Chapter 3

Set Theory and Topology

At its essence, all of mathematics is built on set theory. In this chapter, we will introduce some of the basics of sets and their properties.

3.1 Sets

Definition 3.1. A **set** is a collection of objects called **elements**. If A is a set and x is an element of A, we write $x \in A$. Otherwise, we write $x \notin A$.

Definition 3.2. The set containing no elements is called the **empty set**, and is denoted by the symbol $\emptyset$.

If we think of a set as a box potentially containing some stuff, then the empty set is a box with nothing in it.

Definition 3.3. The language associated to sets is specific. We will often define sets using the following notation, called **set builder notation**:

$$\boxed{S = \{x \in A \mid x \text{ satisfies some condition}\}}$$

The first part "$x \in A$" denotes what type of x is being considered. The statements to the right of the vertical bar (not to be confused with "divides") are the conditions that x must satisfy in order to be members of the set. This notation is read as "The set of all x in A such that x satisfies some condition," where "some condition" is something specific about the restrictions on x relative to A.

Exercise 3.4. Unpack each of the following sets and see if you can find a simple description of the elements that each set contains.

(a) $A = \{x \in \mathbb{N} \mid x = 3k \text{ for some } k \in \mathbb{N}\}$

(b) $B = \{t \in \mathbb{R} \mid t^2 \leq 2\}$

(c) $C = \{t \in \mathbb{Z} \mid t^2 \leq 2\}$

(d) $D = \{m \in \mathbb{R} \mid m = 1 - \frac{1}{n}, \text{ where } n \in \mathbb{N}\}$

Exercise 3.5. Write each of the following sentences using set builder notation.

(a) The set of all real numbers less than $-\sqrt{2}$.

(b) The set of all real numbers greater than -12 and less than or equal to 42.

(c) The set of all even natural numbers.

Definition 3.6. If A and B are sets, then we say that A is a **subset** of B, written $\boxed{A \subseteq B}$, provided that every element of A is also an element of B.

Observe that $A \subseteq B$ is equivalent to "For all x (in the universe of discourse), if $x \in A$, then $x \in B$." Since we know how to deal with "for all" statements and conditional propositions, we know how to go about proving $A \subseteq B$.

Problem 3.7. Suppose A and B are sets. Describe a skeleton proof for proving that $A \subseteq B$.

Every set always has two rather boring subsets.

Theorem 3.8. Let S be a set. Then

(a) $S \subseteq S$

(b) $\emptyset \subseteq S$.

Exercise 3.9. List all of the subsets of $A = \{1, 2, 3\}$.

Theorem 3.10 (Transitivity of subsets)**.** Suppose that A, B, and C are sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Definition 3.11. If $A \subseteq B$, then A is called a **proper subset** provided that $A \neq B$. In this case, we may write $\boxed{A \subset B}$ or $\boxed{A \subsetneq B}$.[1]

The following definitions should look familiar from precalculus.

Definition 3.12 (Interval Notation)**.** For $a, b \in \mathbb{R}$ with $a < b$, we define the following.

(a) $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

(b) $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$

(c) $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$

(d) $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

We analogously define $[a, b)$, $(a, b]$, $[a, \infty)$, and $(-\infty, b]$.

Definition 3.13. Let A and B be sets in some universe of discourse U.

(a) The **union** of the sets A and B is $\boxed{A \cup B} = \{x \in U \mid x \in A \text{ or } x \in B\}$.

(b) The **intersection** of the sets A and B is $\boxed{A \cap B} = \{x \in U \mid x \in A \text{ and } x \in B\}$.

(c) The **set difference** of the sets A and B is $\boxed{A \setminus B} = \{x \in U \mid x \in A \text{ and } x \notin B\}$.

(d) The **complement of** A (relative to U) is the set $\boxed{A^c} = U \setminus A = \{x \in U \mid x \notin A\}$.

Definition 3.14. If two sets A and B have the property that $A \cap B = \emptyset$, then we say that A and B are **disjoint** sets.

[1] *Warning:* Some books use $\subset$ to mean $\subseteq$.

Exercise 3.15. Suppose that the universe of discourse is $U = \{1,2,3,4,5,6,7,8,9,10\}$. Let $A = \{1,2,3,4,5\}$, $B = \{1,3,5\}$, and $C = \{2,4,6,8\}$. Find each of the following.

