Pass on what you have learned. Strength, mastery. But weakness, folly, failure also. Yes, failure most of all. The greatest teacher, failure is.

Yoda, Jedi master

Chapter 3

Set Theory

At its essence, all of mathematics is built on set theory. In this chapter, we will introduce some of the basics of sets and their properties.

3.1 Sets

Definition 3.1. A **set** is a collection of objects called **elements**. If A is a set and x is an element of A, we write $x \in A$. Otherwise, we write $x \notin A$. The set containing no elements is called the **empty set**, and is denoted by the symbol $\emptyset$. Any set that contains at least one element is referred to as a **nonempty set**.

If we think of a set as a box potentially containing some stuff, then the empty set is a box with nothing in it. One assumption we will make is that for any set A, $A \notin A$. The language associated to sets is specific. We will often define sets using the following notation, called **set-builder notation**:

$$S = \{x \in A \mid P(x)\},$$

where $P(x)$ is some predicate statement involving x. The first part "$x \in A$" denotes what type of x is being considered. The predicate to the right of the vertical bar (not to be confused with "divides") determines the condition(s) that each x must satisfy in order to be a member of the set. This notation is read as "The set of all x in A such that $P(x)$." As an example, the set $\{x \in \mathbb{N} \mid x \text{ is even and } x \geq 8\}$ describes the collection of even natural numbers that are greater than or equal to 8.

There are a few sets that are commonly discussed in mathematics and have predefined symbols to denote them. We have already encountered the integers, natural numbers, and real numbers. Notice that our definition of the rational numbers uses set-builder notation.

- **Natural Numbers:** $\mathbb{N} := \{1, 2, 3, \ldots\}$. Some books will include zero in the set of natural numbers, but we do not.

- **Integers:** $\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.

- **Rational Numbers:** $\mathbb{Q} := \{a/b \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$.
- **Real Numbers:** $\mathbb{R}$ denotes the set of real numbers. We are taking for granted that you have some familiarity with this set.

Since the set of natural numbers consists of the positive integers, the natural numbers are sometimes denoted by $\mathbb{Z}^+$.

Problem 3.2. Unpack the meaning of each of the following sets and provide a description of the elements that each set contains.

(a) $A = \{x \in \mathbb{N} \mid x = 3k \text{ for some } k \in \mathbb{N}\}$

(b) $B = \{t \in \mathbb{R} \mid t \leq 2 \text{ or } t \geq 7\}$

(c) $C = \{t \in \mathbb{Z} \mid t^2 \leq 2\}$

(d) $D = \{s \in \mathbb{Z} \mid -3 < s \leq 5\}$

(e) $E = \{m \in \mathbb{R} \mid m = 1 - \frac{1}{n}, \text{ where } n \in \mathbb{N}\}$

Problem 3.3. Write each of the following sentences using set-builder notation.

(a) The set of all real numbers less than $-\sqrt{2}$.

(b) The set of all real numbers greater than -12 and less than or equal to 42.

(c) The set of all even integers.

Parts (a) and (b) of Problem 3.3 are examples of intervals.

Definition 3.4. For $a, b \in \mathbb{R}$ with $a < b$, we define the following sets, referred to as **intervals**.

(a) $(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$

(b) $[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$

(c) $[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}$

(d) $(a, \infty) := \{x \in \mathbb{R} \mid a < x\}$

(e) $(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$

(f) $(-\infty, \infty) := \mathbb{R}$

We analogously define $(a, b]$, $[a, \infty)$, and $(-\infty, b]$. Intervals of the form (a, b), $(-\infty, b)$, (a, ∞), and $(-\infty, \infty)$ are called **open intervals** while $[a, b]$ is referred to as a **closed interval**. A **bounded interval** is any interval of the form (a, b), $[a, b)$, $(a, b]$, and $[a, b]$. For bounded intervals, a and b are called the **endpoints** of the interval.

We will always assume that any time we write $(a,b), [a,b], (a,b], \text{ or } [a,b)$ that $a < b$. We will see where the terminology of "open" and "closed" comes from in Section 5.2.

Problem 3.5. Give an example of each of the following.

(a) An interval that is neither an open nor closed interval.

(b) An infinite set that is not an interval.

Definition 3.6. If A and B are sets, then we say that A is a **subset** of B, written $\boxed{A \subseteq B}$, provided that every element of A is an element of B.

