Problem Collection for Introduction to Mathematical Reasoning

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Problem 1. Three strangers meet at a taxi stand and decide to share a cab to cut down the cost. Each has a different destination but all are heading in more-or-less the same direction. Bob is traveling 10 miles, Sally is traveling 20 miles, and Mike is traveling 30 miles. If the taxi costs $2 per mile, how much should each contribute to the total fare? What do you think is the most common answer to this question?

Problem 2. Christine wants to take yoga classes to increase her strength and flexibility. In her neighborhood, there are two yoga studios: Namaste Yoga and Yoga Spirit. At Namaste Yoga, a student's first class costs $12, and additional classes cost $10 each. At Yoga Spirit, a student's first class costs $24, and additional classes cost $8 each. Because Christine wants to save money, she is interested in comparing the costs of the two studios. For what number of yoga classes do the two studios cost the same amount?

Problem 3. Imagine a hallway with 1000 doors numbered consecutively 1 through 1000. Suppose all of the doors are closed to start with. Then some dude with nothing better to do walks down the hallway and opens all of the doors. Because the dude is still bored, he decides to close every other door starting with door number 2. Then he walks down the hall and changes (i.e., if open, he closes it; if closed, he opens it) every third door starting with door 3. Then he walks down the hall and changes every fourth door starting with door 4. He continues this way, making a total of 1000 passes down the hallway, so that on the 1000th pass, he changes door 1000. At the end of this process, which doors are open and which doors are closed?

Problem 4. The Sunny Day Juice Stand sells freshly squeezed lemonade and orange juice at the farmers' market. The juices are ladled out of large glass jars, each holding exactly the same amount of juice. Linda and Julie set up their stand early one Saturday morning. The first customer of the day ordered orange juice and Linda carefully ladled out 8 ounces into a paper cup. As she was about to hand the cup to the customer, he changed his mind and asked for lemonade instead. Accidentally, Linda dumped the cup of orange juice into the jar of lemonade. She quickly mixed up the juices, ladled out a cup of the mixture (mostly lemonade) and turned to hand it to the customer. "I've decided I don't want anything to drink right now," he said, and frazzled, Linda dumped the cupful of juice mixture into the orange juice jar. Linda's assistant, Julie, watched all of this with amusement. As the man walked away, she wondered aloud, "Now is there more orange juice in the lemonade or more lemonade in the orange juice?"

Problem 5. Imagine you have 25 pebbles, each occupying one square on a 5 by 5 chess board. Tackle each of the following variations of a puzzle.

(a) Variation 1: Suppose that each pebble must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(b) Variation 2: Suppose that all but one pebble (your choice which one) must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(c) Variation 3: Consider Variation 1 again, but this time also allow diagonal moves to adjacent squares. If this is possible, describe a solution. If this is impossible, explain why.

Problem 6. Consider an $n \times n$ chess board and variation 1 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 7. Consider an $n \times n$ chess board and variation 2 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 8. Describe where on Earth from which you can travel one mile south, then one mile east, and then one mile north and arrive at your original location. There is more than one such location. Find them all.

Problem 9. I have 10 sticks in my bag. The length of each stick is an integer. No matter which 3 sticks I try to use, I cannot make a triangle out of those sticks. What is the minimum length of the longest stick?

Problem 10. Suppose there are two bags of candy containing 8 pieces and 6 pieces, respectively. You and your friend are going to play a game and the winner gets to eat all of the candy. Here are the rules for the game:

1. You and your friend will alternate removing pieces of candy from the bags. Let's assume that you go first.
2. On each turn, the designated player selects a bag that still has candy in it and then removes at least one piece of candy. The designated player can only remove candy from a single bag and he/she must remove at least one piece.
3. The winner is the one that removes all the candy from the last remaining bag.

Does one of you have a guaranteed winning strategy? If so, describe that strategy. Can you generalize to handle any number of pieces of candy in either of the two bags?

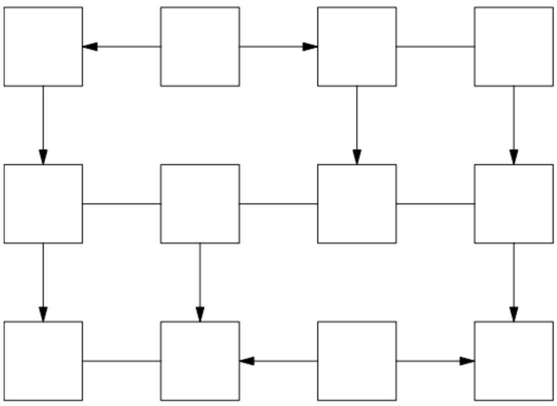
Problem 11. Suppose you have 6 toothpicks that are exactly the same length. Can you arrange the toothpicks so that exactly 4 identical triangles are formed? You cannot cut, break, or bend the toothpicks. Moreover, each vertex of a triangle must be formed when the tips of two toothpicks meet.

Problem 12. An ant is crawling along the edges of a unit cube. What is the maximum distance it can cover starting from a corner so that it does not cover any edge twice?

Problem 13. The grid below has 12 boxes and 15 edges connecting boxes. In each box, place one of the six integers from 1 to 6 such that the following conditions hold:

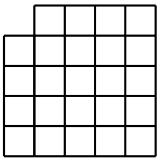
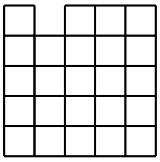
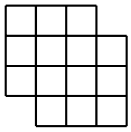
- For each possible pair of distinct numbers from 1 to 6, there is exactly one edge connecting two boxes with that pair of numbers.
- If an edge has an arrow, then it points from a box with a smaller number to a box with a larger number.

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above.



