Chapter 6

Pigeonhole Principle

The **Pigeonhole Principle** is a very natural idea. It says: If a collection of at least $n + 1$ objects is put into n boxes, then there is a box with at least two things in it. The Pigeonhole Principle has surprisingly deep applications. We will start with a few examples.

Example 6.1. Back in Problem 2.11, we implicitly used the Pigeonhole Principle when we argued that if $f : A \to B$ is a function for finite sets A and B, then

(a) If f is an injection, then $|A| \leq |B|$.

(b) If f is a surjection, then $|A| \geq |B|$.

Problem 6.2. A box has blue, green, yellow, red, orange, and white balls. How many must be drawn without looking to be sure of getting at least two of the same color?

Problem 6.3. Prove that if seven distinct numbers are selected from $\{1, 2, \ldots, 11\}$, then some two of these numbers sum to 12.

We would like to generalize the Pigeonhole Principle, but first we need a useful function. The **ceiling function** of a real number x, written $\boxed{\lceil x \rceil}$, is the smallest integer greater than or equal to x. That is, $\lceil x \rceil$ is an integer, $\lceil x \rceil \geq x$, and there is no other integer between $\lceil x \rceil$ and x. You can think of it as the "round-up to an integer" function.

Example 6.4. For example, $\lceil \pi \rceil = 4$, $\lceil -\pi \rceil = -3$, and $\lceil 7 \rceil = 7$.

We can now generalize the Pigeonhole Principle as follows.

Theorem 6.5 (Generalized Pigeonhole Principle)**.** If n objects are placed in m boxes, then there is a box with at least $\lceil \frac{n}{m} \rceil$ objects.

Problem 6.6. If 20 buses seating at most 50 carry 621 passengers to a ball game, then some bus must have at least __________ passengers.

Problem 6.7. How many balls must be drawn from the box in Problem 6.2 in order to be sure of getting at least 4 of the same color?

Problem 6.8. Suppose $n \in \mathbb{N}$ and let $S = \{1, 2, \ldots, 2n\}$. Argue that if we choose $n + 1$ distinct elements from the set S, then among those elements there must exist a pair of numbers such that one is a multiple of the other. *Hint:* Every number in S is of the form $2^k m$, where k is a nonnegative integer and m is odd.

Problem 6.9. Explain why every collection of ten distinct integers $x_1, x_2, \ldots, x_{10}$ must have at least one subset whose sum of the elements in the subset is divisible by 10.