Observe that $A \subseteq B$ is equivalent to "For all x (in the universe of discourse), if $x \in A$, then $x \in B$." Since we know how to deal with "for all" statements and conditional propositions, we know how to go about proving $A \subseteq B$.

Problem 3.7. Suppose A and B are sets. Describe a skeleton proof for proving that $A \subseteq B$.

Every nonempty set always has two subsets.

Theorem 3.8. Let A be a set. Then

(a) $A \subseteq A$, and

(b) $\emptyset \subseteq A$.

Notice that if $A = \emptyset$, then Parts (a) and (b) of the previous theorem say the same thing.

Problem 3.9. List all of the subsets of $A = \{1,2,3\}$.

Theorem 3.10 (Transitivity of Subsets)**.** Suppose that A, B, and C are sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Definition 3.11. Two sets A and B are **equal**, denoted $\boxed{A = B}$, if the sets contain the same elements.

Since the next theorem is a biconditional proposition, you need to write two distinct subproofs, one for "$A = B$ implies $A \subseteq B$ and $B \subseteq A$", and another for "$A \subseteq B$ and $B \subseteq A$ implies $A = B$". Be sure to make it clear to the reader when you are proving each implication.

Theorem 3.12. Suppose that A and B are sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Note that if we want to prove $A = B$, then we have to do two separate subproofs: one for $A \subseteq B$ and one for $B \subseteq A$. Be sure to make it clear to the reader where these subproofs begin and end. One approach is to label each subproof with "($\subseteq$)" and "($\supseteq$)" (including the parentheses), respectively.

Definition 3.13. If $A \subseteq B$, then A is called a **proper subset** provided that $A \neq B$. In this case, we may write $\boxed{A \subset B}$ or $\boxed{A \subsetneq B}$.

Note that some authors use $\subset$ to mean $\subseteq$, so some confusion could arise if you are not reading carefully.

Definition 3.14. Let A and B be sets in some universe of discourse U.

(a) The **union** of the sets A and B is $\boxed{A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\}}$.

(b) The **intersection** of the sets A and B is $\boxed{A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\}}$.

(c) The **set difference** of the sets A and B is $\boxed{A \setminus B := \{x \in U \mid x \in A \text{ and } x \notin B\}}$.

(d) The **complement of** A (relative to U) is the set $\boxed{A^c := U \setminus A = \{x \in U \mid x \notin A\}}$.

Definition 3.15. If two sets A and B have the property that $A \cap B = \emptyset$, then we say that A and B are **disjoint** sets.

Problem 3.16. Suppose that the universe of discourse is $U = \{1,2,3,4,5,6,7,8,9,10\}$. Let $A = \{1,2,3,4,5\}$, $B = \{1,3,5\}$, and $C = \{2,4,6,8\}$. Find each of the following.

(a) $A \cap C$

(b) $B \cap C$

(c) $A \cup B$

(d) $A \setminus B$

(e) $B \setminus A$

(f) $C \setminus B$

(g) B^c

(h) A^c

(i) $(A \cup B)^c$

(j) $A^c \cap B^c$

Problem 3.17. Suppose that the universe of discourse is $U = \mathbb{R}$. Let $A = [-3,-1)$, $B = (-2.5, 2)$, and $C = (-2, 0]$. Find each of the following.

(a) A^c

(b) $A \cap C$

(c) $A \cap B$

(d) $A \cup B$

(e) $(A \cap B)^c$

(f) $(A \cup B)^c$

(g) $A \setminus B$

(h) $A \setminus (B \cup C)$

(i) $B \setminus A$

Problem 3.18. Suppose that the universe of discourse is $U = \{x, y, z, \{y\}, \{x, z\}\}$. Let $S = \{x, y, z\}$ and $T = \{x, \{y\}\}$. Find each of the following.

(a) $S \cap T$

(b) $(S \cup T)^c$

(c) $T \setminus S$

Theorem 3.19. If A and B are sets such that $A \subseteq B$, then $B^c \subseteq A^c$.

Theorem 3.20. If A and B are sets, then $A \setminus B = A \cap B^c$.

In Chapter 2, we encountered De Morgan's Law (see Theorem 2.26 and Problem 2.27), which provided a method for negating compound propositions involving conjunctions and disjunctions. The next theorem provides a method for taking the complement of unions and intersections of sets. This result is also known as De Morgan's Law. Do you see why?

