Supplemental Problems for Exam 3

General Information

Unless your instructor states otherwise, Exam 3 focuses on content in Sections 4.1–4.7, 5.1–5.3. In particular, here is a list of topics/concepts that you must have an understanding of or be able to do:

- Have an intuitive understanding of derivatives, based on your knowledge of rate of change, speed, velocity, and slope of tangent lines.
- Find the local linearization (i.e., tangent line approximation) and the equation of the tangent line to the graph of a function at a particular point.
- Evaluate limits of various indeterminate forms.
- Be able to apply L'Hoôpital's Rule and know when it applies.
- Be able to state the Mean Value Theorem and be able to apply it in appropriate circumstances.
- Know the definition of critical points and be able to find them. *Note:* We will use the terms "critical points" and "critical numbers" interchangeably.
- Understand the concept of local maximum and local minimum.
- Understand the relationship between critical points and local extrema.
- Understand the concept of global maximum and global minimum.
- Understand the relationship between the sign of the derivative and the direction of the original function.
- Understand the relationship between the sign of the second derivative and concavity of the original function.
- Know the First Derivative Test and be able to use it to make conclusions about local extrema.
- Know the Second Derivative Test and be able to use it to make conclusions about local extrema. When is this test inconclusive?
- Know the definition of a point of inflection and be able to find them.
- Be able to find vertical and horizontal asymptotes. For horizontal asymptotes, be prepared to use L'Hôpital's Rule.
- Be able to sketch the graph of a function following the function analysis process.
- Be able to find absolute extrema of a continuous function on a closed interval.
- Be able to solve optimization problems.
- Be able to evaluate a definite integral by interpreting it as a signed area of known geometric shapes (rectangles, triangles, and circles).
- Be able to approximate the value of a definite integral, especially those involving piecewise defined functions, using left and right Riemann sums.
- Be able to find antiderivatives and indefinite integrals.
- Be able to give examples and counter examples to demonstrate different properties of functions.

- Call upon your own mental faculties to respond in flexible, thoughtful and creative ways to problems that may seem unfamiliar on first glance.

The problems that follow will provide you with an opportunity to review the relevant topics. However:

This is not a practice test!

It is possible that problems on your exam will resemble problems on this review, but you should not expect exam problems to be identical to ones found below. This review contains an abundance of problems and it is not the intention that every student will complete every problem. You should complete as many problems in each section below as you think are necessary to solidify your understanding.

In addition, you should review examples done in class, as well as your homework exercises, especially the ones on the Weekly Homework assignments.

Words of Advice

Here are few things to keep in mind when taking your exam:

- Show all work! The thought process and your ability to show how and why you arrived at your answer is more important than the answer itself.
- The exam will be designed so that you can complete it without a calculator. If you find yourself yearning for a calculator, you might be doing something wrong.
- Make sure you have answered the question that you were asked. Also, ask yourself if your answer makes sense.
- If you know you made a mistake, but you can't find it, explain why you think you made a mistake and indicate where the mistake might be. This shows that you have a good understanding of the problem.
- If you write down an "=" sign, then you better be sure that the two expressions on either side are equal. Similarly, if two things are equal and it is necessary that they be equal to make your conclusion, then you better use "=."
- Don't forget to write limits where they are needed.

Linear Approximation

1. Determine the local linearization of each of the following functions at the indicated value.
   (a) $f(x) = \tan(x)$ at $x = \pi/6$.
   (b) $g(x) = \ln(e^x + e^{2x})$ at $x = 0$.
2. Suppose the local linearization of g at 2 is given by $l_2(x) = 3x - 9$. Determine $h'(2)$ if $h = f \circ g$ and $f(x) = e^{4x} - 5$.
3. Use local linearization to approximate each of the following.
   (a) $\ln(0.9)$.
   (b) $\sqrt{101}$.

Critical Numbers, Extrema, & Shape of a Graph

4. Provide an example of each of the following.

   (a) An *equation* of a function f such that f has a critical number at $x = 0$, but f does not have a local maximum or local minimum at $x = 0$.

   (b) An *equation* of a function g such that $g''(0) = 0$, but g does not have an inflection point at $x = 0$.

5. Find the critical numbers for each of the following functions.

   (a) $f(t) = 2t^3 + 3t^2 + 6t + 4$

   (b) $g(r) = \dfrac{r}{r^2 + 1}$

   (c) $h(x) = \sqrt{x}(1 - x)$

   (d) $f(\theta) = \sin^2(2\theta)$

6. Let $f(x) = \frac{6}{5}x^{5/3} - \frac{9}{2}x^{2/3}$. Find all critical numbers of f and then classify each critical number as a *local minimum*, *local maximum*, or *neither*. Sufficient work must be shown.

