Quiz 2

Your Name:

Instructions

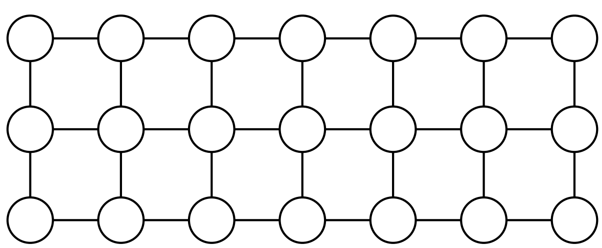
This quiz consists of two parts. In each part complete **two** problems for a total of four problems. You should provide detailed solutions on your own paper to the problems you choose to complete. I expect your solutions to contain sufficient justification. I also expect your solutions to be *well-written, neat, and organized*. Incomplete thoughts, arguments missing details, and scattered symbols and calculations are not sufficient. Each problem is worth 8 points for a total of 32 points. Good luck and have fun!

Part A

Complete **two** of the following problems.

A1. An overfull prison has decided to terminate some prisoners. The jailer comes up with a game for selecting who gets terminated. Here is his scheme. 10 prisoners are to be lined up all facing the same direction. On the back of each prisoner's head, the jailer places either a black or a red dot. Each prisoner can only see the color of the dot for all of the prisoners in front of them and the prisoners do not know how many of each color there are. The jailer may use all black dots, or perhaps he uses 3 red and 7 black, but the prisoners do not know. The jailer tells the prisoners that if a prisoner can guess the color of the dot on the back of their head, they will live, but if they guess incorrectly, they will be terminated. The jailer will call on them in order starting at the back of the line. Before lining up the prisoners and placing the dots, the jailer allows the prisoners 5 minutes to come up with a plan that will maximize their survival. What plan can the prisoners devise that will maximize the number of prisoners that survive? Some more info: each prisoner can hear the answer of the prisoner behind them and they will know whether the prisoner behind them has lived or died. Also, each prisoner can only respond with the word "black" or "red." You must carefully articulate your answer.

A2. In the lattice below, we color 11 vertices points black. Prove that no matter which 11 are colored black, we always have a rectangle with black vertices (and vertical and horizontal sides).



A3. You have 14 coins, dated 1901 through 1914. Seven of these coins are real and weigh 1.000 ounce each. The other seven are counterfeit and weigh 0.999 ounces each. You do not know which coins are real or counterfeit. You also cannot tell which coins are real by look or feel. Fortunately for you, Zoltar the Fortune-Weighing Robot is capable of making very precise measurements. You may place any number of coins in each of Zoltar's two hands and Zoltar will do the following:

- If the weights in each hand are equal, Zoltar tells you so and returns all of the coins.

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- If the weight in one hand is heavier than the weight in the other, then Zoltar takes one coin, at random, from the heavier hand as tribute. Then Zoltar tells you which hand was heavier, and returns the remaining coins to you.

Your objective is to identify a single real coin that Zoltar has not taken as tribute. You must carefully articulate your answer.

Part B

Complete **two** of the following problems.

B1. Let t_n denote the nth triangular number. Find both an algebraic proof and a visual proof of the following fact.

For all $a, b \in \mathbb{N}$, $t_{a+b} = t_a + t_b + ab$.

B2. In this problem, we will explore a modified version of the Sylver Coinage Game. In the new version of the game, a fixed positive integer $n \geq 3$ is agreed upon in advance. Then 2 players, A and B, alternately name positive integers from the set $\{1, 2, \ldots, n\}$ that are not the sum of nonnegative multiples of previously named numbers among $\{1, 2, \ldots, n\}$. The person who is forced to name 1 is the loser! Here is a sample game between A and B using the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (i.e., $n = 10$):

(a) A opens with 4. Now neither player can name $4, 8$.

(b) B names 5. Neither player can name $4, 5, 8, 9, 10$.

(c) A names 6. Neither player can name $4, 5, 6, 8, 9, 10$.

(d) B names 3. Neither player can name $3, 4, 5, 6, 7, 8, 9, 10$.

(e) A names 2. Neither player can name $2, 3, 4, 5, 6, 7, 8, 9, 10$.

(f) B is forced to name 1 and loses.

Suppose player A always goes first. Argue that if there exists an n such that player B is guaranteed to win on the set $\{1, 2, \ldots, n\}$ as long as he or she plays intelligently, then player A is guaranteed to win on the set $\{1, 2, \ldots, n, n+1\}$ as long as he or she plays intelligently. Your argument should describe a strategy for player A.

B3. A certain store sells a product called widgets in boxes of $7, 9$, and 11. A number n is called *widgetable* if one can buy exactly n widgets by buying some number of boxes. What is the largest non-widgetable number?