Theorem 3.21 (De Morgan's Law)**.** If A and B are sets, then

(a) $(A \cup B)^c = A^c \cap B^c$, and

(b) $(A \cap B)^c = A^c \cup B^c$.

The next theorem indicates how intersections and unions interact with each other.

Theorem 3.22 (Distribution of Union and Intersection)**.** If A, B, and C are sets, then

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, and

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Problem 3.23. For each of the statements (a)–(d) on the left, find an equivalent symbolic proposition chosen from the list (i)–(v) on the right. Note that not every statement on the right will get used.

(a) $A \not\subseteq B$.

(b) $A \cap B = \emptyset$.

(c) $(A \cup B)^c \neq \emptyset$.

(d) $(A \cap B)^c = \emptyset$.

(i) $(\forall x)(x \in A \wedge x \in B)$

(ii) $(\forall x)(x \in A \implies x \notin B)$

(iii) $(\exists x)(x \notin A \wedge x \notin B)$

(iv) $(\exists x)(x \in A \vee x \in B)$

(v) $(\exists x)(x \in A \wedge x \notin B)$

In mathematics the art of proposing a question must be held of higher value than solving it.

Georg Cantor, mathematician

3.2 Russell's Paradox

We now turn our attention to the issue of whether there is one mother of all universal sets. Before reading any further, consider this for a moment. That is, is there one largest set that all other sets are a subset of? Or, in other words, is there a set of all sets? To help wrap our heads around this issue, consider the following riddle, known as the **Barber of Seville Paradox**.

> In Seville, there is a barber who shaves all those men, and only those men, who do not shave themselves. Who shaves the barber?

Problem 3.24. In the Barber of Seville Paradox, does the barber shave himself or not?

Problem 3.24 is an example of a **paradox**. A paradox is a statement that can be shown, using a given set of axioms and definitions, to be both true and false. Recall that an axiom is a statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proved. Paradoxes are often used to show the inconsistencies in a flawed axiomatic theory. The term paradox is also used informally to describe a surprising or counterintuitive result that follows from a given set of rules. Now, suppose that there is a set of all sets and call it $\mathcal{U}$. That is, $\mathcal{U} := \{A \mid A \text{ is a set}\}$.

Problem 3.25. Given our definition of $\mathcal{U}$, explain why $\mathcal{U}$ is an element of itself.

If we continue with this line of reasoning, it must be the case that some sets are elements of themselves and some are not. Let X be the set of all sets that are elements of themselves and let Y be the set of all sets that are not elements of themselves.

Problem 3.26. Does Y belong to X or Y? Explain why this is a paradox.

The above paradox is one way of phrasing a paradox referred to as **Russell's Paradox**, named after British mathematician and philosopher Bertrand Russell (1872–1970). How did we get into this mess in the first place?! By assuming the existence of a set of all sets, we can produce all sorts of paradoxes. The only way to avoid these types of paradoxes is to conclude that there is no set of all sets. That is, the collection of all sets cannot be a set itself.

According to naive set theory (i.e., approaching set theory using natural language as opposed to formal logic), any definable collection is a set. As Russell's Paradox illustrates, this leads to problems. It turns out that any proposition can be proved from a contradiction, and hence the presence of contradictions like Russell's Paradox would appear to be catastrophic for mathematics. Since set theory is often viewed as the basis for axiomatic development in mathematics, Russell's Paradox calls the foundations of mathematics into question. In response to this threat, a great deal of research went into developing consistent axioms (i.e., free of contradictions) for set theory in the early 20th century. In 1908, Ernst Zermelo (1871–1953) proposed a collection of axioms for set theory that avoided the inconsistencies of naive set theory. In the 1920s, adjustments to Zermelo's axioms were made by Abraham Fraenkel (1891–1965), Thoralf Skolem (1887–1963), and Zermelo that resulted in a collection of nine axioms, called ZFC, where ZF stands for Zermelo and Fraenkel and C stands for the Axiom of Choice, which is one of the nine axioms. Loosely speaking, the Axiom of Choice states that given any collection of sets, each containing at least one element, it is possible to make a selection of exactly one object from each set, even if the collection of sets is infinite. There was a period of time in mathematics when the Axiom of Choice was controversial, but nowadays it is generally accepted. There is a fascinating history concerning the Axiom of Choice, including its controversy. The Wikipedia page for the Axiom of Choice is a good place to start if you are interested in learning more. There are several competing axiomatic approaches to set theory, but ZFC is considered the canonical collection of axioms by most mathematicians.