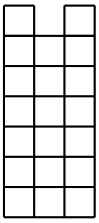
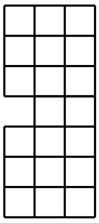
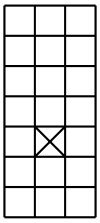
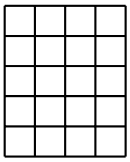
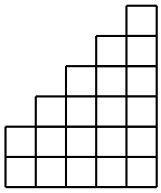
Problem 14. Four red ants and two black ants are walking along the edge of a one meter stick. The four red ants, called Albert, Bart, Debbie, and Edith, are all walking from left to right, and the two black ants, Cindy and Fred, are walking from right to left. The ants always walk at exactly one centimeter per second. Whenever they bump into another ant, they immediately turn around and walk in the other direction. And whenever they get to the end of a stick, they fall off. Albert starts at the left hand end of the stick, while Bart starts 20.2 cm from the left, Debbie is at 38.7cm, Edith is at 64.9cm and Fred is at 81.8cm. Cindy's position is not known—all we know is that he starts somewhere between Bart and Debbie. Which ant is the last to fall off the stick? And how long will it be before he or she does fall off?

Problem 15. Tile the following grids with dominoes. If a tiling is not possible, explain why.

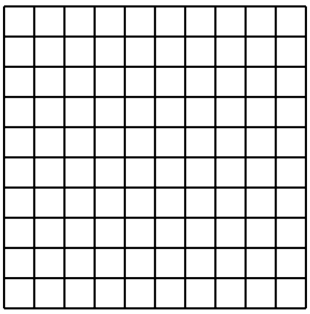
(a)  (b)  (c) 

Problem 16. Find all tetrominoes (polymonio with 4 cells).

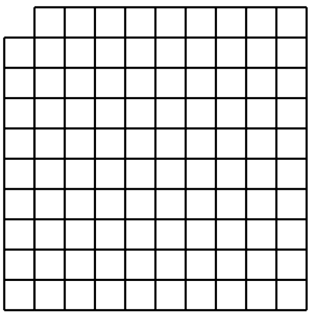
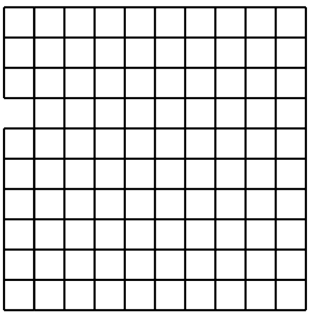
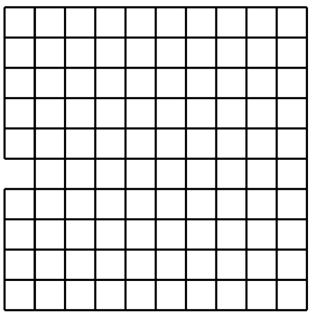
Problem 17. Tile the following grids using every tetromino exactly once. The X in (c) denotes an absence of an available square in the grid. If a tiling is not possible, explain why.

(a)  (b)  (c)  (d)  (e) 

Problem 18. Consider the 10×10 grid of squares below. Show that you can color the squares of the grid with 3 colors so that every consecutive row of 3 squares and every consecutive column of 3 squares uses all 3 colors.



Problem 19. Tile each of the grids below with trominoes that consist of 3 squares in a line. If a tiling is not possible, explain why.

(a)  (b)  (c) 

Problem 20. A mouse eats her way through a $3\times3\times3$ cube of cheese by tunneling through all of the 27 $1\times1\times1$ subcubes. If she starts at one corner and always moves to an uneaten subcube by passing through a face of a subcube, can she finish at the center of the cube?

Problem 21. There is a plate of 40 cookies. You and your friend are going to take turns taking either 1 or 2 cookies from the plate. However, it is a faux pas to take the last cookie, so you want to make sure that you do not take the last cookie. How can you guarantee that you will never be the one taking the last cookie? What about n cookies?

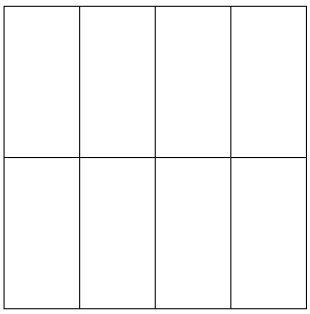
Problem 22. The Sylver Coinage Game is a game in which 2 players alternately name positive integers that are not the sum of nonnegative multiples of previously named integers. The person who names 1 is the loser! Here is a sample game between A and B:

1. A opens with 5. Now neither player can name $5, 10, 15, \ldots$
2. B names 4. Now neither player can name 4, 5, 8, 9, 10, or any number greater than 11.
3. A names 11. Now the only remaining numbers are 1, 2, 3, 6, and 7.
4. B names 6. Now the only remaining numbers are 1, 2, 3, and 7.
5. A names 7. Now the only remaining numbers are 1, 2, and 3.
6. B names 2. Now the only remaining numbers are 1 and 3.
7. A names 3, leaving only 1.
8. B is forced to name 1 and loses.

If player A names 3, can you find a strategy that guarantees that the second player wins? If so, describe the strategy? If such a strategy is not possible, then explain why?

Problem 23. How many factors of 10 are there in 50! (i.e., 50 factorial)?

Problem 24. How many rectangles are in the figure below?



Problem 25. Four prisoners are making plans to escape from jail. Their current plan requires them to cross a narrow bridge in the dark that has no handrail. In order to successfully cross the bridge, they must use a flashlight. However, they only have a single flashlight. To complicate matters, at most two people can be on the bridge at the same time. So, they will need to make multiple trips across the bridge, returning the flashlight back to the first side of the bridge by having someone walk it back. Unfortunately, they can't throw the flashlight. It takes 1, 2, 5, and 10 minutes for prisoner A, prisoner B, prisoner C, and prisoner D to cross the bridge and when two prisoners are walking together with the flashlight, it takes the time of the slower prisoner. What is the minimum total amount of time it takes all four prisoners to get across the bridge?

