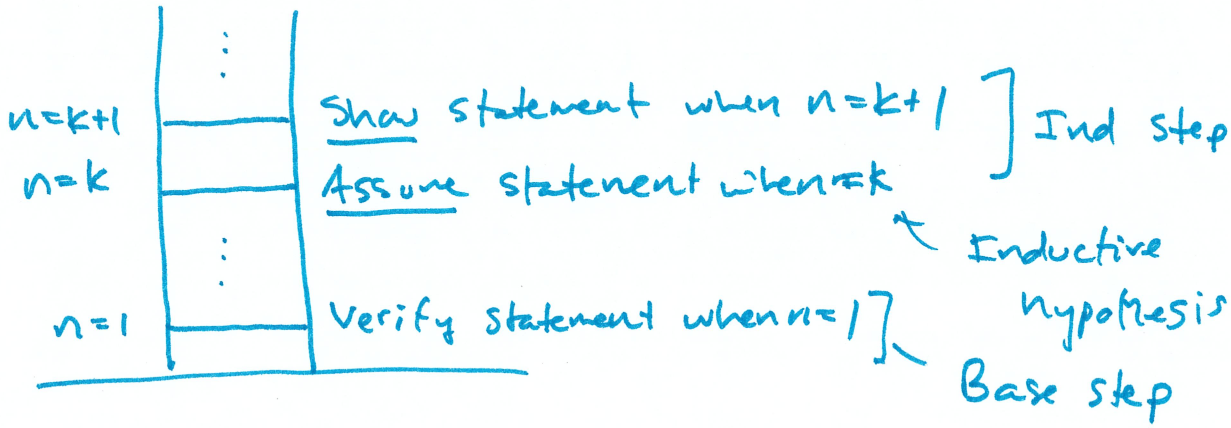
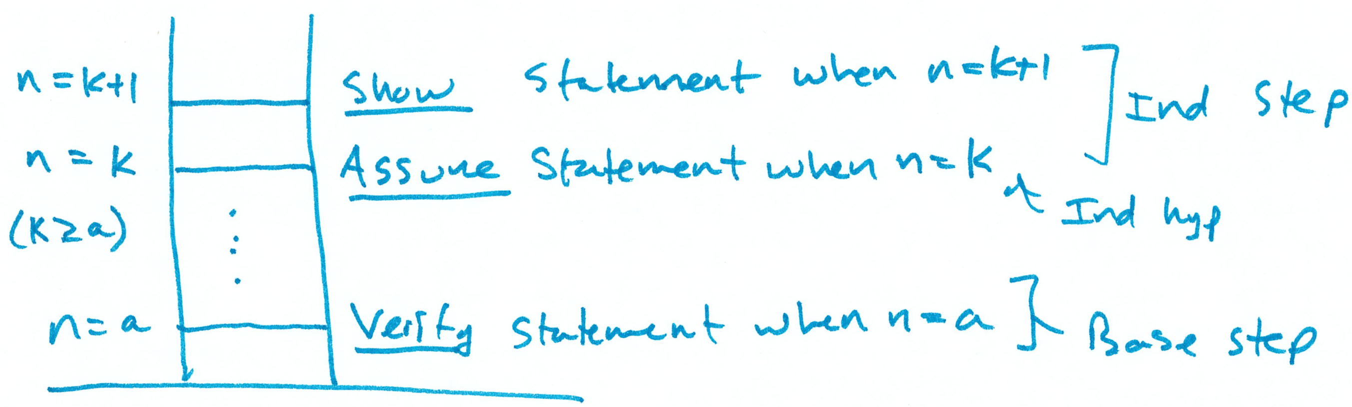
Induction Ladders





Both referred to as "ordinary induction"

Truth sets:

$S = \{k \in \mathbb{N} \mid P(k) \text{ true}\}$ ← n=1 start

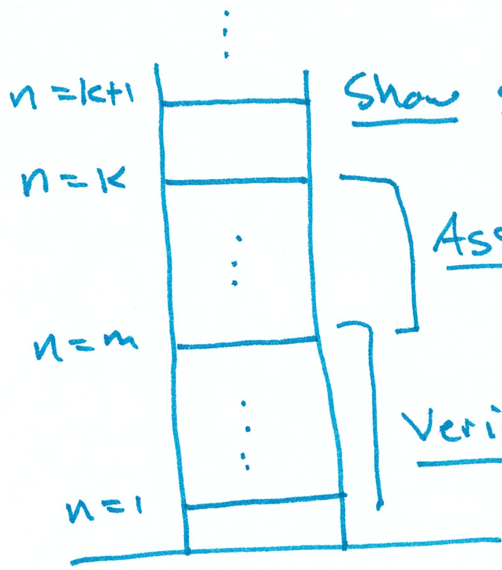
$S = \{k \in \mathbb{N} \mid P(\underbrace{a+k-1}) \text{ true}\}$ ← n=a start

use Axiom of Ind on both

shift start to a

Complete / Strong Induction

Ind step



Show statement true for $n=k+1$

Assume statement true for all $m \le j \le k$

Verify statement true for $n = 1, \dots, m$

Base steps

Important: we use complete induction when we need to "reach back" more than one step during ind step.