(a) $A \cap C$

(b) $B \cap C$

(c) $A \cup B$

(d) $A \setminus B$

(e) $B \setminus A$

(f) $C \setminus B$

(g) B^c

(h) A^c

(i) $(A \cup B)^c$

(j) $A^c \cap B^c$

Exercise 3.16. Suppose that the universe of discourse is $U = \mathbb{R}$. Let $A = [-3,-1)$, $B = (-2.5,2)$, and $C = (-2,0]$. Find each of the following.

(a) A^c

(b) $A \cap C$

(c) $A \cap B$

(d) $A \cup B$

(e) $(A \cap B)^c$

(f) $(A \cup B)^c$

(g) $A \setminus B$

(h) $A \setminus (B \cup C)$

(i) $B \setminus A$

Theorem 3.17. Let A and B be sets. If $A \subseteq B$, then $B^c \subseteq A^c$.

Definition 3.18. Two sets A and B are **equal**, denoted $\boxed{A = B}$, iff $A \subseteq B$ and $B \subseteq A$.

Given two sets A and B, if we want to prove $A = B$, then we have to do two separate mini-proofs: one for $A \subseteq B$ and one for $B \subseteq A$. It is common to label each mini-proof with "($\subseteq$)" and "($\supseteq$)", respectively.

Theorem 3.19. Let A and B be sets. Then $A \setminus B = A \cap B^c$.

For each of the next two theorems, you can choose to prove either part (a) or part (b). Of course, you are welcome to prove both parts, but you do not have to.

Theorem 3.20 (DeMorgan's Law)**.** Let A and B be sets. Then

(a) $(A \cup B)^c = A^c \cap B^c$

(b) $(A \cap B)^c = A^c \cup B^c$.

Theorem 3.21 (Distribution of Union and Intersection)**.** Let A, B, and C be sets. Then

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

3.2 Power Sets and Paradoxes

We've already seen that using union, intersection, set difference, and complement we can create new sets (in the same universe) from existing sets. In this section, we will describe another way to generate new sets; however, the new sets will not "live" in the same universe this time.

Definition 3.22. If S is a set, then the **power set** of S is the set of subsets of S. The power set of S is denoted $\boxed{\mathcal{P}(S)}$.

It follows immediately from the definition that $A \subseteq S$ iff $A \in \mathcal{P}(S)$.[2] For example, if $S = \{a, b\}$, then $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, S\}$.

Exercise 3.23. For each of the following sets, find the power set.

(a) $A = \{\circ, \triangle, \square\}$

(b) $B = \{a, \{a\}\}$

(c) $C = \emptyset$

(d) $D = \{\emptyset\}$

Question 3.24. How many subsets do you think that a set with n elements has? What if $n = 0$? You do not need to prove your conjecture at this time. We will prove this later using mathematical induction.

It is important to realize that the concepts of *element* and *subset* need to be carefully delineated. For example, consider the set $A = \{x, y\}$. The object x is an element of A, but the object $\{x\}$ is both a subset of A and an element of $\mathcal{P}(A)$. This can get confusing rather quickly. Consider the set C from Exercise 3.23. The set $\{a\}$ happens to be an element of C, a subset of C, and an element of $\mathcal{P}(C)$. The upshot is that it is important to pay close attention to whether "$\subseteq$" or "$\in$" is the proper symbol to use.

Theorem 3.25. Let S and T be sets. Then $S \subseteq T$ iff $\mathcal{P}(S) \subseteq \mathcal{P}(T)$.[3]

Theorem 3.26. Let S and T be sets. Then $\mathcal{P}(S) \cap \mathcal{P}(T) = \mathcal{P}(S \cap T)$.

Theorem 3.27. Let S and T be sets. Then $\mathcal{P}(S) \cup \mathcal{P}(T) \subseteq \mathcal{P}(S \cup T)$.

Problem 3.28. Provide a counterexample to show that it is not necessarily true that $\mathcal{P}(S) \cup \mathcal{P}(T) = \mathcal{P}(S \cup T)$. This verifies that the converse of Theorem 3.27 is not true in general. Is it ever true that $\mathcal{P}(S) \cup \mathcal{P}(T)$ and $\mathcal{P}(S \cup T)$ are equal?

