# Homework 4

Discrete Mathematics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition.You are highly encouraged to seek out assistance by asking questions in our Q&A Discussion Board in Canvas. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers. In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. Recall that $[n] := \{1,2,\ldots,n\}$, where $n \in \mathbb{N}$. A **set partition** of $[n]$ is a collection of nonempty disjoint subsets of $[n]$ whose union is $[n]$. Each subset in the set partition is called a **block**. For example, there are 7 set partitions of [4] with 2 blocks, namely:

$$\begin{array}{ll} \{\{1,2,3\},\{4\}\} & \{\{1,2,4\},\{3\}\} \\ \{\{1,3,4\},\{2\}\} & \{\{2,3,4\},\{1\}\} \\ \{\{1,2\},\{3,4\}\} & \{\{1,3\},\{2,4\}\} \\ \{\{1,4\},\{2,3\}\} & \end{array}$$

We define the **Stirling numbers** (of the second kind) via

$$\left\{ {n \atop k} \right\} := \text{number of set partitions of } [n] \text{ with } k \text{ blocks.}$$

I usually pronounce this as "$n$ Stirling $k$". Based on the information above, we know $\left\{ {4 \atop 2} \right\} = 7$.

(a) Compute $\left\{ {4 \atop 3} \right\}$ via brute-force. That is, write down all of the set partitions of [4] with 3 blocks and then count how many you have.

(b) Compute $\left\{ 4 \atop 1 \right\}$ via brute-force.

(c) Compute $\left\{ 4 \atop 4 \right\}$ via brute-force.

(d) Explain why $\left\{ n \atop 1 \right\} = 1 = \left\{ n \atop n \right\}$ for every $n \in \mathbb{N}$.

(e) Explain why $\left\{ n \atop 2 \right\} = 2^{n-1} - 1$ for every $n \in \mathbb{N}$. *Hint:* Every partition of $[n]$ into two sets is determined by choosing a subset for one block since the the second block is necessarily the complement. How many subsets of $[n]$ are there? Are any of these bad choices for one of the blocks? Temporarily pretend the order of the two blocks matters and then use the Division Principle to unorder the bocks.

(f) We define the $n$th **Bell number** $B_n$ to be the total number of set partitions of $[n]$ into any number of blocks. Explain why

$$B_n = \left\{ n \atop 1 \right\} + \left\{ n \atop 2 \right\} + \cdots + \left\{ n \atop n \right\} = \sum_{k=1}^{n} \left\{ n \atop k \right\}.$$

*Note:* You don't need to worry about justifying the expression on the righthand side. That is simply meant to be a reminder about how "Sigma notation" works.

(g) Find $B_4$ using some of your previous answers.

2. How many surjective (i.e., onto) functions are there from [4] to [2]? *Hint:* Utilize information from Problem 1.

3. Imagine we have $n \geq 1$ people competing in a contest where ties are allowed.

(a) Explain why the number of final rankings is given by

$$1!\left\{ n \atop 1 \right\} + 2!\left\{ n \atop 2 \right\} + \cdots + n!\left\{ n \atop n \right\} = \sum_{k=1}^{n} k!\left\{ n \atop k \right\}.$$

*Note:* You don't need to worry about justifying the expression on the righthand side. That is simply meant to be a reminder about how "Sigma notation" works.

(b) Now, imagine that the people are numbered 1 through $n$. How many final rankings are possible under the proviso that person $i$'s ranking is less than or equal to person $(i+1)$'s ranking? Such rankings are called *weakly increasing*.