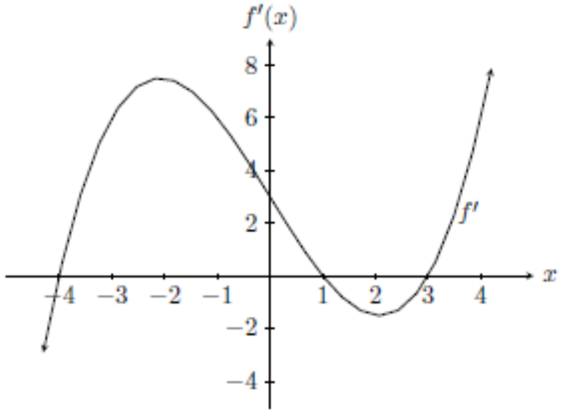
(d) Does f have any local minimums? If so, list the corresponding x-values.

$$f\ achieves\ a\ local\ minimum\ at\ x = 1.\ \ The\ value\ of\ the\ function\ at\ this\ point\ is\ f(1) = -\frac{5}{3}$$

(e) Does f have any local maximums? If so, list the corresponding x-values.

$$f\ achieves\ a\ local\ maximum\ at\ x = -3.\ \ The\ value\ of\ the\ function\ at\ this\ point\ is\ f(-3) = 9$$

8. Suppose f is a differentiable function such that the graph of f' is given below. Note this is the graph of f', NOT f.



(a) List the critical numbers of f.

$$f'(x) = 0 \text{ at the critical numbers } x = -4, x = 1, \text{and } x = 3$$

(b) Find the interval(s) where f is increasing.

$$f' \text{ is positive on } (-4,1) \text{ and on } (3,\infty)$$

$$\text{so } f \text{ is increasing on } (-4,1) \text{ and on } (3,\infty)$$

(c) Find the interval(s) where f is decreasing.

$$f' \text{ is negative on } (-\infty,-4) \text{ and on } (1,3)$$

$$\text{so } f \text{ is decreasing on } (-\infty,-4) \text{ and on } (1,3)$$

(d) Classify whether f has a local maximum, local minimum, or neither at each critical number.

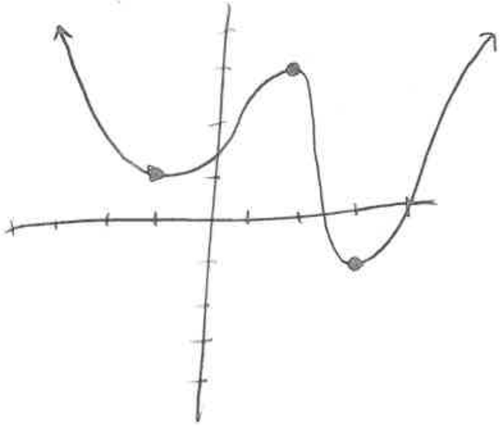
$$f \text{ has a local minimum at } x = -4$$

$$f \text{ has a local maximum at } x = 1$$

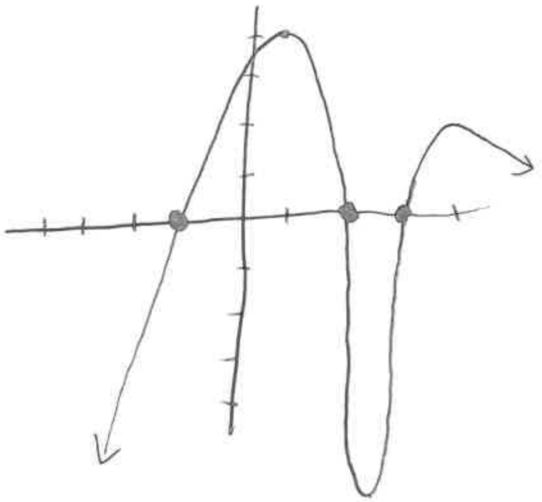
$$f \text{ has a local minimum at } x = 3$$

#9 * local min. at $x = -1$ + $x = 3$ * local max at $x = 2$.

Possible graph of f:



Possible graph of f':



#10 Abs. max/min. of $f(x) = \cos(x) - x$ on $[0, 2\pi]$.

$f'(x) = -\sin(x) - 1$

* $\sin(x) = -1$ at $\frac{3\pi}{2} + 2k\pi$ for any integer k.

$0 = -\sin(x) - 1$

On the interval $[0, 2\pi]$, $\sin(x) = -1$

$\sin(x) = -1$

at the single x-value $x = \frac{3\pi}{2}$.

* check all crit. pnts within $[0, 2\pi]$ and endpoints to find abs. max/min.

$f(0) = 1$ $f(2\pi) = 1 - 2\pi$ * $1 - 2\pi < -\frac{3\pi}{2}$

$f(\frac{3\pi}{2}) = -\frac{3\pi}{2}$

Abs. max of 1 at $x = 0$ (endpnt)

Abs min of $1 - 2\pi$ at $x = 2\pi$ (endpnt)

#11| Abs. max/min of $g(x) = x^3 - 3x + 1$ on $[0,3]$.

$f'(x) = 3x^2 - 3$

$0 = 3x^2 - 3$

$0 = x^2 - 1$

$0 = (x-1)(x+1)$

$x = 1$ and $x = -1$ are crit. pnts.