Appendix C includes a few more examples of paradoxes, which you are encouraged to ponder.

> In times of change, learners inherit the earth, while the learned find themselves beautifully equipped to deal with a world that no longer exists.
>
> Eric Hoffer, moral and social philosopher

3.3 Power Sets

We have already seen that using union, intersection, set difference, and complement we can create new sets (in the same universe) from existing sets. In this section, we will describe another way to generate new sets; however, the new sets will not "live" in the same universe this time. The following set is always a set of subsets. That is, its elements are themselves sets.

Definition 3.27. If S is a set, then the **power set** of S is the set of subsets of S. The power set of S is denoted $\boxed{\mathcal{P}(S)}$.

You can see that a power set of S is not composed of *elements* of S, but rather it is composed of *subsets* of S, and none of these subsets are elements of S.

For example, if $S = \{a, b\}$, then $\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, S\}$. It follows immediately from the definition that $A \subseteq S$ if and only if $A \in \mathcal{P}(S)$.

Problem 3.28. For each of the following sets, find the power set.

(a) $A = \{\circ, \triangle, \square\}$

(b) $B = \{a, \{a\}\}$

(c) $C = \emptyset$

(d) $D = \{\emptyset\}$

Problem 3.29. How many subsets do you think that a set with n elements has? What if $n = 0$? You do not need to prove your conjecture at this time. We will prove this later using mathematical induction.

It is important to realize that the concepts of *element* and *subset* need to be carefully delineated. For example, consider the set $A = \{x, y\}$. The object x is an element of A, but the object $\{x\}$ is both a subset of A and an element of $\mathcal{P}(A)$. This can get confusing rather quickly. Consider the set B from Problem 3.28. The set $\{a\}$ happens to be an element of B, a subset of B, and an element of $\mathcal{P}(B)$. The upshot is that it is important to pay close attention to whether "$\subseteq$" or "$\in$" is the proper symbol to use.

Since the next theorem is a biconditional proposition, you need to write two distinct subproofs, one for "$S \subseteq T$ implies $\mathcal{P}(S) \subseteq \mathcal{P}(T)$", and another for "$\mathcal{P}(S) \subseteq \mathcal{P}(T)$ implies $S \subseteq T$".

Theorem 3.30. Let S and T be sets. Then $S \subseteq T$ if and only if $\mathcal{P}(S) \subseteq \mathcal{P}(T)$.

Problem 3.31. Let S and T be sets. Determine whether each of the following statements is true or false. If the statement is true, prove it. If the statement is false, provide a counterexample.

(a) $\mathcal{P}(S \cap T) \subseteq \mathcal{P}(S) \cap \mathcal{P}(T)$

(b) $\mathcal{P}(S) \cap \mathcal{P}(T) \subseteq \mathcal{P}(S \cap T)$

(c) $\mathcal{P}(S \cup T) \subseteq \mathcal{P}(S) \cup \mathcal{P}(T)$

(d) $\mathcal{P}(S) \cup \mathcal{P}(T) \subseteq \mathcal{P}(S \cup T)$

While power sets provide a useful way of generating new sets, they also play a key role in Georg Cantor's (1845–1918) investigation into the "size" of sets. **Cantor's Theorem** (see Theorem 9.64) states that the power set of a set—even if the set is infinite—is always "larger" than the original set. One consequence of this is that there are different sizes of infinity and no largest infinity. Mathematics is awesome.

> The master has failed more times than the beginner has even tried.
>
> Stephen McCranie, author & illustrator

3.4 Indexing Sets

Suppose we consider the following collection of open intervals:

$$(0,1),(0,1/2),(0,1/4),\ldots,(0,1/2^{n-1}),\ldots$$

This collection has a natural way for us to "index" the sets:

$$I_1=(0,1), I_2=(0,1/2),\ldots,I_n=(0,1/2^{n-1}),\ldots$$

In this case the sets are **indexed** by the set $\mathbb{N}$. The subscripts are taken from the **index set**. If we wanted to talk about an arbitrary set from this indexed collection, we could use the notation I_n.