We now turn out attention to the issue of whether there is one mother of all universal sets. Before reading any further, consider this for a moment. That is, is there one largest set that all other sets are a subset of? Or, in other words, is there a set of all sets? To help wrap our heads around this issue, consider the following riddle, known as the **Barber of Seville Paradox**.

[2] Recall that "iff" is an abbreviation for 'if and only if", which is a statement of the form $A \iff B$ for propositions A and B. Recall that this is short for both $(A \implies B) \wedge (B \implies A)$.

[3] To prove this theorem, you have to write two distinct subproofs: $A \implies B$ and $B \implies A$.

> In Seville, there is a barber who shaves all those men, and only those men, who do not shave themselves. Who shaves the barber?

Problem 3.29. In the Barber of Seville Paradox, does the barber shave himself or not?

Problem 3.29 is an example of a **paradox**. What do you think paradox means? Now, suppose that there is a set of all sets and call it $\mathcal{U}$. That is, $\mathcal{U} := \{A \mid A \text{ is a set}\}$.

Problem 3.30. Given our definition of $\mathcal{U}$, explain why $\mathcal{U}$ is an element of itself.

If we continue with this line of reasoning, it must be the case that some sets are elements of themselves and some are not. Let X be the set of all sets that are elements of themselves and let Y be the set of all sets that are not elements of themselves.

Problem 3.31. Does Y belong to X or Y? Explain why this is a paradox.

The above paradox is one way of phrasing a paradox referred to as **Russell's Paradox**. Okay, how did we get into this mess in the first place?! By assuming the existence of a set of all sets, we can produce all sorts of paradoxes. The only way to avoid the paradoxes is to conclude that there is no set of all sets. Here is some more evidence that we shouldn't assume the existence of a set of all sets.

Problem 3.32. If $\mathcal{U}$ is the set of all sets, then what is the relationship between $\mathcal{U}$ and $\mathcal{P}(\mathcal{U})$? What about $\mathcal{P}(\mathcal{P}(\mathcal{U})$?

The upshot is that the collection of all sets is *not* a set!

Problem 3.33. Pick any two of the paradoxes below and for each one explain why it is a paradox.

(a) **Librarian's Paradox.** A librarian is given the unenviable task of creating two new books for the library. Book A contains the names of all books in the library that reference themselves and Book B contains the names of all books in the library that do not reference themselves. But the librarian just created two new books for the library, so their titles must be in either Book A or Book B. Clearly Book A can be listed in Book B, but where should the librarian list Book B?

(b) **Liar's Paradox.** Consider the statement: this sentence is false. Is it true or false?

(c) **Berry Paradox.** Consider the claim: every natural number can be unambiguously described in fourteen words or less. It seems clear that this statement is false, but if that is so, then there is some smallest natural number which cannot be unambiguously described in fourteen words or less. Let's call it n. But now n is "the smallest natural number that cannot be unambiguously described in fourteen words or less." This is a complete and unambiguous description of n in fourteen words, contradicting the fact that n was supposed not to have such a description. Therefore, all natural numbers can be unambiguously described in fourteen words or less!

(d) **The Naming Numbers Paradox.** Consider the claim: every natural number can be unambiguously described using no more than 50 characters (where a character is a–z, 0–9, and a "space"). For example, we can describe 9 as "9" or "nine" or "the square of the second prime number." There are only 37 characters, so we can describe at most 37^{50} numbers, which is very large, but not infinite. So the statement is false. However, here is a "proof" that it is true. Let S be the set of natural numbers that can be unambiguously described using no more than 50 characters. For the sake of contradiction, suppose it is not all of $\mathbb{N}$. Then there is a smallest number $t \in \mathbb{N} - S$. We can describe t as: the smallest natural number not in S. Thus t can be described using no more than 50 characters. So $t \in S$, a contradiction.

(e) **Euathlus and Protagoras.** Euathlus wanted to become a lawyer but could not pay Protagoras. Protagoras agreed to teach him under the condition that if Euathlus won his first case, he would pay Protagoras, otherwise not. Euathlus finished his course of study and did nothing. Protagoras sued for his fee. He argued:

If Euathlus loses this case, then he must pay (by the judgment of the court).
If Euathlus wins this case, then he must pay (by the terms of the contract).
He must either win or lose this case.
Therefore Euathlus must pay me.