* $x = 1$ is in $[0,3]$, but $x = -1$ is not.

Check crit pnts in $[0,3]$ and endpnts.

$f(0) = 1$ $\quad f(3) = 19$

$f(1) = -1$

* Abs. max. of 19 at $x = 3$ (endpnt.)

* Abs. min. of -1 at $x = 1$ (crit. pnt).

#12| Find x-values of inflection pnts. for $f(x) = \frac{x^5}{20} - \frac{x^4}{6} + \frac{x^3}{6} + 5x + 1$

$f'(x) = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} + 5$

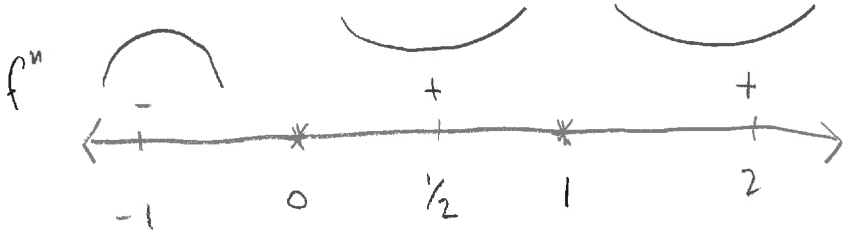
$f''(x) = x^3 - 2x^2 + x$

$0 = x^3 - 2x^2 + x$

$0 = x(x-1)^2$

$x = 0$ and $x = 1$ are *possible* pnts of inflection.

* test values: $x = -1, \frac{1}{2}, 2$



$f''(-1) = -4$ * concave down

$f''(1/2) = \frac{1}{8} - \frac{1}{2} + \frac{1}{2} = \frac{1}{8}$ * concave up

$f''(2) = 2$ * concave up.

* $x = 0$ is only pnt of inflection

⑬. $f(x) = 10 - \frac{16}{x}$ $[2, 8]$

slope of the secant line $= \frac{f(8) - f(2)}{8 - 2} = \frac{6}{6} = 1$

$f'(x) = -16(-x^{-2})$

$= \frac{16}{x^2}$

$\frac{16}{x^2} = 1 \Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

$x = 4$ confirms MVT since, $2 < 4 < 8$.

⑭ a) $f(x)$ is continuous on $[1, 4]$ and differentiable on $(1, 4)$ Thus $f(x)$ satisfies the hypothesis of the MVT.

b) Now; $\frac{f(4) - f(1)}{4 - 1} = \frac{\frac{1}{3}}{3} = \frac{1}{9}$

$f'(x) = \frac{(x+2).1 - x.1}{(x+2)^2} = \frac{2}{(x+2)^2}$

To find the x; $\frac{2}{(x+2)^2} = \frac{1}{9} \Rightarrow (x+2)^2 = 18$

$\Rightarrow (x+2) = \pm\sqrt{18}$

$\Rightarrow x = \pm\sqrt{18} - 2$

Now $x = +\sqrt{18} - 2$ confirms MVT since

$1 < \sqrt{18} - 2 < 4$.

b.) f does not satisfy the hypothesis of MVT since it is not continuous at $x=-2$ and $-2 \in [-8,6]$.

⑮ The average speed of the driver is

$$\frac{158 \text{ miles}}{2 \text{ hours}} = 79 \text{ mph} > 70 \text{ mph}$$

From the MVT we have that the driver reached 79 mph at least once during the 2 hour period.

Assumptions :=

① ~~The driver drove continuously within the period considered~~. The position function is continuous in the period considered.

② The position function is differentiable in the period considered.

L'Hopital's Rule

#16 $$\lim_{x\to\infty} \frac{1-x^3}{2x^3-5x^2+1} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{-3x^2}{6x^2-10x} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{-6x}{} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{-6}{12}$$

$$= -\frac{1}{2}$$

#17 $$\lim_{x\to-\infty} \frac{1-x^3}{2x^3-5x^2+1} \quad \left(\frac{-\infty}{-\infty}\right)$$

$$\overset{L'H}{=} \lim_{x\to-\infty} \frac{-3x^2}{6x^2-10x}$$

$$\overset{L'H}{=} \lim_{x\to-\infty} \frac{-6x}{12x-10}$$

$$\overset{L'H}{=} \lim_{x\to-\infty} \frac{-6}{12}$$

$$= -\frac{1}{2}$$

#18 $$\lim_{x\to0} \frac{1-x^3}{2x^3-5x^2+1}$$

$$= \frac{1-0}{0-0+1}$$

$$= 1$$

#19 $$\lim_{x\to0} \frac{x}{e^{4x}-1} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x\to0} \frac{1}{4e^{4x}}$$

$$= \frac{1}{4\cdot e^{\text{"}0\text{"}}} = \frac{1}{4\cdot1} = \frac{1}{4}$$

#20 $$\lim_{x\to\infty} \frac{x}{e^{4x}-1} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{1}{4e^{4x}}$$

$$= \frac{1}{\text{"}\infty\text{"}}$$

$$= 0$$

#21 $$\lim_{x\to2} \frac{x}{e^{4x}-1}$$

$$= \frac{2}{e^8-1}$$

\# 22 $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{\sin(5x)} \quad \left(\frac{0}{0}\right)$