Ordinary induction $\Longleftrightarrow$ Complete induction !

"$\Leftarrow$" is clear

Why "$\Rightarrow$"? IH ~~rest proof in DS cond later~~

↑ we don't need this, but it's true

Thm 4.6: $\forall n \in \mathbb{N}$, 6 divs $n^3 - n$.

true at other times too!

Pf: We proceed by induction.

Base Step: S'pose $n=1$. Then $n^3 - n = 1^3 - 1 = 0$, which is div by 6 since $6 \cdot 0 = 0$.

"k"

Ind Step: Let $k \in \mathbb{N}$ and assume 6 divs $k^3 - k$. Then $\exists\, l \in \mathbb{Z}$ s.t. $k^3 - k = 6l$. We must show 6 divs $(k+1)^3 - (k+1)$. We see that

$$
\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\
&= k^3 + 3k^2 + 2k \\
&= k^3 - k + 3k^2 + 2k + k \\
&= 6l + 3k(k+1) \quad \text{(by Ind hyp)}.
\end{aligned}
$$

Now, notice that either k or $k+1$ is even. wlog, assume k even, so that $k = 2j$ for some $j \in \mathbb{Z}$. Then we see that

$$6\ell + 3k(k+1)$$

$$= 6\ell + 3 \cdot 2j(k+1)$$

$$= 6(\ell + j(k+1)).$$

By closure, $\ell + j(k+1) \in \mathbb{Z}$, and hence $(k+1)^3 - (k+1)$ is div by 6.

Therefore, by induction $n^3 - n$ is div by 6 for all $n \in \mathbb{N}$. $\boxtimes$

<u>Thm 4.16</u>: For all ints $n \geq 0$, $\overset{2^0}{\overset{\shortparallel}{1}} + 2^1 + \cdots + 2^n = 2^{n+1} - 1$.

<u>Pf</u>: We proceed by Induction.

<u>Base Step</u>: S'pose $n = 0$. Then the LHS becomes

$2^0 = 1$ while RHS ~~becom~~ becomes $2^{n+1} - 1 = 2^1 - 1 = 1$.

This shows statement true when $n = 0$.

<u>Ind Step</u>: Let $k \geq 0$ be an integer and

assume $2^0 + 2^1 + \cdots + 2^k = 2^{k+1} - 1$. We see that

$$2^0 + 2^1 + \cdots + 2^k + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1} \text{ (by ind hyp)}$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+2} - 1,$$

which shows statement is true when $n = k+1$.

Therefore, by induction the claim is true

for all ints $n \geq 0$. $\blacksquare$

Thm 4.28: Define sequence by

$a_1 = 3,\ a_2 = 5,\ a_3 = 9$

and $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 4$. Then

$a_n = 2^n + 1$ for all $n \in \mathbb{N}$.

Pf: We proceed by complete ind.

Base Steps: We see that:

$n=1:\ 2^n + 1 = 2^1 + 1 = 3 = a_1$ ✓

$n=2:\ 2^n + 1 = 2^2 + 1 = 5 = a_2$ ✓

$n=3:\ 2^n + 1 = 2^3 + 1 = 9 = a_3$ ✓

Ind Step: Let $k \geq 3$ and assume

$a_j = 2^j + 1$ for all $3 \leq j \leq k$. We see that

$$a_{k+1} = 2a_k + a_{k-1} - 2a_{k-2} \quad \text{(by def)}$$

$$= 2(2^k + 1) + (2^{k-1} + 1) - 2(2^{k-2} + 1)$$

(by ind hyp)

$$= 2^{k+1} + \cancel{2} + \cancel{2^{k-1}} + 1 - \cancel{2^{k-1}} - \cancel{2}$$

$$= 2^{k+1} + 1,$$

which shows claim is true for $n = k+1$.

Thus, by ind the claim holds for all $n \in \mathbb{N}$.

$\blacksquare$