Problem 26. In order to assess the reasoning skills of a newly developed android robot with artificial intelligence, the android's creator designs the following experiment. On Sunday, the creator describes the details of the experiment to the android and then turns the the android off. Once or twice, during the experiment, the android will be turned on, interviewed, and then turned back off. In addition, the creator will erase the awakening from the android's memory. On Sunday evening, a fair coin will be tossed to determine which experimental procedure to undertake:

- If the coin comes up heads, the android will be awakened and interviewed on Monday only.
- If the coin comes up tails, the android will be awakened and interviewed on both Monday and Tuesday.

In either case, the android will be awakened on Wednesday without interview and the experiment ends. Any time the android is awakened and interviewed, it will not be able to tell which day it is or whether it has been awakened before. During the interview the android is asked: "What is your credence now for the proposition that the coin landed heads?". One way to interpret "credence" in this context is the android's determination of the probability that the coin landed on heads. How should/would the android answer the interviewer's question?

Problem 27. As a broke college student, you agree to take part in a recurring experiment. Each experiment begins on Sunday evening and ends on Wednesday morning. The experiment will be repeated 100 weeks in a row. You are told the details of the experiment in advance. Each Sunday evening, the experimenter describes the details of the experiment and then gives you a drug to put you to sleep. Once or twice, during the experiment, you will be awakened, interviewed, and then put back to sleep using a drug that includes an amnesia-inducing component that makes you forget the awakening. On Sunday evening, a fair coin will be tossed to determine which experimental procedure to undertake:

- If the coin comes up heads, you will be awakened and interviewed on Monday only.
- If the coin comes up tails, you will be awakened and interviewed on both Monday and Tuesday.

In either case, you will be awakened on Wednesday without interview and the experiment ends. Any time you are awakened and interviewed, you will not be able to tell which day it is or whether you have been awakened before. During the interview you will be asked: "Is the coin heads or tails?". You are required to respond with either "heads" or "tails". The experimenter will record whether you were correct or not, but you will not be told whether you guessed correctly. At the end of the 100th run of the experiment, you will be given $10 for each correct answer that you gave. What strategy should you employ in order to optimize your profit?

Problem 28. Take 15 poker chips or coins, divide into any number of piles with any number of chips in each pile. Arrange piles in adjacent columns. Take the top chip off every column and make a new column to the left. Repeat forever. What happens? Make conjectures about what happens when we change the number of chips.

Problem 29. The nth *triangular number* is defined via $t_n := 1+2+\cdots+n$. For example, $t_4 = 1+2+3+4 = 10$. Find a visual proof of the following fact. By "visual proof" we mean a sufficiently general picture that is convincing enough to justify the claim.

For all $n \in \mathbb{N}$, $t_n = \dfrac{n(n+1)}{2}$.

Problem 30. Let t_n denote the nth triangular number. Find both an algebraic proof and a visual proof of the following fact.

For all $n \in \mathbb{N}$, $t_n + t_{n+1} = (n+1)^2$.

Problem 31. Find a visual proof of the following fact. *Warning:* This problem is not about triangular numbers.

For $n \in \mathbb{N}$, $1+3+5+\cdots+(2n-1) = n^2$.

Problem 32. Suppose someone draws 20 distinct random lines in the plane. What is the maximum number of intersections of these lines?

Problem 33. A certain fast-food chain sells a product called "nuggets" in boxes of 6, 9, and 20. A number n is called *nuggetable* if one can buy exactly n nuggets by buying some number of boxes. For example, 21 is nuggetable since you can buy two boxes of six and one box of nine to get 21. Here are the first few nuggetable numbers:

$$6, 9, 12, 15, 18, 20, 21, 24, 26, 27, \ldots$$

and here are the first few non-nuggetable numbers:

$$1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots$$

What is the largest non-nuggetable number?

Problem 34. Let t_n denote the nth triangular number. Find an algebraic and a visual proof of the following fact.

For all $a, b \in \mathbb{N}$, $t_{ab} = t_a t_b + t_{a-1} t_{b-1}$.

Problem 35. How many ways can 110 be written as the sum of 14 different positive integers? *Hint:* First, figure out what the largest possible integer could be in the sum. Note that the largest integer in the sum will be maximized when the other 13 numbers are as small as possible. Finish off the problem by doing an analysis of cases.

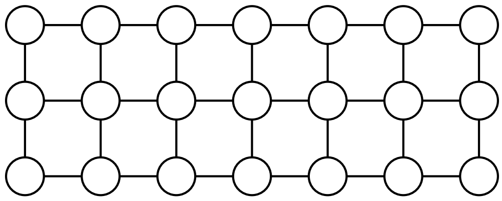
Problem 36. Suppose you randomly cut a stick into 3 pieces. What is the probability that you can form a triangle out of these 3 pieces?

Problem 37. Suppose you randomly pick 3 distinct points on a circle. What is the probability that the center of the circle lies in the interior of the triangle formed by these 3 points?

Problem 38. Consider a gambler who tosses a coin at most 6 times, and if it comes out heads (H), wins a dollar, and if it comes out tails (T), loses a dollar. He is kicked out as soon as he is in the red, i.e., has negative capital. In how many ways can he survive to 6 rounds, but at the end break even?

Problem 39. An overfull prison has decided to terminate some prisoners. The jailer comes up with a game for selecting who gets terminated. Here is his scheme. 10 prisoners are to be lined up all facing the same direction. On the back of each prisoner's head, the jailer places either a black or a red dot. Each prisoner can only see the color of the dot for all of the prisoners in front of them and the prisoners do not know how many of each color there are. The jailer may use all black dots, or perhaps he uses 3 red and 7 black, but the prisoners do not know. The jailer tells the prisoners that if a prisoner can guess the color of the dot on the back of their head, they will live, but if they guess incorrectly, they will be terminated. The jailer will call on them in order starting at the back of the line. Before lining up the prisoners and placing the dots, the jailer allows the prisoners 5 minutes to come up with a plan that will maximize their survival. What plan can the prisoners devise that will maximize the number of prisoners that survive? Some more info: each prisoner can hear the answer of the prisoner behind them and they will know whether the prisoner behind them has lived or died. Also, each prisoner can only respond with the word "black" or "red." What if there are n prisoners?