$\overset{L'H}{=} \lim\limits_{x\to 0} \dfrac{3\cos(3x)}{5\cos(5x)}$

$= \dfrac{3\cdot 1}{5\cdot 1}$

$= \dfrac{3}{5}$

\# 23. $\lim\limits_{x\to 2} \dfrac{2e^{x-2}-x}{x^2-4} \quad \left(\frac{0}{0}\right)$

$\overset{L'H}{=} \lim\limits_{x\to 2} \dfrac{2e^{x-2}-1}{2x}$

$= \dfrac{2e^{2-2}-1}{2\cdot 2}$

$= \dfrac{2-1}{4} = \dfrac{1}{4}$

\# 24. $\lim\limits_{x\to\infty} x^2 e^{-x^2} \quad \overset{\infty\cdot 0}{(\cancel{0\cdot\infty})}$

$\overset{\cancel{L'H}}{=} \lim\limits_{x\to\infty} \dfrac{x^2}{e^{x^2}} \quad \left(\frac{\infty}{\infty}\right)$

$\overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{2x}{e^{x^2}\cdot 2x}$

$= \lim\limits_{x\to\infty} \dfrac{1}{e^{x^2}}$

$= \dfrac{1}{``\infty"} = 0$

\#25. $\lim\limits_{x\to 0} \dfrac{4x^3}{e^x}$ (Not indetermined form)

$= \dfrac{0}{1}$

$= 0$

\# 26. $\lim\limits_{x\to\infty} (\ln(x))^{\frac{1}{x}} \quad (\infty^0) = \boxed{1}$

Sol: Let $y = (\ln(x))^{\frac{1}{x}}$

$\Rightarrow \ln(y) = \ln\left[(\ln(x))^{\frac{1}{x}}\right]$

$= \dfrac{1}{x}\ln[\ln(x)]$

$\Rightarrow \lim\limits_{x\to\infty} \ln(y) = \lim\limits_{x\to\infty} \dfrac{1}{x}\ln[\ln(x)]$.

$\lim\limits_{x\to\infty} \dfrac{1}{x}\ln[\ln(x)] \quad (0\cdot\infty)$

$= \lim\limits_{x\to\infty} \dfrac{\ln(\ln(x))}{x} \quad \left(\frac{\infty}{\infty}\right)$

$\overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{1}{\ln(x)}\cdot\dfrac{1}{x}$

$= \lim\limits_{x\to\infty} \dfrac{1}{x\ln(x)}$

$= \dfrac{1}{``\infty"} = 0$

$\therefore \lim\limits_{x\to\infty} \ln(y) = 0$

$\Rightarrow \lim\limits_{x\to\infty} y = e^0 = \boxed{1}$

$$\#27 \lim_{x\to 0^+} [\sin(x)]^{\frac{1}{x}} = 0^{``\infty"} = 0$$

$$\#28 \lim_{x\to 0^+} \frac{\sin(x)}{\ln(x)} = \frac{``0"}{``-\infty"} = 0.$$

$$\#29 \lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) \quad (\infty - \infty)$$

$$= \lim_{x\to 0^+} \frac{\sin(x) - x}{x\sin(x)} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\cos(x) - 1}{\sin(x) + x(-\cos(x))} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{-\sin(x)}{\cos(x) - \cos(x) + x\sin(x)}$$