Let's consider another example:

$$\{a\},\{a,b\},\{a,b,c\},\ldots,\{a,b,c,\ldots,z\}$$

An obvious way to index these sets is as follows:

$$A_1=\{a\}, A_2=\{a,b\}, A_3=\{a,b,c\},\ldots,A_{26}=\{a,b,c,\ldots,z\}$$

In this case, the collection of sets is indexed by $\{1,2,\ldots,26\}$.

Using indexing sets in mathematics is an extremely useful notational tool, but it is important to keep straight the difference between the sets that are being indexed, the elements in each set being indexed, the indexing set, and the elements of the indexing set.

Any set (finite or infinite) can be used as an indexing set. Often capital Greek letters are used to denote arbitrary indexing sets and small Greek letters to represent elements of these sets. If the indexing set is a subset of $\mathbb{R}$, then it is common to use Roman letters as individual indices. Of course, these are merely conventions, not rules.

- If Δ is a set and we have a collection of sets indexed by Δ, then we may write $\{S_\alpha\}_{\alpha\in\Delta}$ to refer to this collection. We read this as "the set of S-sub-alphas over alpha in Delta."
- If a collection of sets is indexed by $\mathbb{N}$, then we may write $\{U_n\}_{n\in\mathbb{N}}$ or $\{U_n\}_{n=1}^{\infty}$.
- Borrowing from this idea, a collection $\{A_1,\ldots,A_{26}\}$ may be written as $\{A_n\}_{n=1}^{26}$.

Definition 3.32. Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of sets.

(a) The **union of the entire collection** is defined via

$$\boxed{\bigcup_{\alpha\in\Delta} A_\alpha := \{x \mid x \in A_\alpha \text{ for some } \alpha\in\Delta\}}.$$

(b) The **intersection of the entire collection** is defined via

$$\boxed{\bigcap_{\alpha\in\Delta} A_\alpha := \{x \mid x \in A_\alpha \text{ for all } \alpha\in\Delta\}}.$$

In the special case that $\Delta = \mathbb{N}$, we write

$$\bigcup_{n=1}^{\infty} A_n = \{x \mid x \in A_n \text{ for some } n\in\mathbb{N}\} = A_1 \cup A_2 \cup A_3 \cup \cdots$$

and

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid x \in A_n \text{ for all } n\in\mathbb{N}\} = A_1 \cap A_2 \cap A_3 \cap \cdots$$

Similarly, if $\Delta = \{1,2,3,4\}$, then

$$\bigcup_{n=1}^{4} A_n = A_1 \cup A_2 \cup A_3 \cup A_4 \quad\text{and}\quad \bigcap_{n=1}^{4} A_n = A_1 \cap A_2 \cap A_3 \cap A_4.$$

Notice the difference between "$\bigcup$" and "$\cup$" (respectively, "$\bigcap$" and "$\cap$").

Problem 3.33. Let $\{I_n\}_{n\in\mathbb{N}}$ be the collection of open intervals from the beginning of the section. Find each of the following.

(a) $\bigcup_{n\in\mathbb{N}} I_n$ (b) $\bigcap_{n\in\mathbb{N}} I_n$

Problem 3.34. Let $\{A_n\}_{n=1}^{26}$ be the collection from earlier in the section. Find each of the following.

(a) $\bigcup_{n=1}^{26} A_n$ (b) $\bigcap_{n=1}^{26} A_n$

Problem 3.35. Let $S_n = \{x \in \mathbb{R} \mid n-1 < x < n\}$, where $n \in \mathbb{N}$. Find each of the following.

(a) $\bigcup_{n=1}^{\infty} S_n$ (b) $\bigcap_{n=1}^{\infty} S_n$

Problem 3.36. Let $T_n = \{x \in \mathbb{R} \mid -\frac{1}{n} < x < \frac{1}{n}\}$, where $n \in \mathbb{N}$. Find each of the following.

(a) $\bigcup_{n=1}^{\infty} T_n$ (b) $\bigcap_{n=1}^{\infty} T_n$

Problem 3.37. For each $r \in \mathbb{Q}$ (the rational numbers), let N_r be the set containing all real numbers *except* r. Find each of the following.

