Exam 2 (Take-Home Portion)

Your Name:

Names of Any Collaborators:

Instructions

This portion of Exam 2 is worth a total of 18 points and is worth 30% of your overall score on Exam 2. This take-home exam is due at the beginning of class on **Friday, November 15**. Your overall score on Exam 2 is worth 20% of your overall grade. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 5.35, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

This exam requires some new terminology. You can find a more lengthy discussion in Chapter 5 of the course notes.

Let G be a group and let $H \leq G$. On the in-class portion of Exam 2, I defined the relation $\sim_L$ on G via

$$a \sim_L b \text{ if and only if } ab^{-1} \in H.$$

Similarly, we can define $\sim_R$ on G via

$$a \sim_R b \text{ if and only if } a^{-1}b \in H.$$

It turns out that both $\sim_L$ and $\sim_R$ are equivalence relations on G (Theorem 5.9) and I asked you to prove this fact for $\sim_L$ on the in-class exam. This means that both relations are reflexive, symmetric, and transitive.

Since $\sim_L$ and $\sim_R$ are equivalence relations, the corresponding equivalence classes form a partition of G. If $a \in G$, then the "left" and "right" equivalence classes containing a are given by

$$[a]_{\sim_L} = \{g \in G \mid a \sim_L g\}$$

and

$$[a]_{\sim_R} = \{g \in G \mid a \sim_R g\}.$$

It turns out that there is a nice description of the equivalence classes in terms of special sets called cosets. The subsets

$$aH := \{ah \mid h \in H\}$$

and

$$Ha := \{ha \mid h \in H\}$$

are called the **left** and **right cosets of** H **containing** a, respectively. It isn't too hard to prove that $[a]_{\sim_L} = aH$ and $[a]_{\sim_R} = Ha$ (Theorem 5.10). In the interest of time, let's take this for granted.* The upshot is that the left (respectively, right) cosets of H form a partition of G.

It turns out that sometimes the left cosets of H agree with the right cosets of H and sometimes they don't. There are examples of each scenario in Chapter 5. An interesting fact is that if H is visible in a Cayley diagram for G, then the right cosets of H always correspond to clones of H. The situation where the left and right cosets of H coincide is special. If $aH = Ha$ for all $a \in G$, then H is called a **normal subgroup**, and we write $H \trianglelefteq G$.

Regardless of whether the left and right cosets of a subgroup are the same, all of the cosets have the same cardinality, which the next problem asks you to prove.

Problem 1 (2 points each)**.** Define $\phi : H \to aH$ via $\phi(h) = ah$.

(a) Prove that ϕ is one-to-one

(b) Prove that ϕ is onto.

Note that the function ϕ is not intended to satisfy the homomorphic property. In fact, it doesn't (unless a is the identity).

An immediate consequence of the previous result is that all of the left and right cosets of H have the same cardinality as H. In other words $\#(aH) = |H| = \#(Ha)$ for all $a \in G$ (where $\#(aH)$ denotes the cardinality of aH). Note that everything works out just fine even if H has infinite order.

The next problem asks you to prove a result known as Lagrange's Theorem (Theorem 5.16).

*In case you are interested, the proof involves showing two set containments: $[a]_{\sim_L} \subseteq aH$ and $aH \subseteq [a]_{\sim_L}$. Both arguments are straightforward.

Problem 2 (2 points)**.** Prove that if G is a finite group and $H \leq G$, then $|H|$ divides $|G|$.[†]

Lagrange's Theorem tells us what the possible orders of a subgroup are, but if k is a divisor of the order of a group, it does not guarantee that there is a subgroup of order k. It's not too hard to show that the converse of Lagrange's Theorem is true for cyclic groups. However, it's not true, in general. For example, there is no subgroup of order 6 in the alternating group A_4 (introduced in Section 4.4), which has order 12.

Recall that by definition $|a| = |\langle a \rangle|$, and so a special case of Lagrange's Theorem is that if G is a finite group and $a \in G$, then $|a|$ divides $|G|$ (Theorem 5.19).

Problem 3 (4 points each)**.** Prove **three** of the following theorems.

(a) Suppose G is a finite nontrivial cyclic group such that $|G| = n$. Then G has no proper nontrivial subgroups if and only if n is prime.

(b) Suppose p and q are distinct primes. If G is any group of order pq, then G has either an element of order p or an element of order q.[‡]

(c) Suppose $\phi : G_1 \to G_2$ is a function between two groups that satisfies the homomorphic property. If $g \in G_1$ such that g has finite order, then $|\phi(g)|$ divides $|g|$.

(d) Suppose G is a group and let $H \leq G$. If $gHg^{-1} \subseteq H$ for all $g \in G$, then $H \trianglelefteq G$, where $gHg^{-1} := \{ghg^{-1} \mid h \in H\}$.

(e) If $\phi : G_1 \to G_2$ is a function between two groups that satisfies the homomorphic property, then $\ker(\phi) \trianglelefteq G_1$.[§]

It turns out that the converse of Problem 3(d) is also true. That is, $H \trianglelefteq G$ if and only if $gHg^{-1} \subseteq H$ for all $g \in G$. The set gHg^{-1} is often called the **conjugate of** H **by** g. Another way of thinking about normal subgroups is that they are "closed under conjugation." It's not too hard to show that if $gHg^{-1} \subseteq H$ for all $g \in G$, then we actually have $gHg^{-1} = H$ for all $g \in G$. This implies that $H \trianglelefteq G$ if and only if $gHg^{-1} = H$ for all $g \in G$. This seemingly stronger statement is sometimes used as the definition of normal subgroup.

[†] The number of points this problem is worth is a hint that you should not do anything too complicated. Use the fact that the left cosets partition G and that they all have the same cardinality.

[‡] Recall that in mathematics, "or" is inclusive unless specified otherwise. So, this statement allows for both an element of order p and an element of order q. It turns out that G must have both an element of order p and an element of order q, but you don't need to prove this. One approach to tackling this theorem is to first consider the case when G is cyclic.

[§] Recall that the kernel of ϕ is defined via $\ker(\phi) := \{g \in G_1 \mid \phi(g) = e_2\}$. On Weekly Homework 5, you proved that $\ker(\phi)$ is a subgroup of G_1, so all that remains to prove is that $\ker(\phi)$ is normal. To prove this you can either prove that $a\ker(\phi) = \ker(\phi)a$ for all $a \in G_1$ or you can make use of Problem 3(d).