$$= \lim_{x\to 0^+} \frac{-\sin(x)}{x\sin(x)}$$

$$= \cancel{0} \lim_{x\to 0^+} -\frac{1}{x} = -\frac{1}{``0"} = -\infty$$

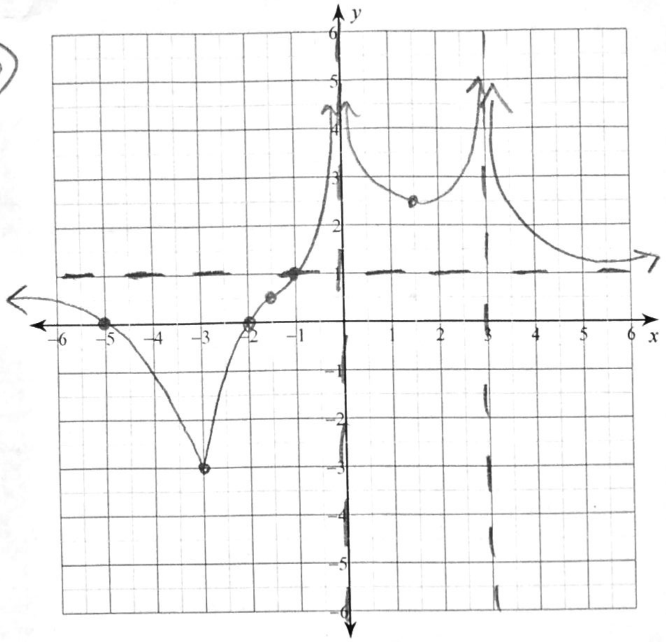
Calculus

Name_______________________________ ID: 1

Assignment

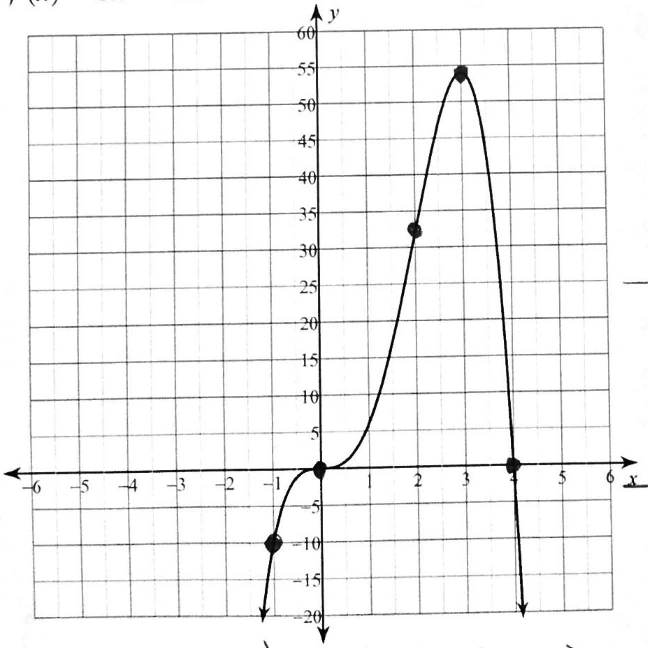
Date_______________ Period____

30)



31a)

$f(x) = 8x^3 - 2x^4$



y-int: 0

x int: $2x^3(4-x)=0$

$x = 0, 4$

no asymptotes

$f'(x) = 24x^2 - 8x^3 = 0$

$8x^2(3-x) = 0$

$x = 0, 3$

f'(x) +, +, −

↗ 0 ↗ 3 ↘

$f''(x) = 48x - 24x^2 = 0$

$24x(2-x) = 0$

$x = 0, 2$

f''(x) −, +, −

∩ 0 ∪ 2

$f(-1) = -8 - 2 = -10$

l. max: (3, 54)

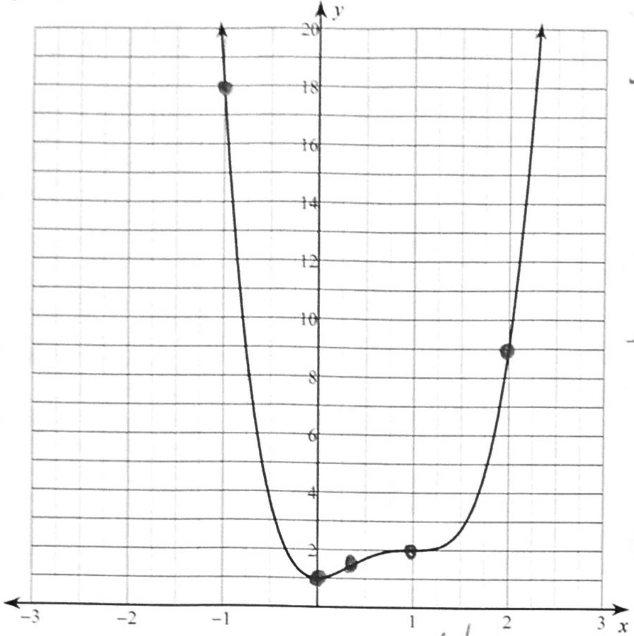
l. min: none

IP: (0, 0)

(2, 32)

31b)

$f(x) = 3x^4 - 8x^3 + 6x^2 + 1$



y int: 1

x int: none

no asymptotes

$f'(x) = 12x^3 - 24x^2 + 12x = 0$

$12x(x^2 - 2x + 1) = 0$

$x = 0, 1$

f'(x) − | + | +

↘ 0 ↗ 1 ↗

$f''(x) = 36x^2 - 48x + 12 = 0$

$12(3x^2 - 4x + 1) = 0$

$12(3x - 1)(x - 1) = 0$

$x = \frac{1}{3}, 1$

f''(x) + | − | +

∪ 1/3 ∩ 1 ∪

---

l. min: (0,1)

l. max: none

IP: $(\frac{1}{3},$

$(1, 2)$

$f(-1) = 18$ $f(2) = 9$

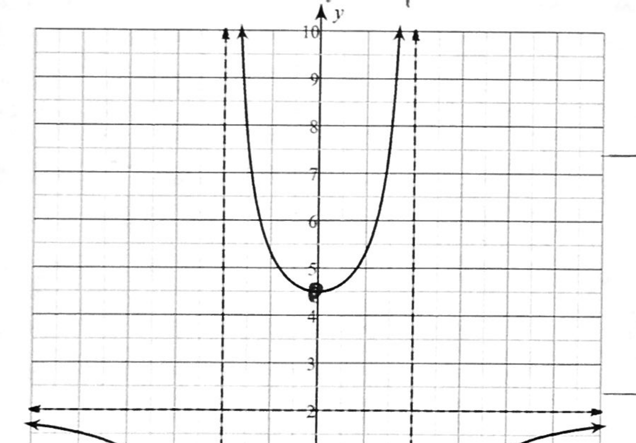
$$f(\tfrac{1}{3}) = \frac{3}{81} - \frac{8}{27} + \frac{6}{9} + 1$$

$$= \frac{3}{81} - \frac{24}{81} + \frac{54}{81} + \frac{81}{81} = \frac{114}{81} = \frac{38}{27}$$

$$= 1\frac{11}{27}$$

31c)

