Exam 1 (Take-Home Portion)

Your Name:

Names of Any Collaborators:

Instructions

This portion of Exam 1 is worth a total of 29 points and is worth 30% of your overall score on Exam 1. This take-home exam is due at the beginning of class on **Wednesday, February 14**. Your overall score on Exam 1 is worth 18% of your overall grade. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 5.35, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

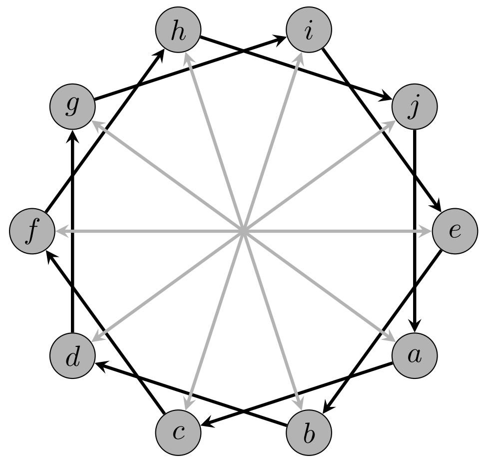
Signature:

Good luck and have fun!

 Written by D.C. Ernst

1. (3 points each) Suppose G is a group with generating set S and assume that $st = ts$ for all $s, t \in S$.
   (a) Prove that $s^{-1}t^{-1} = t^{-1}s^{-1}$ for all $s, t \in S$.
   (b) Prove that $st^{-1} = t^{-1}s$ for all $s, t \in S$.
   (c) Prove that G is abelian.

2. Suppose G is the group given by the Cayley diagram below. Assume that e is the identity.



   (a) (2 points) What generating set was used to create the Cayley diagram for G?
   (b) (2 points) Is this generating set minimal? Briefly justify your answer.
   (c) (4 points) Is G a cyclic group? Justify your answer.

3. For this problem, we will need some new terminology. After digesting the content below, prove **three** of the theorems that follow. You may use an earlier theorem to prove a later theorem even if you did not prove the earlier theorem.

   Let G be a group and let H be a subset of G. Then H is a **subgroup** of G, written $H \leq G$, provided that H is a group in its own right under the binary operation inherited from G. The phrase "under the binary operation inherited from G" means that to combine two elements in H, we should treat the elements as if they were in G and perform the binary operation of G.

   It turns out that $\{e\}$ and G are always subgroups of G (see Theorems 3.7 and 3.9 in the course notes). The subgroup $\{e\}$ is referred to as the **trivial subgroup**. All other subgroups are called **nontrivial**. We refer to subgroups that are not equal to the whole group as **proper subgroups**. If H is a proper subgroup, then we may write $H < G$.

   Recall Theorem 2.50 that states that if G is a group under $*$ and S is a subset of G, then $\langle S \rangle$ is also a group under $*$. It follows immediately from the definition that $\langle S \rangle \leq G$. In particular, $\langle S \rangle$ is the smallest subgroup of G containing S (see Theorem 3.10 in the course notes). The subgroup $\langle S \rangle$ is called the **subgroup generated by** S. In the special case when S equals a single element, say $S = \{a\}$, then

$$\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\},$$

   which is called the (**cyclic**) **subgroup generated by** a. Every subgroup can be written in the "generated by" form. That is, if H is a subgroup of a group G, then there always exists a subset S of G such that $\langle S \rangle = H$. In particular, $\langle G \rangle = G$.

Theorem 1. Suppose G is a group and H is a nonempty subset of G. Then $H \leq G$ if and only if (i) for all $h \in H$, $h^{-1} \in H$, as well, and (ii) H is closed under the binary operation of G.

Theorem 2. If G is a group such that $H, K \leq G$, then $H \cap K \leq G$.

Theorem 3. If G is a cyclic group such that G has exactly one element that generates all of G, then the order of G is at most order 2.

Theorem 4. If G is a group such that G has no proper nontrivial subgroups, then G is cyclic.

Theorem 5. Suppose G is a group and let $g \in G$ such that $\langle g \rangle$ is finite. If n is the smallest positive integer such that $g^n = e$, then $\langle g \rangle = \{e, g, g^2, \ldots, g^{n-1}\}$ and this set contains n distinct elements.[*]

Theorem 6. Suppose G is a group and let $g \in G$ such that $\langle g \rangle$ is finite. If n is the smallest positive integer such that $g^n = e$ and $g^i = g^j$, then n divides $i - j$.[†]

[*] One of the theorems on the in-class portion of the exam guaranteed the existence of such an exponent. To prove this theorem, I suggest you make use of the Division Algorithm, which states that if n is a positive integer and m is any integer, then there exist unique integers q (called the **quotient**) and r (called the **remainder**) such that $m = nq + r$, where $0 \leq r < n$. By the way, the claim that the set contains n distinct elements is not immediate. You need to argue that there are no repeats in the list.

[†] Try using the Division Algorithm.

