Exam 3 (Take-Home Portion)

Your Name:

Names of Any Collaborators:

Instructions

This portion of Exam 3 is worth a total of 20 points and is worth 30% of your overall score on Exam 3. This take-home exam is due at the beginning of class on **Wednesday, April 25**. Your overall score on Exam 3 is worth 18% of your overall grade. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 5.35, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

 Written by D.C. Ernst

The next few problems explore the concept of quotient groups. For additional information, check out Section 6.2 of the course notes.

Suppose G is an arbitrary group and let $H \leq G$. Consider the set of left cosets of H and define

$$(aH)(bH) := (ab)H.$$

The natural question to ask is whether this operation is well-defined. That is, does the result of multiplying two left cosets depend on our choice of representatives? More specifically, suppose $c \in aH$ and $d \in bH$. Then $cH = aH$ and $dH = bH$ by Theorem 5.9(b). According to the operation defined above, $(cH)(dH) = cdH$. It better be the case that $cdH = abH$, otherwise the operation is not well-defined.

On the rest of the exam, you may use an earlier problem in the proof of a later problem (even if you did not complete the earlier problem).

Problem 1. (4 points) Complete **one** of the following.

(a) Let G be a group and let $H \leq G$. Prove that left coset multiplication (as defined above) is well-defined if and only if $H \trianglelefteq G$. *

(b) Let G be a group and let $H \trianglelefteq G$. Prove that the set of left cosets of H in G forms a group under left coset multiplication. †

The group from Problem 1(b) is denoted by G/H, read "G mod H", and is referred to as the **quotient group** of G by H. If G is a finite group, then G/H is exactly the group that arises from "gluing together" the colored blocks in a checkerboard-patterned group table. It's also the group that we get after applying the quotient process to the clones of a subgroup in a Cayley diagram (see Figure 6.1 for an example). It's important to point out once more that this only works properly if H is a normal subgroup.

If $H \trianglelefteq G$, then $|G/H|$ is the number of left cosets of H in G. That is, $|G/H| = [G : H]$. In particular, if G is finite, then $|G/H| = |G|/|H|$.

The next problem is one of my favorites.

Problem 2. (4 points) Let G be a group. Recall that the center of G is the set

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

Since the elements of $Z(G)$ commute with all the elements of G, the left and right cosets of $Z(G)$ will be equal, and hence $Z(G)$ is normal in G. Prove that if $G/Z(G)$ is cyclic, then G is abelian.

Problem 3. (4 points each) Complete **two** of the following.

(a) Prove that if G is a finite cyclic group with generator g such that $|G| = n$, then for all $m \in \mathbb{Z}$, $|g^m| = \dfrac{n}{\gcd(n,m)}$. ‡

(b) Prove that if $|G| = pq$, where p and q are primes (not necessarily distinct), then either $Z(G) = \{e\}$ or G is abelian.

(c) Let H be a normal subgroup of a group G. Prove that if $g \in G$, then the order of gH (in G/H) divides the order of g (in G).

*This is Theorem 6.29. The paragraph above Problem 1 more or less tells you what it is you need to prove. Checking out Figure 6.2 might help with your intuition.

†This is Theorem 6.30.

‡You'll recognize this as Theorem 4.44.

(d) Prove that $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0,2)\rangle \cong \mathbb{V}_4$.[§]

Problem 4. (4 points) Suppose $\phi : G_1 \to G_2$ is a function between two groups that satisfies the homomorphic property (but may or may not be 1-1 or onto). Define the **kernel** of ϕ via

$$\ker(\phi) := \{g \in G_1 \mid \phi(g) = e_2\},$$

where e_1 and e_2 are the identities of G_1 and G_2, respectively. Note that we denoted the kernel by K_ϕ on the take-home-portion of Exam 2. By the take-home portion of Exam 2, $\ker(\phi)$ is a subgroup of G_1. Complete **one** of the following.

(a) Prove that $\ker(\phi)$ is a normal subgroup of G_1.[¶]

(b) Prove that if $K_\phi = \{e_1\}$, then the function ϕ given above is one-to-one.[‖]

[§]Note that $\langle(0,2)\rangle$ is a normal subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_4$ since $\mathbb{Z}_2 \times \mathbb{Z}_4$ is abelian, and hence $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0,2)\rangle$ is a sensible quotient group.

[¶]This is Theorem 7.11. You may use Theorem 5.35 if you'd like, but it isn't required.

[‖]This is one direction of Theorem 7.13. It turns out that the converse is also true, but you do not need to worry about proving it.