$f(x) = \frac{2(x^2 - 9)}{x^2 - 4} = \frac{2x^2 - 18}{x^2 - 4}$



y int: $\frac{18}{4} = 4\frac{1}{2}$

y int: $2x^2 - 18 = 0$

$x = \pm 3$

VA: $x^2 - 4 = 0$

$x = \pm 2$

HA: $y = 2$

$$f'(x) = \frac{\cancel{4x^3} - 16x - \cancel{4x^3} + 36x}{(x^2-4)^2} = \frac{20x}{(x^2-4)^2} = 0$$

$x = 0$, DNE at $x = \pm 2$

f'(x) − | − | + | +

−2 0 2

$$f''(x) = \frac{20(x^2-4)^2 - 4x(x^2-4)(20x)}{(x^2-4)^4}$$

$$= \frac{20(x^2-4) - 80x^2}{(x^2-4)^3} = 0$$

$20x^2 - 80 - 80x^2 = 0$

$-60x^2 - 80 = 0$

$-20(3x^2 + 4)$ no solution

DNE at $x = \pm 2$ f''(x) − | + | −

∩ −2 ∪ 2 ∩

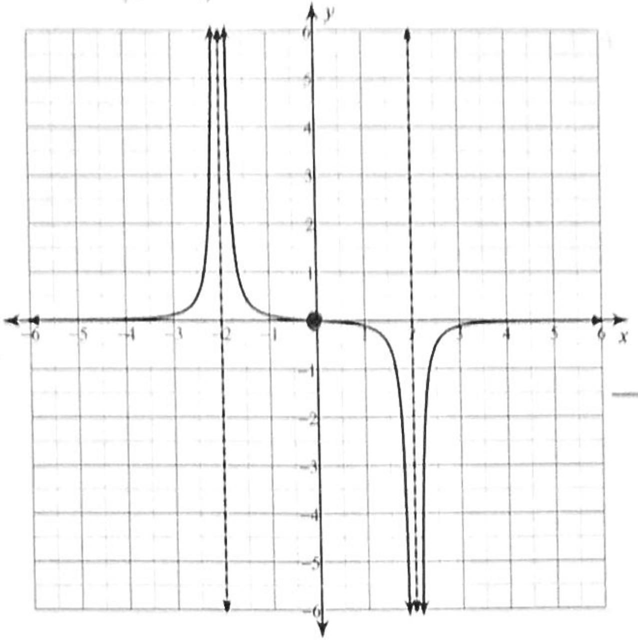
l. max: none

l. min: $(0, 4\frac{1}{2})$

IP: none

$f(x) = -\frac{x}{(x^2-4)^2}$

31d)



y int : 0 VA : $x^2-4=0$ HA : $y=0$

x int : 0 $x = \pm 2$

$$f'(x) = \frac{-(x^2-4)^2 + 4x^2(x^2-4)}{(x^2-4)^4} = \frac{-x^2+4+4x^2}{(x^2-4)^3} = 0$$

$3x^2+4=0$ no solution

DNE at $x = \pm 2$ $f'(x)$ + | − | +

$\nearrow$ −2 $\searrow$ 2 $\nearrow$

$$f''(x) = \frac{6x(x^2-4)^3 - 6x(x^2-4)^2(3x^2+4)}{(x^2-4)^6}$$

$$= \frac{6x(x^2-4) - 6x(3x^2+4)}{(x^2-4)^4} = \frac{-12x^3-48x}{(x^2-4)^4}$$

$-12x(x^2+4)=0$

$x=0$ DNE at $x=\pm 2$ $f''(x)$ + | + | − | −

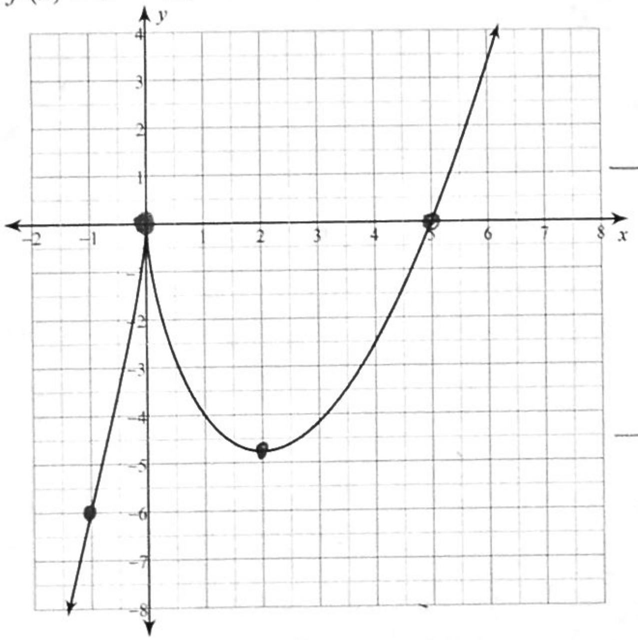
$\cup$ −2 $\cup$ 0 $\cap$ 2 $\cap$

l. max: none IP: (0,0)

l. min: none

$f(x) = x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$

31e)



y int : 0 no asymptotes

x int: $x^{\frac{2}{3}}(x-5)=0$

$x = 0, 5$

$f'(x) = \frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}} = 0$

$\frac{1}{3}x^{-\frac{1}{3}}(5x-10)=0$ $f'(x)$ + | − | +

$x=2$ DNE at $x=0$ $\nearrow$ 0 $\searrow$ 2 $\nearrow$

$f''(x) = \frac{10}{9}x^{-\frac{1}{3}} + \frac{10}{9}x^{-\frac{4}{3}} = 0$ $f''(x)$ − | + | +