(a) $\bigcup_{r\in\mathbb{Q}} N_r$ (b) $\bigcap_{r\in\mathbb{Q}} N_r$

Definition 3.38. A collection of sets $\{A_\alpha\}_{\alpha\in\Delta}$ is **pairwise disjoint** if $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$.

Problem 3.39. Provide an example of a collection of sets $\{A_\alpha\}_{\alpha\in\Delta}$ that is not pairwise disjoint even though $\bigcap_{\alpha\in\Delta} A_\alpha = \emptyset$.

Problem 3.40. For each of the following, provide an example of a collection of sets with the stated property.

(a) A collection of three subsets of $\mathbb{R}$ such that the collection is not pairwise disjoint, the union equals $\mathbb{R}$, and the intersection of the collection is empty.

(b) A collection of infinitely many subsets of $\mathbb{R}$ such that the collection is not pairwise disjoint, the union equals $\mathbb{R}$, and the intersection of the collection is empty.

(c) A collection of infinitely many subsets of $\mathbb{R}$ such that the collection is pairwise disjoint, the union equals $\mathbb{R}$, and the intersection of the collection is empty.

Theorem 3.41 (Generalized Distribution of Union and Intersection)**.** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of sets and let B be any set. Then

(a) $B \cup \left(\bigcap_{\alpha\in\Delta} A_\alpha\right) = \bigcap_{\alpha\in\Delta}(B \cup A_\alpha)$, and

(b) $B \cap \left(\bigcup_{\alpha \in \Delta} A_\alpha \right) = \bigcup_{\alpha \in \Delta} (B \cap A_\alpha)$.

Theorem 3.42 (Generalized De Morgan's Law)**.** Let $\{A_\alpha\}_{\alpha \in \Delta}$ be a collection of sets. Then

(a) $\left(\bigcup_{\alpha \in \Delta} A_\alpha \right)^C = \bigcap_{\alpha \in \Delta} A_\alpha^C$, and

(b) $\left(\bigcap_{\alpha \in \Delta} A_\alpha \right)^C = \bigcup_{\alpha \in \Delta} A_\alpha^C$.

At the end of Section 3.2, we mentioned the Axiom of Choice. Using the language of indexing sets, we can now state this axiom precisely.

Axiom 3.43 (Axiom of Choice)**.** For every indexed collection $\{A_\alpha\}_{\alpha \in \Delta}$ of nonempty sets, there exists an indexed collection $\{a_\alpha\}_{\alpha \in \Delta}$ of elements such that $a_\alpha \in A_\alpha$ for each $\alpha \in \Delta$.

Intuitively, the Axiom of Choice guarantees the existence of mathematical objects that are obtained by a sequence of choices. It applies to both the finite and infinite setting. As an analogy, we can think of each A_α as a drawer in a dresser and each a_α as an article of clothing chosen from the drawer identified with A_α. The Axiom of Choice is surprisingly powerful, sometimes leading to unexpected consequences. It often gets used in subtle ways that mathematicians are not always explicit with. We will require the Axiom of Choice when proving Theorems 9.31 and 9.47. When proving these theorems, be on the lookout for where you are invoking the Axiom of Choice.

> All sorts of things can happen when you're open to new ideas and playing around with things.
>
> Stephanie Kwolek, chemist

3.5 Cartesian Products of Sets

Given a collection of sets, we can form new sets by taking unions, intersections, complements, and set differences. In this section, we introduce a type of "product" of sets. You have already encountered this concept when you learned to plot points in the plane. You also crossed paths with this notion if you have taken a course in linear algebra.

Definition 3.44. For each $n \in \mathbb{N}$, we define an ***n*-tuple** to be an ordered list of n elements of the form $(a_1, a_2, \ldots, a_n)$. We refer to a_i as the ith **component** (or **coordinate**) of $(a_1, a_2, \ldots, a_n)$. Two n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equal if $a_i = b_i$ for all $1 \leq i \leq n$. A 2-tuple (a, b) is more commonly referred to as an **ordered pair** while a 3-tuple (a, b, c) is often called an **ordered triple**.

Occasionally, other symbols are used to surround the components of an n-tuple, such as square brackets "[ ]" or angle brackets "⟨ ⟩". In some programming languages, curly braces "{ }" are used to specify arrays. However, we avoid this convention in mathematics since curly braces are the standard notation for sets. The term "tuple" can also occur when discussing other mathematical objects, such as vectors.