Problem 40. In the lattice below, we color 11 vertices points black. Prove that no matter which 11 are colored black, we always have a rectangle with black corners.



Problem 41. Each point of the plane is colored red or blue. Show that there is a rectangle whose corners are all the same color.

Problem 42. You have 14 coins, dated 1901 through 1914. Seven of these coins are real and weigh 1.000 ounce each. The other seven are counterfeit and weigh 0.999 ounces each. You do not know which coins are real or counterfeit. You also cannot tell which coins are real by look or feel. Fortunately for you, Zoltar the Fortune-Weighing Robot is capable of making very precise measurements. You may place any number of coins in each of Zoltar's two hands and Zoltar will do the following:

- If the weights in each hand are equal, Zoltar tells you so and returns all of the coins.
- If the weight in one hand is heavier than the weight in the other, then Zoltar takes one coin, at random, from the heavier hand as tribute. Then Zoltar tells you which hand was heavier, and returns the remaining coins to you.

Your objective is to identify a single real coin that Zoltar has not taken as tribute.

Problem 43. Our space ship is at a Star Base with coordinates $(1,2)$. Our hyper drive allows us to jump from coordinates (a,b) to either coordinates $(a, a+b)$ or to coordinates $(a+b, b)$. How can we reach the impending enemy attack at coordinates $(8,13)$?

Problem 44. Consider our Star Base from the previous problem. Recall that our hyper drive allows us to jump from coordinates (a,b) to either coordinates $(a,a+b)$ or to coordinates $(a+b,b)$. If we start at $(1,0)$, which points in the plane can we get to by using our hyper drive? Justify your answer.

Problem 45. Find all integers a,b,c,d, and e, such that

$$\begin{aligned} a^2 &= a+b-2c+2d+e-8 \\ b^2 &= -a-2b-c+2d+2e-6 \\ c^2 &= 3a+2b+c+2d+2e-31 \\ d^2 &= 2a+b+c+2d+2e-2 \\ e^2 &= a+2b+3c+2d+e-8. \end{aligned}$$

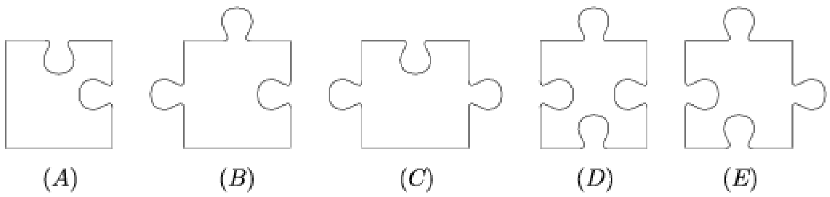
Problem 46. There are 30 red, 40 yellow, 50 blue, and 60 green balls in a box. We take out balls from the box with closed eyes. On the first turn we take out 1 ball, on the second turn we take out 2, and so on. On the nth turn we take out n balls. What is the minimum number of balls we need to take out to guarantee the following:

(a) We have a blue ball;

(b) We have a red and a green ball;

(c) We have all four colors.

Problem 47. A rectangular puzzle that says "850 pieces" actually consists of 851 pieces. Each piece is identical to one of the 5 samples shown in the diagram. How many pieces of type (E) are there in the puzzle?



Problem 48. In the game Turnaround, you are given a permutation of the numbers from 1 to n. Your goal is to get them in the natural order $12\cdots n$. At each stage, your only option is to reverse the order of the first k places (you get to pick k at each stage). For instance, given 6375142, you could reverse the first four to get 5736142 and then reverse the first six to get 4163752. Solve the following sequence in as few moves as possible: 352614.

Problem 49. A signed permutation of the numbers 1 through n is a fixed arrangement of the numbers 1 through n, where each number can be either be positive or negative. For example, $(-2,1,-4,5,3)$ is a signed permutation of the numbers 1 through 5. In this case, think of positive numbers as being right-side-up and negative numbers as being upside-down. A *reversal* of a signed permutation is the act of performing a 180-degree rotation to some consecutive subsequence of the permutation. That is, a reversal swaps the order of a subsequence of numbers while changing the sign of each number in the subsequence. Performing a reversal to a signed permutation results in a new signed permutation. For example, if we perform a reversal on the second, third, and fourth entries in $(-2,1,-4,5,3)$, we obtain $(-2,-5,4,-1,3)$. The *reversal distance* of a signed permutation of 1 through n is the minimum number of reversals required to transform the given signed permutation into $(1,2,\ldots,n)$. It turns out that the reversal distance of $(3,1,6,5,-2,4)$ is 5. Find a sequence of 5 reversals that transforms $(3,1,6,5,-2,4)$ into $(1,2,3,4,5,6)$.

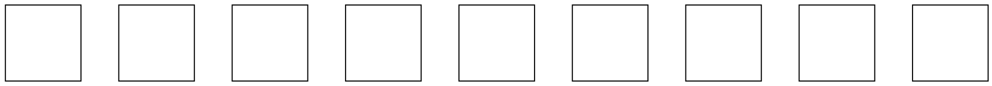
Problem 50. Consider a tournament with 15 teams. If every team plays every other team, how many games were played?

Problem 51. Two different positive numbers a and b each differ from their reciprocal by 1. What is $a+b$?

Problem 52. Let X be the intersection of the diagonals of the trapezoid $ABCD$ with parallel sides AB and CD. Show that the areas of triangles AXD and BXC are the same.