$\frac{10}{9}x^{-\frac{1}{3}}(1+x^{-1})=0$ −1 0

$x=-1$ DNE at $x=0$

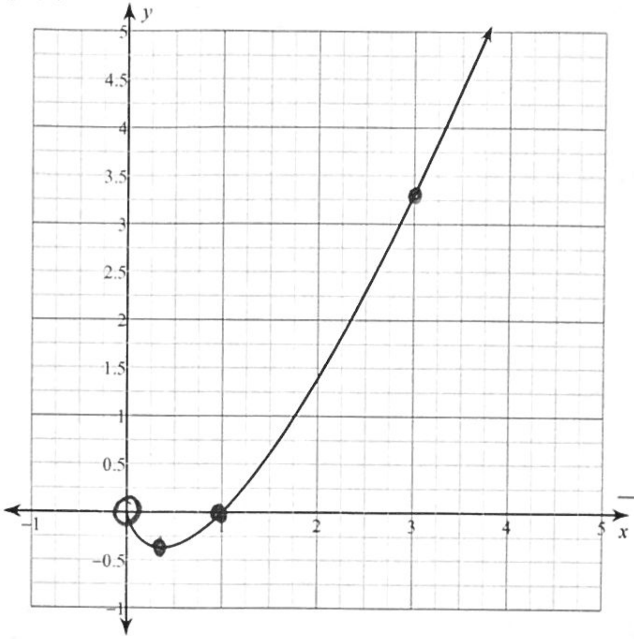
l. max: (0,0)

l. min: (2,−4.76)

IP: (−1,−6)

31f)

$f(x) = x\ln x$



y int: DNE   no asymptotes

x int: $x \ln x = 0$

$x = 0, 1$

$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$ form $\frac{-\infty}{\infty}$

$\overset{L}{=} \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+} -x$

$= 0$

$f'(x) = \ln x + 1 = 0$

$\ln x = -1$

$e^{-1} = x$

$f'(x)$ (− | +

0 $\frac{1}{e}$

l. max: none

l. min: $(\frac{1}{e}, -\frac{1}{e})$

IP: none

$f(3) = 3\ln 3 = 3.3$

$f''(x) = \frac{1}{x} = 0$

DNE at $x = 0$

$f''(x)$ (+

0

Applied Optimization

32) Find 2 positive #s such that their product is 192 and the sum of the first and 3 times the second is as small as possible.

Looking for x, y where x & y are positive Numbers

Equations $xy = 192$ Secondary $y = \frac{192}{x}$

$f(x) = x + 3y$ Primary $f(x) = x + 3\left(\frac{192}{x}\right) = x + 576x^{-1}$

Domain $x > 0$ (and $y > 0$)

$f'(x) = 1 - 576x^{-2} = 1 - \frac{576}{x^2} =$

CP when $f'(x)$ DNE at $x = 0$ (but not in domain)

or when $f'(x) = 0 = 1 - \frac{576}{x^2}$

$\frac{576}{x^2} = 1$ $x^2 = 576$ $x = 24$ (Note, ignore $x = -24$ b/c $x > 0$)

Behavior of f ↘ ↗

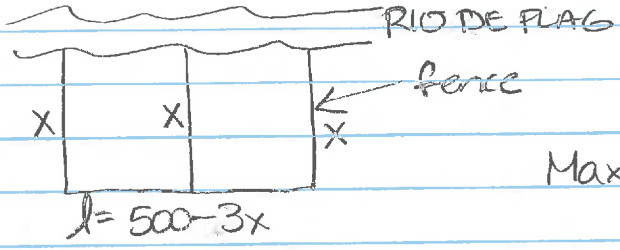
Sign of f' − − +

-1 0 1 24 25

$1 - \frac{576}{x^2}$ ↖ local min.

$x + 3y$ is minimized at $x = 24$ and $y = \frac{192}{24} = 8$

33)



Maximize Area

$A(x) = (500-3x)x = 500x - 3x^2$ Domain $(x, \frac{500}{3})$

$A'(x) = 500 - 6x$

CP at $A'(x) = 0 = 500 - 6x$

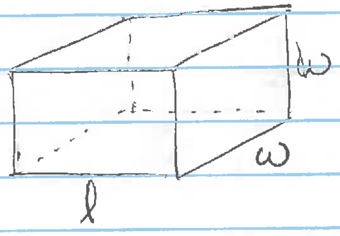
$6x = 500 \quad x = \frac{500}{6} = \frac{250}{3}$ ft

