Homework 8

Discrete Mathematics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition.You are highly encouraged to seek out assistance by asking questions on our Discord server. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers. In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. Find the alternating row sums in Pascal's Triangle. That is, for $n \geq 0$, find a formula for

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} := \binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}.$$

   Briefly justify your answer. *Hint:* The answer for $n = 0$ is different than for $n > 0$. For $n > 0$, you can either use the Binomial Theorem or make use of Pascal's Identity together with careful algebraic bookkeeping.

2. A cash box contains eight \$1 bills, six \$5 bills, five \$10 bills, and three \$20 bills. **No detailed justification required.**

   (a) If I pull bills out at random, how many bills must I pull to be guaranteed at least 4 of one denomination?

   (b) If I pull bills out at random, how many bills must I pull to be guaranteed at least 5 of one denomination?

3. Use the Pigeonhole Principle to **explain why** if five points are placed inside an equilateral triangle of side length 1, then at least two of them are within 1/2 unit of each other.

4. Suppose a soccer team scores at least one goal in 20 consecutive games. If it scores a total of 30 goals in those 20 games, **prove** that in some sequence of consecutive games it scores exactly 9 goals.

Hint: This one is tricky! Let g_i be the total number of goals scored in games 1 through i. Consider the list of 40 numbers $g_1, \ldots, g_{20}, g_1 + 9, \ldots, g_{20} + 9$. Notice that since that at least one goal is scored in each game, the values $g_1, \ldots, g_{20}$ are distinct. This implies that the values $g_1 + 9, \ldots, g_{20} + 9$ are also distinct. What's the smallest possible value that could appear in the list of all 40 values? What's the largest value? Apply the Pigeonhole Principle to conclude that there must be a repeated value amongst the 40 values. Then use this information to construct a consecutive sequence of games where exactly 9 goals are scored.

5. Of 200 aliens at an interplanetary convention, 83 had pointed ears, 101 had bony brows and 73 ridged noses. If 32 had pointed ears and ridged noses, 52 had pointed ears and bony brows, 34 had bony brows and ridged noses and 22 had all three features, how many did not have any of the three features? **No detailed justification required.**

6. Of 47 students in a dorm, 34 like hiking, 23 like mountain biking and 15 like rock climbing. If 12 like to hike and bike, 13 like to hike and rock climb, 7 enjoy all of these things and everyone likes at least one thing. **No detailed justification required.**

   (a) How many like to mountain bike and rock climb?

   (b) How many like to mountain bike but not rock climb?

7. At a restaurant, 6 people place different orders for dinner. The server brings their meals to the table. How many ways could the server distribute the meals so that no one receives the meal they ordered? **For this problem, find the exact value of the answer and provide some justification.**