Homework 7
Discrete Mathematics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition.You are highly encouraged to seek out assistance by asking questions on our Discord server. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers. In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. On the last question on the previous exam, there were 10 items in the left column and 10 items in the right column. How many arrangements of answers have exactly half the answers correct? Assume each item in the second is used exactly once. **No detailed justification required**.

2. How many strings using the digits $0, 1, \ldots, 9$ of length n are there such that the digits arranged in weakly increasing (nondecreasing) order (i.e., as we read the string from left to right, each digit we encounter is greater than or equal to the digit preceding it)? **Explain** your answer.

3. Define a **multiset** M to be an unordered collection of elements that may be repeated. To distinguish between sets and multisets, we will use the notation $\{\}$ versus $\{\!\{\ \}\!\}$. For example, $M = \{\!\{a,a,a,b,c,c\}\!\}$ is a multiset (not to be confused with the set containing the set $\{a,b,c\}$; mathematical notation is both awesome and frustrating at times). The primes occurring in the prime factorization of a natural number greater than 1 is an example of multiset. The cardinality of a multiset is its number of elements counted with multiplicity. So, in our example, $|M| = 3 + 1 + 2 = 6$. If S is a set, then M is a **multiset on** S if every element of M is an element of S. For $n \geq 1$ and $k \geq 0$, define

$$\left(\!\!\binom{n}{k}\!\!\right) := \text{number of multisets on } [n] \text{ of size } k.$$

   Explain why

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}.$$

Hint: Here is one possible approach. Assume the elements of each multiset is written in weakly increasing order. Use sticks and stones. You'll need to be careful with what n and k represent.

4. Use the Binomial Theorem to expand $(x+1)^{10}$. **No detailed justification required**.

5. What is the coefficient of $x^3 y^5$ in the expansion of $(2x - 3y)^8$? No justification required. **No detailed justification required**.

6. Use the Binomial Theorem to determine what the sum $\sum_{k=0}^{n} (-2)^k \binom{n}{k}$ is equal to. **No detailed justification required**.

7. At the end of the semester a professor must award all students a grade of A, B, C, D, or F. How big must the class be to guarantee that at least 5 students will earn the same grade? **Briefly explain** your answer.