$\ell = 500 - 3(\frac{250}{3}) = 250$ ft

Dim: $\frac{250}{3}$ ft x 250 ft

Check: $\frac{250}{3} + \frac{250}{3} + \frac{250}{3} + 250 \overset{?}{=} 500$ ✓

34)



A box for shipping

Girth = 4w

Maximize Volume

Equations Domain: $(0, \frac{108}{4})$

$V(w) = \ell \cdot w^2$ Primary Eqn

$\ell + 4w \leq 108$ Secondary Eqn $\ell = 108 - 4w$

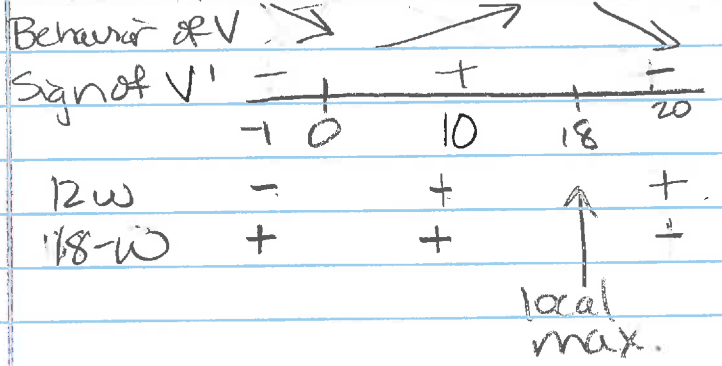
↖want to maximize this to maximize Volume

$V(w) = (108-4w)w^2 = 108w^2 - 4w^3$

$V'(w) = 216w - 12w^2 = 12w(18-w)$

CP at $V'(w) = 0 = 12w(18-w)$

CP at $w=0$ (not in domain) and $w=18$

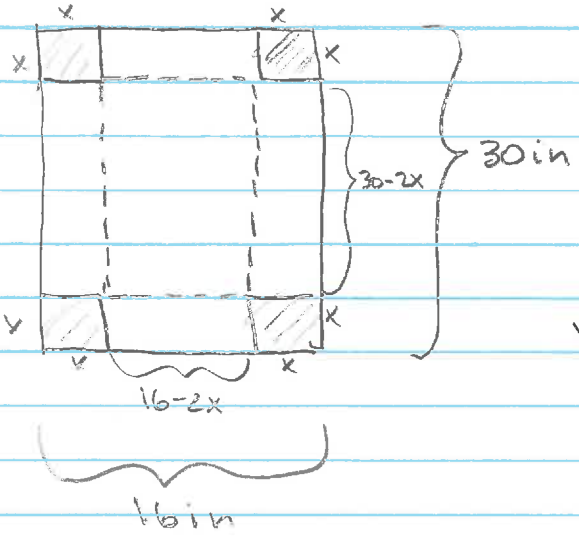


$\ell = 108 - 4(18)$

$\ell = 36''$

$$\boxed{\begin{array}{l} w = 18\text{ in} \\ \ell = 36\text{ in} \end{array}}$$

35.



$V = L \times W \times H$

$V(x) = (30-2x)(16-2x)x$

$V(x) = (480 - 92x + 4x^2)x$

$V(x) = 4x^3 - 92x^2 + 480x$

$0 \leq x \leq 8$

$V'(x) = 12x^2 - 184x + 480$

$4(3x^2 - 46x + 120) = 0$

$4(3x - 10)(x - 12) = 0$

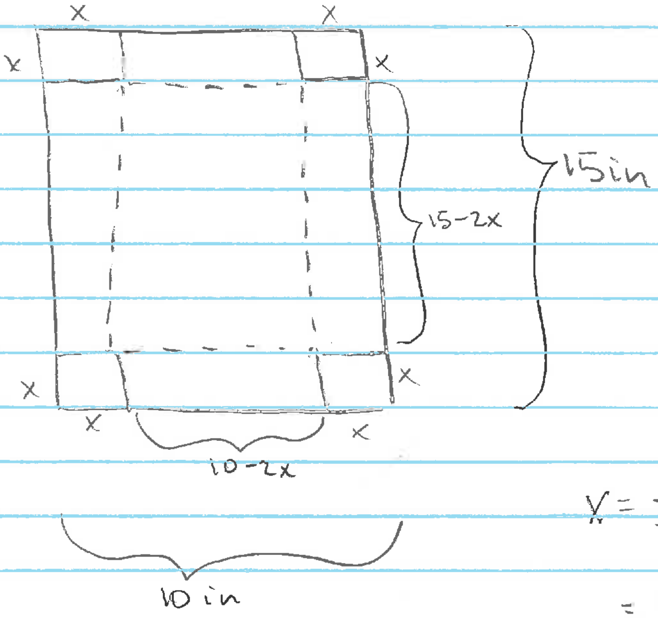
$x = \frac{10}{3}, 12$

↳ not possible for box!

- $V(0) = 0$
- $V(8) = 4(8)^3 - 92(8)^2 + 480(8) = 0$
- $V(\frac{10}{3}) = 4(\frac{10}{3})^3 - 92(\frac{10}{3})^2 + 480(\frac{10}{3}) = \frac{19600}{27} \approx 725.93$ max

The squares should be $\frac{10}{3}$ in by $\frac{10}{3}$ in.

36.