Problem 53. There are 8 frogs and 9 rocks on a field. The 9 rocks are laid out in a horizontal line. The 8 frogs are evenly divided into 4 green frogs and 4 brown frogs. The green frogs sit on the first 4 rocks facing right and the brown frogs sit on the last 4 rocks facing left. The fifth rock is vacant for now. Switch the places of the green and brown frogs by using the following rules:

- A frog can only jump forward
- A frog can hop to an vacant rock one place ahead
- A frog can leap over its neighbor frog to a vacant rock two places ahead



Can we generalize this problem and find how many jumps are necessary to switch n green and n brown frogs?

Problem 54. Four people are lined up on some steps. They are all looking down the steps and a wall separates the fourth person from the other three. In particular:

- Person 1 can see persons 2 and 3.
- Person 2 can see person 3.
- Person 3 cannot see anyone.
- Person 4 cannot see anyone.

All four people are wearing hats. They are told that there are two white hats and two black hats. Initially, no one knows what color hat they are wearing. They are told to shout out the color of the hat that they are wearing as soon as they know for certain what color it is. Additional constraints:

- They are not allowed to turn around or move.
- They are not allowed to talk to each other.
- They are not allowed to take their hats off.

Who is the first person to shout out the color of his/her hat and why?

Problem 55. Which of the following statements is/are true?

1. Exactly one of the statements in this list is false.
2. Exactly two of the statements in this list are false.
3. Exactly three of the statements in this list are false.
4. Exactly four of the statements in this list are false.
5. Exactly five of the statements in this list are false.
6. Exactly six of the statements in this list are false.
7. Exactly seven of the statements in this list are false.
8. Exactly eight of the statements in this list are false.
9. Exactly nine of the statements in this list are false.
10. Exactly ten of the statements in this list are false.

Problem 56. Which answer in the list is the correct answer to this question?

1. All of the below.
2. None of the below.
3. All of the above.
4. One of the above.
5. None of the above.
6. None of the above.

Problem 57. Annie, Bob, and Cristy are sitting by a campfire when Cristy announces that she is thinking of a 3-digit number. She then tells Annie and Bob that the number she is thinking of is one of the following:

$$515, 516, 519, 617, 618, 714, 716, 814, 815, 817.$$

Next, Cristy whispers the leftmost digit in Annie's ear and then whispers the remaining two digits in Bob's ear. The following conversation then takes place:

Annie: I don't know what the number is, but I know Bob doesn't know too.

Bob: At first I didn't know what the number was, but now I know.

Annie: Ah, then I know the number, too.

From that information, determine Cristy's number.

Problem 58 (Two Deep)**.** Consider the equation below. If a is a number, what number is it?

$$a = \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cdots}}}}}$$

Problem 59. You and your two friends Thor and Valkyrie are captured by Loki. In order to gain your freedom, Loki sets you the following challenge. The three of you are put in adjacent cells. In each cell is a quantity of stones. Each of you can count the number of stones in your own cell, but not in anyone else's. You are told that each cell has at least one stone but at most nine stones, and no two cells have the same number of stones. The rules of the challenge are as follows: The three of you will ask Loki a single question each, which he will answer truthfully "Yes" or "No". Every one hears the questions and the answers. Loki will set all of you free only if one of you can correctly determine the total number of stones in all the cells. Here is the initial conversation.

Thor: Is the total an even number?

Loki: No.

Valkyrie: Is the total a prime number?

Loki: No

You have five stones in your cell. What question will you ask? You should assume that Thor and Valkyrie are just as good at logic as you are.

Problem 60. We have the following information about three integers:

(a) Their product is an integer;

(b) Their product is a prime;

(c) One of them is the average of the other two.

What are these numbers? *Hint*: You need to find all such triples and show that there are no others.

Problem 61. Three boxes, one with black, one with white, and one with black and white balls. Each of the boxes is labeled B, W, and BW, but unfortunately, *all* the boxes are labeled incorrectly. Moreover, you cannot see inside each of the boxes, but you can reach in and pull a ball out. What is the minimum number of balls that need to be pulled before you can relabel all the boxes correctly?

Problem 62. We have two strings of pyrotechnic fuse. The strings do not look homogeneous in thickness but both of them have a label saying 4 minutes. So we can assume that it takes 4 minutes to burn through either of these fuses. How can we measure a one minute interval?

Problem 63. My Uncle Robert owns a stable with 25 race horses. He wants to know which three are the fastest. He owns a race track that can accommodate five horses at a time. What is the minimum number of races required to determine the fastest three horses?

Problem 64. A father has 20 one dollar bills to distribute among his five sons. He declares that the oldest son will propose a scheme for dividing up the money and all five sons will vote on the plan. If a majority agree to the plan, then it will be implemented, otherwise dad will simply split the money evenly among his sons. Assume that all the sons act in a manner to maximize their monetary gain but will opt for evenly splitting the money, all else being equal. What proposal will the oldest son put forth, and why?

Problem 65. Imagine that in the scenario of the previous problem the father decides that after the oldest son's plan is unveiled, the second son will have the opportunity to propose a different division of funds. The sons will then vote on which plan they prefer. Assume that the sons still act to maximize their monetary gain, but will vote for the older son's plan if they stand to receive the same amount of money either way. What will transpire in this case, and why?

Problem 66. A box contains two red hats and three green hats. Azalea, Barnaby, and Caleb close their eyes, take a hat from the box and put it on. When they open their eyes they can see each other's hats but not their own. They do not know which hats are left in the box. We can assume that all the protagonists are perfect logicians who tell the truth. They know all the information in the above paragraph. In addition, one of them is colorblind. They all know who the colorblind person is.