But Euathlus had learned well the art of rhetoric. He responded:

If I win this case, I do not have to pay (by the judgment of the court).
If I lose this case, I do not have to pay (by the contract).
I must either win or lose the case.
Therefore, I do not have to pay Protagoras.

3.3 Indexing Sets

Suppose we consider the following collection of open intervals:

$$(0,1),(0,1/2),(0,1/4),\ldots,(0,1/2^{n-1}),\ldots$$

This collection has a natural way for us to "index" the sets:

$$I_1 = (0,1), I_2 = (0,1/2),\ldots, I_n = (0,1/2^{n-1}),\ldots$$

In this case the sets are **indexed** by the set $\mathbb{N}$. The subscripts are taken from the **index set**. If we wanted to talk about an arbitrary set from this indexed collection, we could use the notation I_n.

Let's consider another example:

$$\{a\},\{a,b\},\{a,b,c\},\ldots,\{a,b,c,\ldots,z\}$$

An obvious way to index these sets is as follows:

$$A_1 = \{a\}, A_2 = \{a,b\}, A_3 = \{a,b,c\}, \ldots, A_{26} = \{a,b,c,\ldots,z\}$$

In this case, the collection of sets is indexed by $\{1,2,\ldots,26\}$.

Using indexing sets in mathematics is an extremely useful notational tool, but it is important to keep straight the difference between the sets that are being indexed, the elements in each set being indexed, the indexing set, and the elements of the indexing set.

Any set (finite or infinite) can be used as an indexing set. Often capital Greek letters are used to denote arbitrary indexing sets and small Greek letters to represent elements of these sets. If the indexing set is a subset of $\mathbb{R}$, then it is common to use Roman letters as individual indices. Of course, these are merely conventions, not rules.

- If Δ is a set and we have a collection of sets indexed by Δ, then we may write $\{S_\alpha\}_{\alpha\in\Delta}$ to refer to this collection. We read this as "the set of S-alphas over alpha in Delta."
- If a collection of sets is indexed by $\mathbb{N}$, then we may write $\{U_n\}_{n\in\mathbb{N}}$ or $\{U_n\}_{n=1}^{\infty}$.
- Borrowing from this idea, a collection $\{A_1,\ldots,A_{26}\}$ may be written as $\{A_n\}_{n=1}^{26}$.

Definition 3.34. Suppose we have a collection $\{A_\alpha\}_{\alpha\in\Delta}$.

(a) The **union of the entire collection** is defined via

$$\boxed{\bigcup_{\alpha\in\Delta} A_\alpha = \{x \mid x \in A_\alpha \text{ for some } \alpha \in \Delta\}.}$$

(b) The **intersection of the entire collection** is defined via

$$\boxed{\bigcap_{\alpha\in\Delta} A_\alpha = \{x \mid x \in A_\alpha \text{ for all } \alpha \in \Delta\}.}$$

In the special case that $\Delta = \mathbb{N}$, we write

$$\bigcup_{n=1}^{\infty} A_n = \{x \mid x \in A_n \text{ for some } n \in \mathbb{N}\} = A_1 \cup A_2 \cup A_3 \cup \cdots$$

and

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid x \in A_n \text{ for all } n \in \mathbb{N}\} = A_1 \cap A_2 \cap A_3 \cap \cdots$$

Similarly, if $\Delta = \{1,2,3,4\}$, then

$$\bigcup_{n=1}^{4} A_n = A_1 \cup A_2 \cup A_3 \cup A_4 \quad \text{and} \quad \bigcap_{n=1}^{4} A_n = A_1 \cap A_2 \cap A_3 \cap A_4.$$

Notice the difference between "$\bigcup$" and "$\cup$" (respectively, "$\bigcap$" and "$\cap$").

Exercise 3.35. Let $\{I_n\}_{n\in\mathbb{N}}$ be the collection of open intervals from the beginning of the section. Find each of the following.

(a) $\displaystyle\bigcup_{n\in\mathbb{N}} I_n$ (b) $\displaystyle\bigcap_{n\in\mathbb{N}} I_n$

Exercise 3.36. Let $\{A_n\}_{n=1}^{26}$ be the collection from earlier in the section. Find each of the following.