$V = L \times W \times H$

$V(x) = (15-2x)(10-2x)x$

$V(x) = (150 - 50x + 4x^2)x$

$V(x) = 4x^3 - 50x^2 + 150x$

$1 \leq x \leq 3$

$V'(x) = 12x^2 - 100x + 150$

$2(6x^2 - 50x + 75) = 0$

$x = \frac{-(-50) \pm \sqrt{(-50)^2 - 4(6)(75)}}{2(6)}$

$= \frac{50 \pm \sqrt{700}}{12} = \frac{50 \pm 10\sqrt{7}}{12}$

$x = \frac{25 + 5\sqrt{7}}{6}$, $x = \frac{25 - 5\sqrt{7}}{6}$

≈ 6.371 not in domain

≈ 1.962

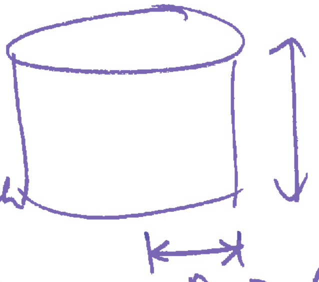
- $V(1) = 4(1)^3 - 50(1)^2 + 150(1)$

  $= 104 \text{ in}^3$ min volume
- $V(3) = 4(3)^3 - 50(3)^2 + 150(3)$

  $= 108$
- $V(\frac{25 - 5\sqrt{7}}{6}) \approx 132.038 \text{ in}^3$ max volume

37. A soup can:

Sides cost $ 0.015 per square inch



h = height, (inches). $h > 0$

r = radius, (inches). $r > 0$.

Top and bottom lids cost $ 0.027 per in^2

want volume $V = 16 \ in^3$.

what dimensions minimize cost?

$V = \pi r^2 h$ is volume of cylinder

$A_s = 2\pi r h$ is the area of the sides

$A_\ell = 2 \cdot \pi r^2$ is the area of the two lids

$C = 0.015 A_s + 0.027 A_\ell$ is the cost (in dollars)

$C = 0.015 \cdot 2\pi r h + 0.027 \cdot 2\pi r^2$, from substituting A_s and A_ℓ.

Eliminate r or h from the constraint $V = \pi r^2 h = 16$

To avoid $\sqrt{\ }$, eliminate $h = \frac{16}{\pi r^2}$

So $C = f(r) = 0.015 \cdot 2\pi r \cdot \frac{16}{\pi r^2} + 0.027 \cdot 2\pi r^2$

factor $0.015 = 5 \cdot 0.003$, $0.027 = 9 \cdot 0.003$

$= 0.003\left[5 \cdot 2 \cdot \frac{16}{r} + 9 \cdot 2\pi \cdot r^2\right]$

factor out 2

$$\begin{array}{r} 16 \\ \cdot 5 \\ \hline 80 \end{array}$$

$C = f(r) = 0.006\left[\frac{80}{r} + 9\pi r^2\right]$

Domain of f is $0 < r < \infty$, and $\lim_{r \to 0^+} f(r) = \infty$, and $\lim_{r \to \infty} f(r) = \infty$.

37, pg2. Find the critical point(s) (AKA critical number(s)) of f:

$$f'(r) = 0.006\left[-80r^{-2} + 18\pi r\right]$$

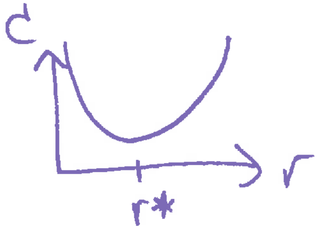
$$= 0.012\left[-\frac{40}{r^2} + 9\pi r\right] \overset{\text{set}}{=} 0$$

$$-\frac{40}{r^2} + 9\pi r = 0 \quad \therefore \quad 9\pi r = \frac{40}{r^2} \quad \therefore \quad r^3 = \frac{40}{9\pi}$$

The unique critical point is $r^* = \left(\frac{40}{9\pi}\right)^{1/3}$

Note that $f'(r) < 0$ for $0 < r < r^*$, and $f'(r) > 0$ for $r > r^*$.

Therefore, f has a global minimum at r^*.



The corresponding height is $h^* = \frac{16}{\pi (r^*)^2} = \frac{16}{\pi \left(\frac{40}{9\pi}\right)^{2/3}}$

We could leave it like that,

but let's simplify: $(40)^{2/3} = (8 \cdot 5)^{2/3} = 8^{2/3} \cdot 5^{2/3} = (8^{1/3})^2 \cdot 5^{2/3}$

$$= 2^2 \cdot 5^{2/3} = 4 \cdot 5^{2/3}$$

$\frac{\pi}{\pi^{2/3}} = \pi^{1/3}$, and $9^{2/3}$ can't be simplified.

So $h^* = \frac{16 \cdot 9^{2/3}}{\pi^{1/3} \cdot 4 \cdot 5^{2/3}} = \frac{4}{\pi^{1/3}} \cdot \left(\frac{9}{5}\right)^{2/3}$

Note that we are asked for the dimensions, not the cost.

To minimize the cost, make the radius $r^* = \left(\frac{40}{9\pi}\right)^{1/3}$ and the height $h^* = \frac{4}{\pi^{1/3}} \cdot \left(\frac{9}{5}\right)^{2/3}$