> Azalea says: "I don't know the color of my hat."
>
> Barnaby says: "I don't know the color of my hat."
>
> Caleb says: "I don't know the color of my hat."
>
> Azalea says: "I don't know the color of my hat."

Who is the colorblind person, and what color is their hat?

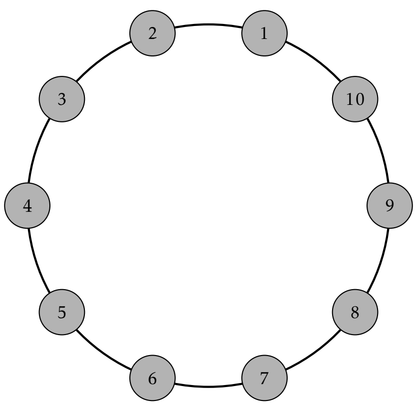
Problem 67. Given enough space, the population of a certain type of bacteria doubles every minute. Suppose one bacterium is placed in a bottle at 11:00AM and an hour later, the bottle is full.

(a) At what time is the bottle half full?

(b) Suppose that at 11:15AM an intelligent bacterium recognizes the space limitations her fellow bacteria are going to have in 45 minutes. The bacteria look around the room and notice 3 empty bottles nearby. Shortly thereafter, the bacteria start emigrating to the empty bottles in an attempt to prolong their existence. At what time will all 4 bottles be full?

Problem 68. The first vote counts of the papal conclave resulted in 33 votes each for candidates A and B and 34 votes for candidate C. The cardinals then discussed the candidates in pairs. In the second round each pair of cardinals with differing first votes changed their votes to the third candidate they did not vote for in the first round. The new vote counts were 16, 17 and 67. They were about to start the smoke signal when Cardinal Ordinal shouted "wait". What was his reason?

Problem 69. Ten people form a circle. Each picks a number and tells it to the two neighbors adjacent to him/her in the circle. Then each person computes and announces the average of the numbers of his/her two neighbors. The figure shows the average announced by each person. What is the number picked by the person who announced 6?



Problem 70. Alice and Brenda both ran in a 100-meter race. When Alice crossed the finish line, Brenda was 10 meters behind her. The girls then repeated the race with Alice starting 10 meters behind Brenda.

(a) If each girl ran the same rate in both races, then who won the second race? By how many meters?

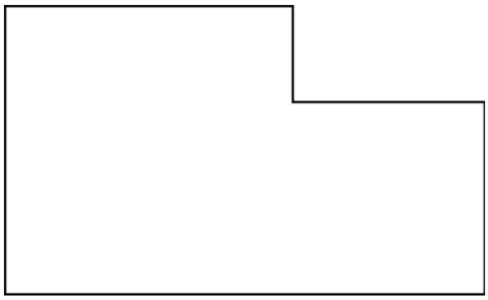
(b) Assuming the girls run the same rate, how many meters behind Brenda should Alice start in order for them to finish in a tie?

Problem 71. In the game Light Up, two players alternately choose unlit squares from an $m \times n$ grid of light-up squares. The objective of the game is to be the first to light up the entire grid. At the beginning of the game, all squares are turned off. On each player's turn, the player selects any square that is currently off and then the selected square gets lit up. Moreover, additional squares get lit up if at least two of its immediate neighbors (horizontal or vertical) are lit up. This process continues until no new squares are lit up and then it is the next player's turn. The loser of the game is the player that no longer has an available square to light up. Determine which player has a winning strategy for the following grid sizes: 1×3, 1×4, 1×5, 2×2, 2×3, 3×3.

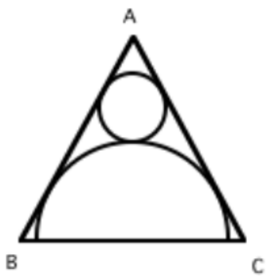
Problem 72. Find a solution to the equation $28x + 30y + 31z = 365$, where x, y, and z are positive whole numbers.

Problem 73. A rectangle that is not a square is folded along a diagonal. Prove that the perimeter of the resulting pentagon is smaller than the perimeter of the original rectangle.

Problem 74. This shape below is made by joining two squares, one 3×3, one 2×2. Divide it into a few pieces which can be re-assembled to make a square.



Problem 75. The figure below shows an equilateral triangle ABC with an inscribed semicircle of radius R that is tangent to sides AB and AC, and inscribed circle of radius r that is tangent to the triangle and the semicircle. Find the value of r/R.

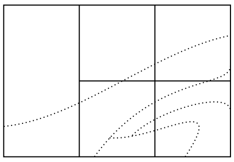
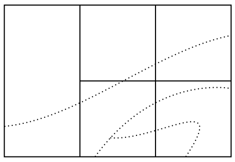
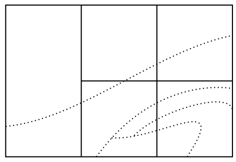


Problem 76. During a class period students used their cell phones once. In fact, for any two students there was a time when both of the students used their phones. Show that there was a time when nobody listened to the instructor.

Problem 77. Consider the regular hexagon $ABCDEF$. Let X be the midpoint of CD and let Y be the midpoint of DE. Let Z be the common point of AX and BY. Which polygon has larger area, ABZ or $DXZY$?

Problem 78. A colony of chameleons on an island currently comprises 13 green, 15 blue, and 17 red individuals. When two chameleons of different colors meet, they both change their colors to the third color. Is it possible that all chameleons in the colony eventually have the same color?

Problem 79. There are five countries on an island. The island also has several species of frogs. The frog territories do not overlap. During an international conference of frog experts, each country wants to create a frog exhibit featuring one of the frog species that live in the country. No two countries want to pick the same kind of frog. How should they choose between the frog species? The map of the island is shown below. Solid lines represent the border between the countries, while dotted lines are the boundaries between the frog territories.

(a)  (b)  (c) 

Problem 80. There are five students at a party. We ask how many friends they have in the group. Here are the answers:

Alex: I have 4 friends.

