Russell's Paradox (Sec 3.2)

Is there a set of all sets?

$\mathcal{U} := \{A \mid A \text{ is a set}\}$

$\mathcal{U}$ is a set $\Rightarrow \mathcal{U} \in \mathcal{U}$ (Prob 3.25)

Hmm ... some sets are elmts of themselves?

X = set of all sets that are elmts of themselves

$= \{A \mid A \in A\}$

$Y = X^c = \{A \mid A \notin A\}$

Notice $\mathcal{U} = X \cup Y$, $X \cap Y = \emptyset$ ... right?

wait, but Y is a set ... so does it belong to X or Y???

$Y \in Y \Rightarrow Y \notin Y$ (by def of Y)

$Y \in X \Rightarrow Y \in Y$ (by def of X) $\Rightarrow Y \notin Y$

Sec 3.4 : Indexing Sets

- Disuuss I_n, A_n

Def 3.32 union, intersection of family

Prob 3.33 $I_n = (0, \frac{1}{n})$ for $n \in \mathbb{N}$

(a) $\bigcup_{n \in \mathbb{N}} I_n = (0,1)$

(b) $\bigcap_{n \in \mathbb{N}} I_n = \emptyset$

Prob 3.34 : $A_n = \{a, \ldots, \text{"nth letter"}\}$

(a) $\bigcup_{n=1}^{26} A_n = \{a, b, \ldots, z\}$

(b) $\bigcap_{n=1}^{26} A_n = \{a\}$

Prob 3.35 $S_n = \{x \in \mathbb{R} \mid n-1 < x < n\} = (n-1, n)$ for $n \in \mathbb{N}$

(a) $\bigcup_{n \in \mathbb{N}} S_n = \mathbb{R}^+ \setminus (\mathbb{N} \cup \{0\})$

(b) $\bigcap_{n = \mathbb{N}} S_n = \emptyset$

3.36, 3.37 : HW

Def 3.38 : pairwise disjoint

Prob 3.39 :

$A_1 = \{a\}$, $A_2 = \{a,b\}$, $A_3 = \{b\}$

collection not pairwise disjoint,

but $\bigcap_{n=1}^{3} A_n = \phi$

Pairwise disjoint stronger than empty full intersection

Prob 3.40 : HW

Thm 3.41

(a) Proof : Let $\{A_\alpha\}_{\alpha \in \Delta}$ be collection of sets and

B any set.

($\subseteq$) Let $x \in B \cup \left(\bigcap_{\alpha \in \Delta} A_\alpha \right)$. Then $x \in B$ or

$x \in \bigcap_{\alpha \in \Delta} A_\alpha$.

Case 1: S'pose $x \in B$. Then $x \in B \cup A_\alpha$ for all $\alpha \in \Delta$. This implies that $x \in \bigcap_{\alpha\in\Delta} (B \cup A_\alpha)$.

Case 2: S'pose $x \in \bigcap_{\alpha\in\Delta} A_\alpha$. Then $x \in A_\alpha$ for all $\alpha \in \Delta$. But then $x \in B \cup A_\alpha$ for all $\alpha \in \Delta$. Hence $x \in \bigcap_{\alpha\in\Delta} (B \cup A_\alpha)$.

In either case, we have

$$B \cup \left(\bigcap_{\alpha\in\Delta} A_\alpha\right) \subseteq \bigcap_{\alpha\in\Delta} (B \cup A_\alpha).$$

($\supseteq$) Now, let $x \in \bigcap_{\alpha\in\Delta} (B \cup A_\alpha)$. Then $x \in B \cup A_\alpha$ for all $\alpha \in \Delta$.

Case 1: S'pose $x \in B$. Then $x \in B \cup \left(\bigcap_{\alpha\in\Delta} A_\alpha\right)$.

Case 2: S'pose $x \notin B$. Since $x \in B \cup A_\alpha$ for all $\alpha \in \Delta$ but $x \notin B$, it must be case that $x \in A_\alpha$ for all $\alpha \in \Delta$. Hence $x \in \bigcap_{\alpha\in\Delta} A_\alpha$, which implies that $x \in B \cup \left(\bigcap_{\alpha\in\Delta} A_\alpha\right)$.

In either case,

$$\bigcap_{\alpha\in\Delta}(B\cup A_\alpha) \subseteq B\cup\left(\bigcap_{\alpha\in\Delta} A_\alpha\right).$$

Therefore, the two sets are equal. $\boxtimes$

Thm 3.42

(a) Proof: Let $\{A_\alpha\}_{\alpha\in\Delta}$ be collection of sets.

($\subseteq$) Let $x \in \left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^c$. Then $x \notin \bigcup_{\alpha\in\Delta} A_\alpha$.

By logical version of De Morgan's, $x \notin A_\alpha$ for all $\alpha \in \Delta$.

So, $x \in A_\alpha^c$ for all $\alpha\in\Delta$. Thus, $x \in \bigcap_{\alpha\in\Delta} A_\alpha^c$.

($\supseteq$) Now, let $x \in \bigcap_{\alpha\in\Delta} A_\alpha^c$. Then $x \in A_\alpha^c$ for all $\alpha\in\Delta$, and so $x \notin A_\alpha$ for all $\alpha\in\Delta$. By logical De Morgan's, $x \notin \bigcup_{\alpha\in\Delta} A_\alpha$. Thus, $x \in \left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^c$.