7. Let $f(x) = \dfrac{x^3}{3} + x^2 - 3x$.

   (a) Find the critical numbers of f.

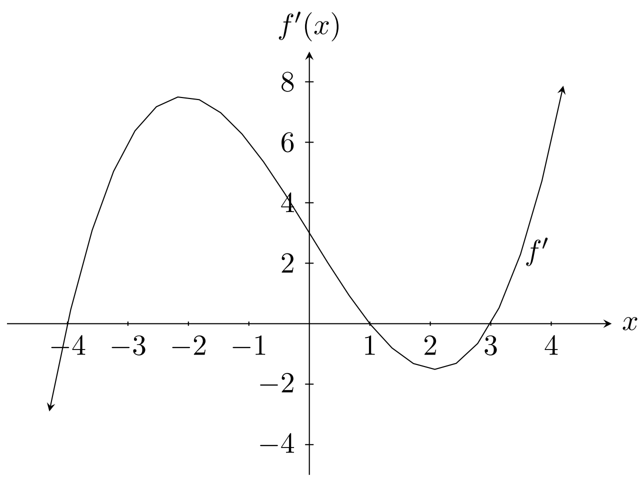
   (b) List the intervals where f is increasing.

   (c) List the intervals where f is decreasing.

   (d) Does f have any local minimums? If so, list the corresponding x-values.

   (e) Does f have any local maximums? If so, list the corresponding x-values.

8. Suppose f is a differentiable function such that the graph of f' is given below. Note this is the graph of f', NOT f.



   (a) List the critical numbers of f.

   (b) Find the interval(s) where f is increasing.

   (c) Find the interval(s) where f is decreasing.

   (d) Classify whether f has a local maximum, local minimum, or neither at each critical number.

9. A function f has a local minimum at $x = -1$ and $x = 3$ and a local maximum at $x = 2$. Sketch possible graphs for both f and f'.

10. Find the absolute maximum and minimum of $f(x) = \cos(x) - x$ on the interval $[0, 2\pi]$.

11. Find the absolute maximum and minimum of $g(x) = x^3 - 3x + 1$ on the interval $[0, 3]$.

12. Let $f(x) = \dfrac{x^5}{20} - \dfrac{x^4}{6} + \dfrac{x^3}{6} + 5x + 1$. Find the x-values of all inflection points for the graph of f.

Mean Value Theorem

13. If $f(x) = 10 - \frac{16}{x}$, then f satisfies the hypotheses of the Mean Value Theorem on the interval $[2, 8]$. Find the number c that the Mean Value Theorem guarantees exists.

14. Consider the function $f(x) = \dfrac{x}{x+2}$.

    (a) Verify that f satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 4]$ and then find the number c that the Mean Value Theorem guarantees exists.

    (b) Why does f not satisfy the hypotheses of the Mean Value Theorem on the interval $[-8, 6]$?

15. A truck driver handed in a ticket at a toll booth showing that in 2 hours he had covered 158 miles on a toll road with speed limit 70 mph. The driver was cited for speeding. Use the Mean Value Theorem to explain why. State the assumptions that we have to make about the position function $p(t)$ of the truck to be able to apply the Mean Value Theorem.

L'Hôpital's Rule

Evaluate each of the following limits. If you make use of L'Hôpital's Rule, indicate where.

