Homework 10

Discrete Mathematics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition.You are highly encouraged to seek out assistance by asking questions on our Discord server. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers. In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. Whoziwhatzits come in boxes of 6, 9, and 20. **Prove** that for any natural number $n \geq 44$, it is possible to buy exactly n Whoziwhatzits with a combination of these boxes.

2. Suppose n lines are drawn in the plane so that no two lines are parallel and no three lines intersect at any one point. Such a collection of lines is said to be in **general position**. Every collection of lines in general position divides the plane into disjoint regions, some of which are polygons with finite area (bounded regions) and some of which are not (unbounded regions).

   (a) Let $R(n)$ be the number of regions the plane is divided into by n lines in general position. Conjecture a formula for $R(n)$ and **prove** that your conjecture is correct using induction.

   (b) Suppose we color each of the regions (bounded and unbounded) so that no two adjacent regions (i.e., share a common edge) have the same color. What is the fewest colors we could use to accomplish this? **Prove** your assertion using induction.

3. Use induction to **prove** that the number of binary strings of length n that never have two consecutive 1's is the Fibonacci number f_{n+2}. See our textbook for the definition of the Fibonacci numbers.

4. How many compositions of n have all their parts greater than 1, except possibly the last entry. For example, for $n = 9$, (3,4,2) and (3,3,2,1) are acceptable, but not (3,3,1,2).

5. Recall the definition of Dyck path on a previous homework and let d_n be the number of Dyck paths from $(0,0)$ to (n,n) (not to be confused with the number of derangements). **Explain why** d_n satisfies the following recurrence for $n \geq 1$:

$$d_n = \sum_{i=0}^{n-1} d_i d_{n-1-i}.$$

Hint: Consider the collection of Dyck paths that hit the line $y = x$ at $(i+1, i+1)$ for the first time after leaving $(0,0)$. Think about how many ways you can draw the Dyck path to get to $(i+1, i+1)$ versus how many ways you can draw the Dyck path from $(i+1, i+1)$ to (n,n). The first case is trickier to think about. Notice that the portion of the Dyck path from $(0,0)$ to $(i+1, i+1)$ never hits the line $y = x$ along the way. Moreover, this portion necessarily starts with a North step and ends with an East step. What are the possible values for i? Add up all the cases.