Chapter 4

Combinations

The notion of k-permutations captures arrangements of distinct objects where order matters. But what should we do if we want to capture a situation where the order of the objects does not matter? Since the order of the objects in a set does not matter, this is the model we should use.

If A is a set and $B \subseteq A$ with $|B| = k$, we refer to B as a **k-subset** of A. The collection of all k-subsets of A is defined via

$$\binom{A}{k} := \{B \subseteq A \mid |B| = k\}.$$

The **binomial coefficient** is defined via

$$\binom{n}{k} := \text{number of } k\text{-subsets of an } n\text{-element set.}$$

In particular, if $|A| = n$, then $|\binom{A}{k}| = \binom{n}{k}$. We read "$\binom{n}{k}$" as "$n$ choose k". Alternate notations for binomial coefficients include $C(n, k)$ and ${}_nC_k$. We will see later why $\binom{n}{k}$ is referred to as a binomial coefficient.

Example 4.1. If $A = \{a, b, c, d\}$, then

$$\binom{A}{2} = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\},$$

which implies that $\binom{4}{2} = 6$.

Problem 4.2. For any A, including the empty set, what is $\binom{A}{0}$? For $n \geq 0$, what is $\binom{n}{0}$ equal to?

Problem 4.3. For $n \geq 0$, what is $\binom{n}{n}$ equal to?

If we let n and k vary, we can organize the binomial coefficients in a triangular array, often referred to as **Pascal's Triangle**. See Table 4.1.

$n \setminus k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Table 4.1: Pascal's Triangle of binomial coefficients.

Problem 4.4. Suppose you have a pool of 6 applicants for a job opening. Let's assume you believe the values in Table 4.1.

(a) How many ways can you choose 3 of the 6 applicants to interview?

(b) How many ways can you hire 3 of the 6 applicants for 3 distinct jobs?

Problem 4.5. What are the row sums in Pascal's Triangle? That is, what is the following sum equal to for any $n \geq 0$?

$$\sum_{k=0}^{n} \binom{n}{k} := \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

Problem 4.6. Using the meanings of k-subset and k-permutation, explain why

$$P(n,k) = \binom{n}{k} \cdot k!.$$

Using the previous problem, we immediately get the following handy formula for computing binomial coefficients.

Theorem 4.7. For $0 \leq k \leq n$, we have

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{P(n,k)}{k!}.}$$

In the last expression above, the numerator of $\frac{P(n,k)}{k!}$ is counting how many distinct arrangements (order matters) there are of k objects taken from n objects and the denominator is essentially unordering arrangements (by Division Principle) that consist of the same objects.

Problem 4.8. A state senate consists of 19 Republicans and 14 Democrats. In how many ways can a committee be chosen if:

(a) The committee contains 6 senators without regard to party?

(b) The committee contains 3 Republicans and 3 Democrats?

Problem 4.9. How many bit strings of length 10 have exactly three 1's?

Problem 4.10. How many bit strings of length 6 have an odd number of 0's?

Problem 4.11. As we noted earlier, we did quite a bit of brute force to determine how many paths I could take to get coffee in Problem 1.15. Find a solution that utilizes binomial coefficients.

Problem 4.12. How many strings of 10 lower-case English letters have exactly two g's and exactly three v's?

Problem 4.13. Assume $1 \leq k \leq n$.

(a) Using the definition of $\binom{n}{k}$ in terms of k-subsets (as opposed to the formula in Theorem 4.7), explain why

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}.$$

This identity is often called **Pascal's Identity** (or **Pascal's Recurrence**).

(b) Connect the formula above with Problem 1.15 involving my walk to get coffee.

Problem 4.14. Assume $1 \leq k \leq n$. It turns out that

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}.$$

(a) Prove the identity above using the formula for $\binom{n}{k}$ given in Theorem 4.7.

(b) Explain why the identity is true by using the definition of $\binom{n}{k}$ in terms of k-subsets.

The upshot is that each row of Pascal's Triangle is a palindrome.

Problem 4.15. Explain why

$$\boxed{1 + 2 + \cdots + n = \binom{n+1}{2}}$$

by counting the number of handshakes that could occur among a group of $n+1$ people in two different ways.

By the way, the number defined by $\boxed{t_n := 1 + 2 + \cdots + n}$ is called the nth **Triangular number** (due to the shape we get by representing each number in the sum by a stack of balls).

Problem 4.16. Without appealing to the previous problem, find a visual proof of the following:

$$\boxed{1 + 2 + \cdots + n = \frac{n(n+1)}{2}}.$$

Hint: Consider a rectangular array of balls that has n rows and $n+1$ columns.

Problem 4.17. Consider the linear equation $x_1 + x_2 + x_3 = 11$. How many *integer* solutions are there if:

(a) $x_1, x_2, x_3 \geq 0$?

(b) $x_1, x_2, x_3 > 0$?

(c) $x_1 \geq 1, x_2 \geq 0, x_3 \geq 2$?

Problem 4.18. How many ways can you distribute 5 identical lollipops to 6 kids?

These last two problems illustrate a technique known as **stars and bars**. In general, n stars tally the number of objects and $k-1$ bars separate them into k distinct categories.

Theorem 4.19. The number of possible collections of n objects of k different types is

$$\boxed{\binom{n+k-1}{k-1} = \binom{n+k-1}{n}.}$$

Problem 4.20. Zittles come in 5 colors: green, yellow, red, orange, and purple. How many different collections of 32 Zittles are possible?