16. $\displaystyle\lim_{x\to\infty} \frac{1-x^3}{2x^3-5x^2+1}$

17. $\displaystyle\lim_{x\to-\infty} \frac{1-x^3}{2x^3-5x^2+1}$

18. $\displaystyle\lim_{x\to 0} \frac{1-x^3}{2x^3-5x^2+1}$

19. $\displaystyle\lim_{x\to 0} \frac{x}{e^{4x}-1}$

20. $\displaystyle\lim_{x\to\infty} \frac{x}{e^{4x}-1}$

21. $\displaystyle\lim_{x\to 2} \frac{x}{e^{4x}-1}$

22. $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)}$

23. $\displaystyle\lim_{x\to 2} \frac{2e^{x-2}-x}{x^2-4}$

24. $\displaystyle\lim_{x\to\infty} x^2 e^{-x^2}$

25. $\displaystyle\lim_{x\to 0} \frac{4x^3}{e^x}$

26. $\displaystyle\lim_{x\to\infty} (\ln(x))^{\frac{1}{x}}$

27. $\displaystyle\lim_{x\to 0^+} [\sin(x)]^{\frac{1}{x}}$

28. $\displaystyle\lim_{x\to 0+} \frac{\sin(x)}{\ln(x)}$

29. $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

Curve Sketching

30. Sketch the graph of a function f that has the following properties.

    (1) $f(-5) = 0, f(-3) = -3, f(-2) = 0$

    (2) $f(-1.5) = .5, f(-.5) = 1, f(1.5) = 2.5$

    (3) $\displaystyle\lim_{x\to 0} f(x) = \infty$ and $\displaystyle\lim_{x\to 3} f(x) = \infty$

    (4) $\displaystyle\lim_{x\to\infty} f(x) = 1$ and $\displaystyle\lim_{x\to-\infty} f(x) = 1$

    (5) $f'(-3)$ undefined

    (6) $f'(1.5) = 0, f'(-1.5) = 0$

    (7) $f'(x) > 0$ on $(-3, -1.5)$, $(-1.5, 0)$, $(1.5, 3)$

    (8) $f'(x) < 0$ on $(-\infty, -3), (0, 1.5)$, $(3, \infty)$

    (9) $f''(x) > 0$ on $(-1.5, 0), (0, 3)$, $(3, \infty)$

    (10) $f''(x) < 0$ on $(-\infty, -3)$, $(-3, -1.5)$

31. Sketch the graph of each of the following functions.

(a) $f(x) = 8x^3 - 2x^4$

(b) $g(x) = 3x^4 - 8x^3 + 6x^2 + 1$

(c) $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$

(d) $g(x) = \dfrac{-x}{(x^2 - 1)^2}$

(e) $h(x) = x^{5/3} - 5x^{2/3}$

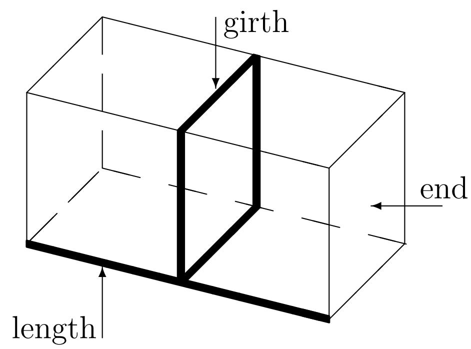
(f) $g(x) = x \ln(x)$

Applied Optimization

32. Find two positive numbers such that their product is 192 and the sum of the first and three times the second is as small as possible.

33. A farmer has 500 feet of fencing with which to enclose a pasture for grazing nuggets. The farmer only need to enclose 3 sides of the pasture since the remaining side is bounded by a river (no, nuggets can't swim). In addition, some of the nuggets don't get along with some of the other nuggets. He plans to separate the troublesome nuggets by forming 2 adjacent corrals. Determine the dimensions that would yield the maximum area for the pasture.

34. The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 inches. What dimensions (length and width) will give a box with a square end the largest possible volume? *Note:* Since the box has a square end, the girth is four times the width in this case.



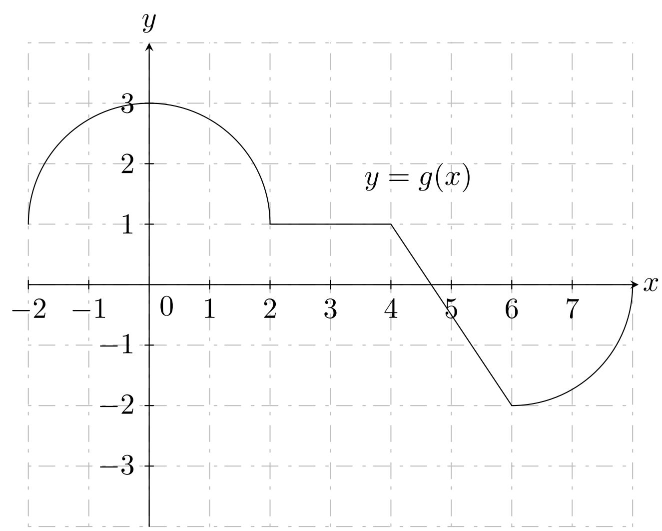
35. An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from each of the four corners and bending up the sides. What size should the squares be to obtain a box with the largest volume?