(a) $\displaystyle\bigcup_{n=1}^{26} A_n$ (b) $\displaystyle\bigcap_{n=1}^{26} A_n$

Exercise 3.37. Let $S_n = \{x \in \mathbb{R} \mid n-1 < x < n\}$, where $n \in \mathbb{N}$. Find each of the following.

(a) $\displaystyle\bigcup_{n=1}^{\infty} S_n$ (b) $\displaystyle\bigcap_{n=1}^{\infty} S_n$

Exercise 3.38. Let $T_n = \{x \in \mathbb{R} \mid -\frac{1}{n} < x < \frac{1}{n}\}$, where $n \in \mathbb{N}$. Find each of the following.

(a) $\displaystyle\bigcup_{n=1}^{\infty} T_n$ (b) $\displaystyle\bigcap_{n=1}^{\infty} T_n$

Exercise 3.39. For each $r \in \mathbb{Q}$ (the rational numbers), let N_r be the set containing all real numbers *except* r. Find each of the following.

(a) $\displaystyle\bigcup_{r\in\mathbb{Q}} N_r$ (b) $\displaystyle\bigcap_{r\in\mathbb{Q}} N_r$

Definition 3.40. A collection of sets $\{A_\alpha\}_{\alpha\in\Delta}$ is **pairwise disjoint** if $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$.

Exercise 3.41. Draw a Venn diagram of a collection of 3 sets that are pairwise disjoint.

Exercise 3.42. Provide an example of a collection of three sets, say $\{A_1, A_2, A_3\}$, such that the collection is *not* pairwise disjoint, but $\bigcap_{n=1}^{3} A_n = \emptyset$.

For each of the next two theorems, you can choose to prove either part (a) or part (b).

Theorem 3.43 (Generalized Distribution of Union and Intersection)**.** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of sets and let B be any set. Then

(a) $\displaystyle B \cup \left(\bigcap_{\alpha\in\Delta} A_\alpha\right) = \bigcap_{\alpha\in\Delta}(B \cup A_\alpha)$, (b) $\displaystyle B \cap \left(\bigcup_{\alpha\in\Delta} A_\alpha\right) = \bigcup_{\alpha\in\Delta}(B \cap A_\alpha)$.

Theorem 3.44 (Generalized DeMorgan's Law)**.** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of sets. Then

(a) $\displaystyle \left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^C = \bigcap_{\alpha\in\Delta} A_\alpha^C$, (b) $\displaystyle \left(\bigcap_{\alpha\in\Delta} A_\alpha\right)^C = \bigcup_{\alpha\in\Delta} A_\alpha^C$.

3.4 Topology of $\mathbb{R}$

For this entire section, our universe of discourse is the set of real numbers. You may assume all the usual basic algebraic properties of the real numbers (addition, subtraction, multiplication, division, commutative property, distribution, etc.).

Recall that an **axiom** is a statement that we *assume* to be true. Here are some useful axioms of the real numbers.

Axiom 3.45. If p and q are two different real numbers in $\mathbb{R}$, then there is a number between them.

Exercise 3.46. Given real numbers p and q with $p < q$, construct a real number x such that $p < x < q$. We know such a point must exist by the previous axiom, but this exercise is asking you to produce an actual candidate.

Axiom 3.47. (Linear ordering) If a, b, and c are real numbers, then:

(a) If $a < b$ and $b < c$, then $a < c$;

(b) Exactly one of the following is true: (i) $a < b$, (ii) $a = b$, or (iii) $a > b$.

Axiom 3.48. If p is a real number, then there exists $q, r \in \mathbb{R}$ such that $q < p < r$.

Axiom 3.49. (Archimedean Property) If x is a real number, then either (i) x is an integer or (ii) there exists an integer n, such that $n < x < n+1$.

Definition 3.50. Suppose $a, b \in \mathbb{R}$ such that $a < b$. The intervals $(a,b), (-\infty, b), (a, \infty)$ are called **open intervals** while the interval $[a,b]$ is called a **closed interval**. An interval like $[a,b)$ is neither open nor closed.

We will always assume that any time we write $(a,b), [a,b], (a,b]$, or $[a,b)$ that $a < b$.

Exercise 3.51. Give an example of each of the following.

(a) An open interval.