Bob: I have fewer friends than Alex has.

Camille: I have as many friends as Doug.

Doug: Edit has one more friend than I have.

Edit: I have an odd number of friends.

Are Camille and Doug friends?

Problem 81. Rodd and Deven are pi-ous ninth century monks. It is the summer of 888 AD, and they have agreed they will share the job of writing the town records every day. Rodd does every day that contains an ODD digit in the date. Deven does all other days. They begin

- 20.08.888 Deven
- 21.08.888 Rodd
- 22.08.888 Deven
- 23.08.888 Rodd
- 24.08.888 Deven
- 25.08.888 Rodd
- 26.08.888 Deven
- 27.08.888 Rodd
- 28.08.888 Deven

When is the next day when Deven has to work?

Problem 82. Show that in any group of 6 students there are 3 students who know each other or 3 students who do not know each other.

Problem 83. Show that in any set of seven different positive integers there are three numbers such that the greatest common divisor of any two of them leaves the same remainder when divided by three.

Problem 84. Let P be a point inside the triangle ABC. Show that $PA + PB < CA + CB$.

Problem 85. In a PE class, everyone has 5 friends. Friendships are mutual. Two students in the class are appointed captains. The captains take turns selecting members for their teams, until everyone is selected. Prove that at the end of the selection process there are the same number of friendships within each team.

Problem 86. After Thor is captured by Loki, Loki sets Thor the following challenge in order to gain his freedom. Thor is presented three closed doors, numbered 1–3. Thor's hammer (which he is unable to summon due to a spell Loki cast on the hammer) is behind one of the doors and there are wolves behind the other two doors. If Thor can guess which door his hammer is behind, Loki will return the hammer and let Thor go. Otherwise, Loki will cast a spell that turns Thor into a goat. Thor picks door number 1. Because Loki is mischievous and knows what is behind each door, he decides to show Thor what is behind door number 3, which happens to be a wolf. Loki says, "Do you want to pick door number 2 or stick with your original choice of door 1?" Is it to Thor's advantage to switch his choice?

Problem 87. Consider the scenario of the previous problem, except now assume that there are $n \geq 4$ doors, behind one of which is Thor's hammer and there are wolves behind the remaining $n-1$ doors. Moreover, assume that Loki shows Thor $k \geq 2$ incorrect choices after Thor's initial guess. Loki says, "Do you want to pick a different door or stick with your original choice?" Should Thor modify his initial guess or not? In particular, explain what Thor should do in the extreme case when Loki opens $k = n-2$ incorrect doors.

Problem 88. In the senate of the Klingon home world no senator has more than three enemies. Show that the senate can be separated into two houses so that nobody has more than one enemy in the same house.

Problem 89 (One Overs)**.** Find positive odd integers $A < B < C$ such that

$$\frac{1}{3} = \frac{1}{A} + \frac{1}{B} + \frac{1}{C}.$$

Problem 90. There are $2n$ Federation ambassadors invited to a Ferengi banquet. Every ambassador has at most $n-1$ enemies. Show that the ambassadors can be seated around a round table avoiding enemies sitting next to each other.

Problem 91. 100 prisoners are isolated in individual jail cells with no way to communicate. They are currently serving life sentences. Due to an overcrowded prison, the jailer decides to offer the prisoners the following deal. There is a a room with nothing in it except a light switch (that starts in the off position). At random, the jailer will escort a single prisoner into the room with the light switch. After 5 seconds, the jailer will escort the prisoner back to his/her jail cell. The jailer will repeat this over and over again. He tells each of the prisoners that if one of the prisoners can indicate when every prisoner has been in the room with the light switch at least once, he will let all the prisoners go. However, if a prisoner erroneously states that each prisoner has been in the room with the light switch, then all the prisoners will be executed. Before beginning, the jailer gets all 100 prisoners together and gives them 5 minutes to come up with a plan. What should their plan be? It's important to note that the jailer is choosing prisoners at random to take in the room. That is, by chance, the same prisoner may be escorted to the room several times in a row. Also, your task is to devise a scheme for the prisoners to communicate with the light switch. You shouldn't bother searching for other ways for the prisoners to communicate.

Problem 92 (The Good Teacher)**.** You are teaching Calculus I, and you wish to give the students a cubic polynomial and have them find its three x-intercepts, its two critical points and its one inflection point. Because you are a kind person, you want these 6 points to all have integer coordinates and you want the cubic to have integer coefficients that are not too horribly large. Find one.

Problem 93. Suppose you have 12 coins, all identical in appearance and weight except for one that is either heavier or lighter than the other 11 coins. What is the minimum number of weighings one must do with a two-pan scale in order to identify the counterfeit?

Problem 94. Consider the situation in the previous problem, but suppose you have n coins. For which n is it possible to devise a procedure for identifying the counterfeit coin in only 3 weighings with a two-pan scale?

Problem 95. Let's revisit the counterfeit coin problem presented in Problems 93 and 94. In Problem 93, we discovered that we could detect the counterfeit coin in at most 3 weighings regardless of whether we knew in advance whether the counterfeit was heavier or lighter than the non-counterfeit coins. One feature of our algorithm was that after our 3 weighings, we could not only tell which coin was the counterfeit but also whether it was in fact heavier or lighter. It's certainly believable that 3 weighings is the best we can guarantee with 12 coins, but we did not prove this.

In Problem 94, we were asked to determine which number of coins we could start with and guarantee that we could identify which coin is counterfeit in at most 3 weighings. We know we can handle 12 coins. How about fewer? What if we have more than 12 coins? It certainly seems believable that if we could handle 12 coins in 3 weighings, we could handle less. But is this true? It's not obvious at all what happens with more than 12 coins.