36. A rectangular piece of cardboard that is 10 inches by 15 inches is being made into a box without a top. To do so, squares are cut from each corner of the box and the remaining sides are folded up. If the box needs to be at least 1 inch deep and no more than 3 inches deep, what is the maximum possible volume of the box? what is the minimum volume?

37. A soup can in the shape of a right circular cylinder is to be made from two materials. The material for the side of the can costs \$0.015 per square inch and the material for the lids costs \$0.027 per square inch. Suppose that we desire to construct a can that has a volume of 16 cubic inches. What dimensions minimize the cost of the can?

Intuitive Definite Integral

1. Consider the graph of the function g that is given below. Assume that the graph is built from line segments, semi-circles, and quarter-circles.



Using the graph above, complete each of the following.

(a) Find $\int_0^4 g(x)\, dx$

(b) Find $\int_{-2}^8 g(x)\, dx$

2. Evaluate the following definite integrals using the graph of the function and basic area formulas.

(a) $\int_0^3 x + 2\, dx$

(b) $\int_{-2}^8 |x - 3| + 2\, dx$

(c) $\int_{-3}^3 -\sqrt{9 - x^2}\, dx$

(d) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\tan(x^5)\, dx$

Riemann Sums

3. Consider the function g given in Problem 1. Find the Riemann sum that approximates $\int_0^8 g(x)\, dx$ using 4 equal width subdivisions and:

(a) left end points.

(b) right end points.

4. Find the Riemann sum that approximates $\int_0^1 \sqrt{1 + x^3}\, dx$ using 5 equal width subdivisions and right endpoints. Do *not* evaluate your expression.

5. The following sum is a Riemann sum that approximates the definite integral $\int_2^b f(x)\, dx$ using a partition of the interval $[2, b]$ into 4 subintervals of equal width:

$$3(\sqrt{5} + 1) + 3(\sqrt{8} + 1) + 3(\sqrt{11} + 1) + 3(\sqrt{14} + 1).$$

(a) What is b?

(b) What is $f(x)$?

6. The following sum is a Riemann sum that approximates the definite integral $\int_1^b f(x)\ dx$ using a partition of the interval $[1, b]$ into n subintervals of equal width:

$$\frac{1}{1+\frac{2}{n}} \cdot \frac{2}{n} + \frac{1}{1+\frac{4}{n}} \cdot \frac{2}{n} + \frac{1}{1+\frac{6}{n}} \cdot \frac{2}{n} + \cdots + \frac{1}{1+\frac{2n}{n}} \cdot \frac{2}{n}.$$

   (a) What is b?
   (b) What is $f(x)$?

7. Evaluate the following definite integral using a limit of Riemann sums and right endpoints.

$$\int_0^1 x^2 - x\ dx$$

Indefinite Integrals

Compute each of the following integrals.

8. $\int 5\ dx$
9. $\int 0\ dx$
10. $\int 2x^3 + x^2 - 5x + 5\ dx$
11. $\int -2\sqrt{x}\ dx$
12. $\int \frac{x+1}{\sqrt{x}}\ dx$
13. $\int \frac{1}{x^3}\ dx$
14. $\int \frac{x+5}{x^2}\ dx$
15. $\int \frac{\sin(x)}{\cos^2(x)}\ dx$
16. $\int \sin(x)\ dx$
17. $\int \cos(2x)\ dx$
18. $\int e^{x/3}\ dx$
19. $\int \frac{x^3 - 2\sqrt{x}}{x}\ dx$
20. $\int \frac{4}{\sqrt{1-x^2}}\ dx$
21. $\int (3x-1)^2\ dx$
22. $\int (3x-1)^{99}\ dx$

Miscellaneous

23. Determine whether each of the following statements is true or false. Circle the correct answer.

   (a) **True** or **False**: $\int_a^b f(x)g(x)\ dx = \int_a^b f(x)\ dx \cdot \int_a^b g(x)\ dx$
   (b) **True** or **False**: $\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$
   (c) **True** or **False**: If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$.
   (d) **True** or **False**: If $f'(x) = g'(x)$, then $f(x) = g(x)$.
   (e) **True** or **False**: The formula $\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$ works for *all* values of n.

24. Find f that satisfies $f'(x) = \sqrt{x}$ and $f(4) = 0$.

25. Find f that satisfies $f''(x) = x^2 + 4$, $f'(3) = 1$, and $f(1) = 6$.

26. A zombie moves in a straight line with velocity $v(t) = -t + 4$ mph after t hours of his start. How far is he from his original position after 6 hours?