Chapter 2

Permutations and Combinations

A k**-permutation** of a set A is an injective function $w : [k] \to A$. The set of all k-permutations of A is denoted by $\boxed{S_{A,k}}$. If A happens to be the set $[n]$, we use the notation $\boxed{S_{n,k}}$. And if $n = k$, we write $\boxed{S_n := S_{n,n}}$ and refer to each n-permutation in S_n as a **permutation**. Let $\boxed{P(n,k) := |S_{n,k}|}$. By convention, $\boxed{P(n,0) = 1}$.

We can denote a k-permutation as string $w = w(1)w(2)\cdots w(k)$, where each entry $w(i)$ that appears in the string is unique (since w is an injection). In other words, we can think of a k-permutation as a linear ordered arrangement of k of n objects.

Problem 2.1. Complete the following.

(a) Write down all of the elements in S_3. What is $P(3,3)$?

(b) Write down all of the elements in $S_{4,3}$. What is $P(4,3)$?

Recall that for $n \in \mathbb{N}$, the **factorial** of n is defined $\boxed{n! := n \cdot (n-1) \cdots 2 \cdot 1}$, and we define $\boxed{0! := 1}$ for convenience.

Problem 2.2. Consider the collection of k-permutations in $S_{n,k}$ with $1 \leq k \leq n$. Explain why $P(n,k)$ is equal to the number of nonattacking rook arrangements on an $n \times k$ chess board. *Hint:* Establish a bijection between the collection of nonattacking rook arrangements on an $n \times k$ chess board and the collection of k-permutations.

Theorem 2.3. For $1 \leq k \leq n$, we have

$$\boxed{P(n,k) = n \cdot (n-1) \cdots (n+1-k) = \frac{n!}{(n-k)!}.}$$

Note that as a special case of the formula above, we have $|S_n| = P(n,n) = n!$. For convenience, we can extend the formula above to obtain

$$\boxed{P(0,0) = \frac{0!}{(0-0)!} = 1} \quad \text{and} \quad \boxed{P(n,0) = \frac{n!}{(n-0)!} = 1.}$$

Problem 2.4. How many strings of length three are there using letters from $\{a, b, c, d, e, f, g\}$ if the letters in the string are not repeated?

Problem 2.5. There are 8 finalists at the Olympic Games 100 meters sprint. Assume there are no ties.

(a) How many ways are there for the runners to finish?

(b) How many ways are there for the runners to get gold, silver, bronze?

(c) How many ways are there for the runners to get gold, silver, bronze given that Usain Bolt is sure to get the gold medal?

Problem 2.6. If $1 \le k \le n$, prove that $\boxed{P(n,n) = P(n,k)P(n-k,n-k)}$, both using the formula in Theorem 2.3, and separately using a combinatorial argument or by using the definition of k-permutations together with the bijection principle.

Problem 2.7. If $1 \le k \le n$, prove that $\boxed{P(n,k) = P(n-1,k) + kP(n-1,k-1)}$, both using the formula in Theorem 2.3, and separately using a combinatorial argument or by using the definition of k-permutations together with the bijection principle.

Problem 2.8. How many ways can the letters of the word PRESCOTT be arranged?

Problem 2.9. How many ways can the letters of the word POPPY be arranged? Try to solve this problem in two different ways.

Consider a set of n objects that are not necessarily distinct, with p different objects and n objects of type i (for $i = 1, 2, \ldots p$), so that $n = n_1 + \cdots + n_p$. An ordered arrangement of these n objects is called a **generalized permutation** and the number of such arrangements is denoted by $\boxed{P(n; n_1, \ldots, n_p)}$. For example, the number of words we can make out of the letters of POPPY is $P(5; 3, 1, 1)$.

Theorem 2.10. For $n, n_1, \ldots, n_p \in \mathbb{N}$ such that $n = n_1 + \cdots + n_p$, we have

$$\boxed{P(n; n_1, \ldots, n_p) = \frac{n!}{n_1! \cdots n_p!}.}$$

Problem 2.11. How many ways can the letters of the word MISSISSIPPI be arranged?

Problem 2.12. In Professor X's class of 9 graduate students she will give two A's, one B, and six C's. How many possible ways are there to do this?

Problem 2.13. Let's revisit Problem 1.14, which involved my walk to get coffee. When we attacked that problem, we did a lot of brute force. Do we now have an easier method?

Problem 2.14. Six friends sit around a circle to play a game. Rotations of the group do not constitute different seating orders.

(a) How many circular seating arrangements are there?

(b) How many circular seating arrangements are there if Sally and Maria always sit next to each other?

The above problem involves what are sometimes called **circular permutations**.

Problem 2.15. How many circular permutations are there involving n objects?

The notion of k-permutations captures arrangements of distinct objects where order matters. But what should we do if we want to capture a situation where the order of the objects does not matter? Since the order of the objects in a set does not matter, this is the model we should use.

