Homework 2

Discrete Mathematics

Please review the *Rules of the Game* from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition.You are highly encouraged to seek out assistance by asking questions in our Q&A Discussion Board in Canvas. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to three late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers. In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. Holly has five different mathematics books, 3 different computer science books, and 2 different history books. She will arrangement on a shelf in an order that keeps the books in the same subject together. In how many ways can this be done?

2. For convenience, define $[n] := \{1,2,\ldots,n\}$, where $n \in \mathbb{N}$. For example, $[4] = \{1,2,3,4\}$. A **set partition** of $[n]$ is a collection of nonempty disjoint subsets of $[n]$ whose union is $[n]$. Each subset in the set partition is called a **block**. For example, there are 7 set partitions of [4] with 2 blocks, namely:

$$\begin{array}{ll} \{\{1,2,3\},\{4\}\} & \{\{1,2,4\},\{3\}\} \\ \{\{1,3,4\},\{2\}\} & \{\{2,3,4\},\{1\}\} \\ \{\{1,2\},\{3,4\}\} & \{\{1,3\},\{2,4\}\} \\ \{\{1,4\},\{2,3\}\} & \end{array}$$

   We define the **Stirling numbers** (of the second kind) via

$$\left\{ {n \atop k} \right\} := \text{number of set partitions of } [n] \text{ with } k \text{ blocks.}$$

   I usually pronounce this as "n Stirling k". Based on the information above, we know $\left\{ {4 \atop 2} \right\} = 7$.

(a) Compute $\left\{ 4 \atop 3 \right\}$ via brute-force.

(b) Explain why $\left\{ n \atop 1 \right\} = 1 = \left\{ n \atop n \right\}$.

(c) Explain why $\left\{ n \atop 2 \right\} = 2^{n-1} - 1$.

(d) Explain why the Stirling numbers satisfy the following for $1 \leq k \leq n$:

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\}.$$

Hint: Consider two disjoint sets, namely, set partitions of $[n]$ with k blocks where the element n is in a block by itself, and set partitions of $[n]$ with k blocks where the element n is not in a block by itself. To count one of the sets, you'll need to use the Product Principle.

3. A **composition** of n with k parts is an ordered list of k positive integers whose sum is n, denoted $\alpha = (\alpha_1, \ldots, \alpha_k)$. We say that α_i is the ith part. How many compositions of n are there? *Hint:* Start by collecting some data and then conjecture a formula. To prove that your proposed formula is correct, consider using a "sticks and stones" model, where the ith part consists of α_i many stones and each part is separated by a stick. For example, the composition $(1,3,2)$ on $n = 6$ corresponds to $\circ \mid \circ\circ\circ \mid \circ\circ$.

4. A k**-permutation** of a set A is an injective function $w : [k] \to A$. The set of all k-permutations of A is denoted by $S_{A,k}$. If A happens to be the set $[n]$, we use the notation $S_{n,k}$. And if $n = k$, we write $S_n := S_{n,n}$ and refer to each n-permutation in S_n as a **permutation**. Let $P(n,k) := |S_{n,k}|$. By convention, $P(n,0) = 1$.

   (a) For $1 \leq k \leq n$, explain why $P(n,k)$ is equal to the number of nonattacking rook arrangements on an $n \times k$ chess board. *Hint:* Establish a bijection between the collection of nonattacking rook arrangements on an $n \times k$ chess board and the collection of k-permutations.

   (b) Recall that for $n \in \mathbb{N}$, the **factorial** of n is defined $n! = n \cdot (n-1) \cdots 2 \cdot 1$, and we define $0! = 1$ for convenience. For $1 \leq k \leq n$, explain why

   $$P(n,k) = n \cdot (n-1) \cdots (n+1-k) = \frac{n!}{(n-k)!}.$$

   Note that as a special case of the formula above, we have $|S_n| = P(n,n) = n!$.