(b) A closed interval.

(c) An interval that is neither open nor closed.

(d) An infinite set that is not an interval.

Definition 3.52. A set U is called an **open set** iff for every $t \in U$, there exists an open interval containing t such that the open interval is a subset of U. We define the empty set to be open.

Problem 3.53. Prove that the set $I = (1,2)$ is an open set.

Theorem 3.54. Every open interval is an open set.

Theorem 3.55. The real numbers form an open set.

Exercise 3.56. Provide an example of an open set that is not a single open interval.

Theorem 3.57. Every closed interval is not an open set.

Theorem 3.58. Let $x \in \mathbb{R}$. Then the set $\{x\}$ is not open.

Exercise 3.59. Determine whether $\{4, 17, 42\}$ is an open set. Briefly justify your assertion.

Theorem 3.60. Let A and B be open sets. Then

(a) $A \cup B$ is an open set

(b) $A \cap B$ is an open set.

Theorem 3.61. Let $\{U_\alpha\}_{\alpha \in \Delta}$ be a collection of open sets. Then $\bigcup_{\alpha \in \Delta} U_\alpha$ is an open set.

Exercise 3.62.

(a) Find a collection of open sets $\{U_\alpha\}_{\alpha \in \Delta}$ such that $\bigcap_{\alpha \in \Delta} U_\alpha$ is not an open set.

(b) Find a collection of open sets $\{B_\alpha\}_{\alpha \in \Delta}$ such that $\bigcap_{\alpha \in \Delta} B_\alpha$ is an open set.

Remark 3.63. Taken together, Theorems 3.60–3.61 and Exercise 3.62 tell us that the union of any collection of open sets is open, but that the intersection of open sets may or may not be open. However, if we are taking the intersection of finitely many open sets, then the intersection will be open.

Exercise 3.64. Determine whether each of the following sets is open or not open.

(a) $W = \displaystyle\bigcup_{n=2}^{\infty} \left(n - \frac{1}{2}, n\right)$

(b) $X = \displaystyle\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$

Definition 3.65. A point p is a **limit point of the set** S iff for every open interval I containing p, there exists a point $q \in I$ such that $q \in S$ with $q \neq p$.

Problem 3.66. Consider the open interval $S = (1, 2)$. Prove each of the following.

(a) The points 1 and 2 are limit points of S.

(b) If $p \in S$, then p is a limit point of S.

(c) If $p < 1$ or $p > 2$, then p is not a limit point of S.

Theorem 3.67. A point p is a limit point of (a, b) iff $p \in [a, b]$.

Problem 3.68. Prove that the point $p = 0$ is a limit point of $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Are there any other limit points?

Exercise 3.69. Provide an example of a set S such that 1 is a limit point of S, $1 \neq S$, and S contains no intervals.

Exercise 3.70. Provide an example of a set T with exactly two limit points.

Theorem 3.71. If $p \in \mathbb{R}$, then p is a limit point of $\mathbb{Q}$.

Definition 3.72. A set is called **closed** iff it contains all of its limit points.

Exercise 3.73. Provide an example of each of the following. You do not need to prove that your answers are correct.

(a) A closed set.

(b) A set that is not closed.

(c) A set that is open and closed.

(d) A set that neither open nor closed.

Theorem 3.74. The set $[a,b]$ is closed.

Theorem 3.75. The set U is open iff U^C is closed.

Theorem 3.76. Every finite set is closed.

Problem 3.77. Prove or provide a counterexample: If a set S is not open, then it is closed.

Theorem 3.78. The set of real numbers is both open and closed.

Theorem 3.79. The set of real numbers is neither open nor closed.

Theorem 3.80. The empty set is both open and closed.

Theorem 3.81. Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of closed sets. Then $\bigcap_{\alpha\in\Delta} A_\alpha$ is a closed set.

Problem 3.82. Prove or provide a counterexample: If A and B are closed sets, then $A \cup B$ is also closed.

Exercise 3.83. Provide an example of a collection of closed sets $\{A_\alpha\}_{\alpha\in\Delta}$ such that $\bigcup_{\alpha\in\Delta} A_\alpha$ is a *not* closed set.

Remark 3.84. You should compare what happened in Theorem 3.81 and Exercise 3.83 to what we stated in Remark 3.63.