We can use the notion of n-tuples to construct new sets from existing sets.

Definition 3.45. If A and B are sets, the **Cartesian product** (or **direct product**) of A and B, denoted $A \times B$ (read as "A times B" or "A cross B"), is the set of all ordered pairs where the first component is from A and the second component is from B. In set-builder notation, we have

$$\boxed{A \times B := \{(a,b) \mid a \in A, b \in B\}}.$$

We similarly define the Cartesian product of n sets, say $A_1, \ldots, A_n$, by

$$\boxed{\prod_{i=1}^{n} A_i := A_1 \times \cdots \times A_n := \{(a_1, \ldots, a_n) \mid a_j \in A_j \text{ for all } 1 \leq j \leq n\}},$$

where A_i is referred to as the ith **factor** of the Cartesian product. As a special case, the set

$$\underbrace{A \times \cdots \times A}_{n \text{ factors}}$$

is often abbreviated as A^n.

Cartesian products are named after French philosopher and mathematician René Descartes (1596–1650). Cartesian products will play a prominent role in Chapter 7.

Example 3.46. If $A = \{a, b, c\}$ and $B = \{☺, ☹\}$, then

$$A \times B = \{(a, ☺), (a, ☹), (b, ☺), (b, ☹), (c, ☺), (c, ☹)\}.$$

Example 3.47. The standard two-dimensional plane $\mathbb{R}^2$ and standard three space $\mathbb{R}^3$ are familiar examples of Cartesian products. In particular, we have

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$$

and

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

Problem 3.48. Consider the sets A and B from Example 3.46.

(a) Find $B \times A$.

(b) Find $B \times B$.

Problem 3.49. If A and B are sets, why do you think that $A \times B$ is referred to as a type of "product"? Think about the area model for multiplication of natural numbers.

Problem 3.50. If A and B are both finite sets, then how many elements will $A \times B$ have?

Problem 3.51. Let $A = \{1,2,3\}$, $B = \{1,2\}$, and $C = \{1,3\}$. Find $A \times B \times C$.

Problem 3.52. Let $X = [0,1]$ and $Y = \{1\}$. Write each of the following using set-builder notation and then describe the set geometrically (e.g., draw a picture).

(a) $X \times Y$

(b) $Y \times X$

(c) $X \times X$

(d) $Y \times Y$

Problem 3.53. If A is a set, then what is $A \times \emptyset$ equal to?

Problem 3.54. Given sets A and B, when will $A \times B$ be equal to $B \times A$?

Problem 3.55. Write $\mathbb{N} \times \mathbb{R}$ using set-builder notation and then describe this set geometrically by interpreting it as a subset of $\mathbb{R}^2$.

We now turn our attention to subsets of Cartesian products.

Theorem 3.56. *Let A, B, C, and D be sets. If $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.*

Problem 3.57. Is it true that if $A \times B \subseteq C \times D$, then $A \subseteq C$ and $B \subseteq D$? Do not forget to think about cases involving the empty set.

Problem 3.58. Is every subset of $C \times D$ of the form $A \times B$, where $A \subseteq C$ and $B \subseteq D$? If so, prove it. If not, find a counterexample.

Problem 3.59. If A, B, and C are nonempty sets, is $A \times B$ a subset of $A \times B \times C$?

Problem 3.60. Let $A = [2,5]$, $B = [3,7]$, $C = [1,3]$, and $D = [2,4]$. Compute each of the following.

(a) $(A \cap B) \times (C \cap D)$

(b) $(A \times C) \cap (B \times D)$

(c) $(A \cup B) \times (C \cup D)$

(d) $(A \times C) \cup (B \times D)$

(e) $A \times (B \cap C)$

(f) $(A \times B) \cap (A \times C)$

(g) $A \times (B \cup C)$

(h) $(A \times B) \cup (A \times C)$

Problem 3.61. Let A, B, C, and D be sets. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

(b) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$

(c) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(d) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(e) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

Problem 3.62. If A and B are sets, conjecture a way to rewrite $(A \times B)^C$ in a way that involves A^C and B^C and then prove your conjecture.

> If there is no struggle, there is no progress.
>
> Frederick Douglass, writer & statesman