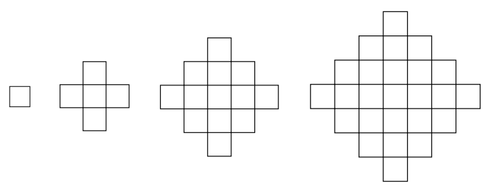
Homework 11

Discrete Mathematics

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers. In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. Show that 4^n is a solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$.
2. Solve the recurrence $a_n = 7a_{n-1} - 10a_{n-2}$ with $a_0 = 2$ and $a_1 = 3$.
3. Solve the recurrence relation $a_n = a_{n-1} + (2n - 1)$ with $a_1 = 1$.
4. Solve the recurrence relation $a_n = a_{n-1} + 2^n$ with $a_0 = 5$.
5. Consider the sequence of figures whose first few terms are as follows.



Let a_n be the area of the nth figure. Assume we start at $n = 1$, so that $a_1 = 1$, $a_2 = 5$, as so on.

(a) Find a recursive definition for the sequence. Don't forget any necessary initial conditions.

(b) Is the sequence arithmetic or geometric? If not, is it the sequence of partial sums of an arithmetic or geometric sequence?

(c) Find a closed formula for the nth term of the sequence.

6. Develop a recurrence relation with initial conditions for the number of ways to climb a staircase of n stairs by taking steps of one, two, or three stairs at a time. How many ways can a person climb a flight of eight stairs under these constraints?
7. Recall the definition of Dyck path on the previous homework and let d_n be the number of Dyck paths from $(0,0)$ to (n,n). Show that d_n satisfies the following recurrence for $n \geq 1$:

$$d_n = \sum_{i=0}^{n-1} d_i d_{n-1-i}.$$

Hint: Consider the collection of Dyck paths that hit the line $y = x$ at $(i+1, i+1)$ for the first time after leaving $(0,0)$. Think about how many ways you can draw the Dyck path to get to $(i+1, i+1)$ versus how many ways you can draw the Dyck path from $(i+1, i+1)$ to (n,n).

The first case is trickier to think about. Notice that the portion of the Dyck path from $(0,0)$ to $(i+1,i+1)$ never hits the line $y=x$ along the way. Moreover, this portion necessarily starts with a North step and ends with an East step. What are the possible values for i? Add up all the cases.