If A is a set and $B \subseteq A$ with $|B| = k$, we refer to B as a k-**subset** of A. The collection of all k-subsets of A is defined via

$$\binom{A}{k} := \{B \subseteq A \mid |B| = k\}.$$

The **binomial coefficient** is defined via

$$\binom{n}{k} := \text{number of } k\text{-subsets of an } n\text{-element set.}$$

We read $\binom{n}{k}$ as "n choose k". In particular, if $|A| = n$, then $|\binom{A}{k}| = \binom{n}{k}$. Alternate notations for binomial coefficients include $C(n,k)$ and ${}_nC_k$. We will see later why $\binom{n}{k}$ is referred to as a binomial coefficient.

Example 2.16. If $A = \{a, b, c, d\}$, then

$$\binom{A}{2} = \{\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}\},$$

which implies that $\binom{4}{2} = 6$.

Problem 2.17. For any A, including the empty set, what is $\binom{A}{0}$? For $n \geq 0$, what is $\binom{n}{0}$ equal to?

Problem 2.18. For $n \geq 0$, what is $\binom{n}{n}$ equal to?

If we let n and k vary, we can organize the binomial coefficients in a triangular array, often referred to as **Pascal's Triangle**. See Table 2.1.

Problem 2.19. Suppose you have a pool of 6 applicants for a job opening. Let's assume you believe the values in Table 2.1.

(a) How many ways can you choose 3 of the 6 applicants to interview?

(b) How many ways can you hire 3 of the 6 applicants for 3 distinct jobs?

$n \setminus k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Table 2.1: Pascal's Triangle of binomial coefficients.

Problem 2.20. What are the row sums in Pascal's Triangle? That is, what is the following sum equal to for any $n \geq 0$?

$$\sum_{k=0}^{n} \binom{n}{k} := \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

Problem 2.21. Using the meaning of k-subset and k-permutation, explain why

$$P(n,k) = \binom{n}{k} \cdot k!.$$

Using the previous problem, we immediately get the following handy formula for computing binomial coefficients.

Theorem 2.22. For $0 \leq k \leq n$, we have

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}.}$$

Notice that the previous formula is equal to $\frac{P(n,k)}{k!}$. The numerator is counting how many distinct arrangements (order matters) there are of k objects taken from n objects and the denominator is essentially unordering arrangements that consist of the same objects.

Problem 2.23. A state senate consists of 19 Republicans and 14 Democrats. In how many ways can a committee be chosen if:

(a) The committee contains 6 senators without regard to party?

(b) The committee contains 3 Republicans and 3 Democrats?

Problem 2.24. How many bit strings of length 10 have exactly three 1's?

Problem 2.25. How many bit strings of length 6 have an odd number of 0's?

Problem 2.26. As we noted earlier, we did quite a bit of brute force to determine how many paths I could take to get coffee in Problem 1.14. Find a solution that utilizes binomial coefficients.

Problem 2.27. How many strings of 10 lower-case English letters have exactly two g's and exactly three v's?

Problem 2.28. Assume $1 \leq k \leq n$.

(a) Using the definition of $\binom{n}{k}$ in terms of k-subsets (as opposed to the formula in Theorem 2.22), explain why

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.}$$

This identity is often called **Pascal's Identity** (or **Pascal's Recurrence**).

(b) Connect the formula above with Problem 1.14 involving my walk to get coffee.

Problem 2.29. Assume $1 \leq k \leq n$. It turns out that

$$\boxed{\binom{n}{k} = \binom{n}{n-k}.}$$

(a) Prove the identity above using the formula for $\binom{n}{k}$ given in Theorem 2.22.

(b) Explain why the identity is true by using the definition of $\binom{n}{k}$ in terms of k-subsets.

The upshot is that each row of Pascal's Triangle is a palindrome.

Problem 2.30. Explain why $1 + 2 + \cdots + n = \binom{n+1}{2}$.

By the way, the number defined by $\boxed{T_n := 1 + 2 + \cdots + n}$ is called the nth **Triangular number** (due to the shape we get by representing each number in the sum by a stack of balls).

Problem 2.31. Consider the linear equation $x_1 + x_2 + x_3 = 11$. How many *integer* solutions are there if:

(a) $x_1, x_2, x_3 \geq 0$?

(b) $x_1, x_2, x_3 > 0$? *Note:* ChatGPT did not get correct answer for me!

(c) $x_1 \geq 1, x_2 \geq 0, x_3 \geq 2$?

Problem 2.32. How many ways can you distribute 5 identical lollipops to 6 kids?

These last two problems illustrate a technique known as **stars and bars**. In general, n stars tally the number of objects and $r-1$ bars separate them into r distinct categories.

Theorem 2.33. The number of possible collections of n objects of r different types is

$$\boxed{\binom{n+r-1}{r-1} = \binom{n+r-1}{n}.}$$

Problem 2.34. Zittles come in 5 colors: green, yellow, red, orange, and purple. How many different collections of 32 Zittles are possible?