Let's do some exploring. Let n be the number of coins. Assume that exactly one coin is counterfeit so that the remaining $n-1$ coins are not counterfeit. Further suppose that we do not know whether the counterfeit coin is heavier or lighter than the others but we do know that the counterfeit coin is one of these. Suppose our goal is to determine which coin is counterfeit and whether this coin is heavier or lighter than the remaining coins. Let k denote the number of weighing used to detect the counterfeit coin and its relative weight. We will attempt to find a relationship between n and k.

(a) Argue that $n \geq 3$.

(b) Suppose that on the first weighing, you take two piles of m coins where $2m < n$ and weigh them. There are two possibilities. Either the two sets of m coins balance on the scale or they don't. Let's first consider the case where the scales balance on the first weighing. In this case, the counterfeit must be one of the remaining $n-2m$ coins. We must be able to detect the counterfeit in the remaining $k-1$ weighings.

  (i) Argue that the number of possibilities for the counterfeit coin together with its relative weight is $2(n-2m)$.

  (ii) Argue that the number of possible sequences of outcomes for the remaining $k-1$ weighings is 3^{k-1}.

  (iii) Argue that $2(n-2m) \leq 3^{k-1}$ and then using the fact that 3^{k-1} is odd, conclude that

$$2(n-2m) \leq 3^{k-1} - 1. \tag{1}$$

(c) Now, let's assume that the scale was unbalanced on the first weighing when we weighed the two piles of m coins.

  (i) Argue that the number of possibilities for the counterfeit coin together with its relative weight is $4m$.

  (ii) Argue that the number of possible sequences of outcomes for the remaining $k-1$ weighings is $2 \cdot 3^{k-1}$.

  (iii) Argue that $2m \leq 3^{k-1}$ and then using the fact that 3^{k-1} is odd, conclude that $2m \leq 3^{k-1} - 1$.

  (iv) Finally, justify that

$$4m \leq 2 \cdot 3^{k-1} - 2. \tag{2}$$

(d) Prove that

$$n \leq \frac{3^k - 3}{2} \tag{3}$$

by adding inequalities (1) and (2) and simplifying.

(e) Use inequality (3) to show that the number of coins must be less than or equal to 12 if we are only allowed $k = 3$ weighings. Just because we found out that the number of coins must be less than or equal to 12 if we are only allowed 3 weighings does not guarantee that we can actually pull this off. However, we've already seen that we could handle $n = 12$ coins in $k = 3$ weighings. This shows that the bound given in inequality (3) is optimal when $k = 3$. In this case, we say that the bound is "sharp."

(f) Sort out which numbers of coins we can handle when $k = 2$. Verify that your answer is correct.

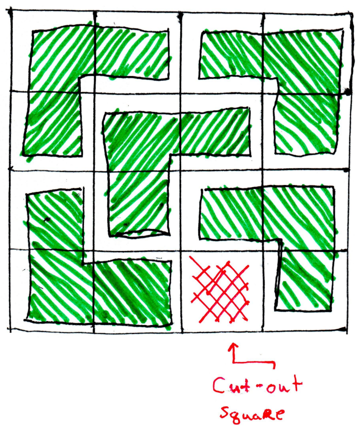
Problem 96 (The Martian Artifacts). Recent archaeological work on Mars discovered a site containing a pile of white spheres, each about the size of a tennis ball. A plaque near the mound states that each sphere contains a jewel that come in many different colors while strictly more than half of the spheres contain jewels of the same color. When two spheres are brought together, they both glow white if their internal jewels are the same color; otherwise, no glow. In how few tests can you find a sphere that you are certain holds a jewel of the majority color if the number of spheres in the pile is 2, 3, 4, 5, 6, 7, 8, or 9? You should provide an answer with justification for each of the different values.

Problem 97. The Infinite Hotel has rooms numbered $1,2,3,4,\ldots$. Every room in the Infinite Hotel is currently occupied. Is it possible to make room for one more guest (assuming they want a room all to themselves)? An infinite number of new guests, say $g_1, g_2, g_3, \ldots$, show up in the lobby and each demands a room. Is it possible to make room for all the new guests even in the hotel is already full?

Problem 98. Suppose we draw n lines in the plane that have the maximum number of unique intersections. This partitions the plane into disjoint regions (some of which are polygons with finite area and some are not). Suppose we color each of the regions so that no two adjacent regions (i.e., share a common edge) have the same color. What is the fewest colors we could use to accomplish this? Justify your answer.

Problem 99. Prove that every natural number can be written as the sum of distinct powers of two.

Problem 100. Consider a grid of squares that is 2^n squares wide by 2^n squares tall such that one of the squares has been cut out, but you don't know which one! You have a bunch of L-shaped trominoes made up of 3 squares. Prove that you can perfectly cover this grid with trominoes (with no overlap) for any $n \in \mathbb{N}$. The figure below depicts one possible covering for the case involving $n = 2$. *Hint:* Use induction.



Problem 101. In a certain kind of tournament, every player plays every other player exactly once and either wins or loses (there are no ties). Define a *top player* to be a player who, for every other player x, either beats x or beats a player y who beats x. (There may be more than one top player.) Prove that every n-player tournament has a top player. *Hint:* Use induction. For the inductive step, start with a tournament with $k+1$ players and remove a single player that has the lowest number of wins. There might be lots of players tied for lowest number of wins, in which case just pick one of them at random to remove.

Problem 102. Each integer on the number line is colored with exactly one of three possible colors—red, green, or blue—according to the following two rules:

- The negative of a red number must be colored blue;
- The sum of two blue numbers (not necessarily distinct) must be colored red.

Using this information, answer the following questions.

(a) Show that the negative of a blue number must be colored red.

(b) Show that the sum of two red numbers must be colored blue.

(c) Show that 0 must be colored green.

(d) Show that the sum of a red number and a blue number must be colored green.

(e) Determine all possible colorings of the integers that satisfy these rules.