Exam 2 (Part 2)

Your Name:

Names of Any Collaborators:

Instructions

Submit your solutions to the following questions by the start of class on **Friday, April 15**.

This part of Exam 2 is worth a total of 27 points and is worth 40% of your overall score on Exam 2. Your overall score on Exam 2 is worth 20% of your overall grade. I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts. Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing the following problems, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. Further information:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Problem 3.16, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.** To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

Below is a summary of some content from Chapters 5 and 6 of the textbook that you will need to make use of on most of the problems that follow. Feel free to take a look at Chapters 5 and 6 if you would like additional exposure.

Recall that on Part 2 of Exam 1, you proved that if G is a finite group and $H \leq G$, then $|H|$ divides $|G|$. This result is known as **Lagrange's Theorem** (see Theorem 5.16 in textbook). Also, recall that if $g \in G$, then $\langle g \rangle \leq G$. Since $|g| = |\langle g \rangle|$, a special case of Lagrange's Theorem says that $|g|$ divides $|G|$ whenever G is finite (see Theorem 5.19). You may freely use these results on this exam.

The following is a summary of some content from Sections 5.1 and 5.3 of the textbook. Let G be a group and $H \leq G$. For each $a \in G$, define

$$aH := \{ah \mid h \in H\}$$

and

$$Ha := \{ha \mid h \in H\}.$$

These sets are called the **left** and **right cosets of H containing** a, respectively. On Part 2 of Exam 1, you also encountered the right cosets. We proved that the right cosets form a partition of G (and used this fact to prove Lagrange's Theorem). It turns out that the right cosets coincide with the clones of H in the Cayley diagram for G (see Problem 5.15 in the textbook). The left cosets also form a partition of G, but they may or may not agree with the clones of H in G. That is, the left and right cosets may or may not be equal. Certainly, if G is abelian, then the left and right cosets will coincide. However, if G is not abelian, sometimes the left and right cosets agree and sometimes they do not.

For example, consider the dihedral group D_4. It is easy to verify that the left cosets of $\langle s \rangle$ are different from the right cosets of $\langle s \rangle$. In particular, observe that $r\langle s \rangle = \{r, rs\}$ while $\langle s \rangle r = \{r, sr\}$. However, it turns out that the left and right cosets of $\langle r \rangle$ are the same. In this case, the left and right cosets end up being $\{e, r, r^2, r^3\}$ and $\{s, sr, sr^2, sr^3\}$.

The situation where the left and right cosets of H coincide is special. If $aH = Ha$ for all $a \in G$, then H is called a **normal subgroup**, and we write $H \trianglelefteq G$. For example, $\langle s \rangle$ is not normal in D_4, but $\langle r \rangle$ is normal in D_4.

Let G be a group and let $H \leq G$. The **index** of H in G is the number of cosets (left or right) of H in G. Equivalently, if G is finite, then the index of H in G is equal to $|G|/|H|$ (this makes sense in light of our proof of Lagrange's Theorem on the take-home part of Exam 1). We denote the index via $[G : H]$. It's important to recognize that the index can be finite even if G is infinite. For example, take $G = \mathbb{Z}$ and $H = 2\mathbb{Z}$ (i.e., the even integers). Notice that since $\mathbb{Z}$ is abelian, the left cosets of $2\mathbb{Z}$ in $\mathbb{Z}$ coincide with the right cosets of $2\mathbb{Z}$ in $\mathbb{Z}$. In particular, there are two cosets of $2\mathbb{Z}$ in $\mathbb{Z}$, namely $2\mathbb{Z}$ in $\mathbb{Z}$ and $1 + 2\mathbb{Z}$ (i.e., the odd integers). Thus, $[\mathbb{Z} : 2\mathbb{Z}] = 2$.

The following is a summary of some content from Section 6.1 of the book. Suppose $(G, *)$ and $(H, \odot)$ are two groups. Recall that the Cartesian product of G and H is defined to be

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

Using the binary operations for the groups G and H, we can define a binary operation on the set $G \times H$. Define $\star$ on $G \times H$ via

$$(g_1, h_1) \star (g_2, h_2) = (g_1 * g_2, h_1 \odot h_2).$$

This looks fancier than it is. We're just doing the operation of each group in the appropriate component. It is easy to prove that $(G \times H, \star)$ is a group, where $\star$ is defined as above. Moreover, (e, e') is the identity of $G \times H$ and the inverse of $(g, h) \in G \times H$ is given by $(g, h)^{-1} = (g^{-1}, h^{-1})$ (see Theorem 6.1 in the book). We will take this fact for granted.

We refer to $G \times H$ as the **direct product** of the groups G and H. In this case, each of G and H is called a **factor** of the direct product. We often abbreviate $(g_1, h_1) \star (g_2, h_2) = (g_1 * g_2, h_1 \odot h_2)$ by $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$. One exception to this is if we are using the operation of addition in each component. For example, consider $\mathbb{Z}_4 \times \mathbb{Z}_2$ under the operation of addition mod 4 in the first component and addition mod 2 in the second component. Then

$$\mathbb{Z}_4 \times \mathbb{Z}_2 = \{(0,0), (1,0), (2,0), (3,0), (0,1), (1,1), (2,1), (3,1)\}.$$

In this case, we will use additive notation in $\mathbb{Z}_4 \times \mathbb{Z}_2$. For example, in $\mathbb{Z}_4 \times \mathbb{Z}_2$ we have

$$(2,1) + (3,1) = (1,0)$$

and

$$(1,0) + (2,1) = (3,1).$$

Moreover, the identity of the group is $(0,0)$. As an example, the inverse of $(1,1)$ is $(3,1)$ since $(1,1)+(3,1) = (0,0)$. There is a very natural generating set for $\mathbb{Z}_4 \times \mathbb{Z}_2$, namely, $\{(1,0),(0,1)\}$ since $1 \in \mathbb{Z}_4$ and $1 \in \mathbb{Z}_2$ generate $\mathbb{Z}_4$ and $\mathbb{Z}_2$, respectively. The corresponding Cayley diagram is shown in Figure 6.1 of the book.

Here are some facts about direct products of groups, each of which is easy to prove:

- The group $G \times H$ is finite if and only if G and H are finite (see Theorems 6.6 and 6.7).
- If G and H are finite, then $|G \times H| = |G| \cdot |H|$ (see Theorem 6.6).
- $G \times H \cong H \times G$ (see Theorem 6.3).
- The group $G \times H$ is abelian if and only if G and H are abelian (see Theorem 6.8).
- If $K \leq G$ and $L \leq H$, then $K \times L \leq G \times H$. However, it is not necessarily true that every subgroup of $G \times H$ is of this form (see Theorem 6.20 and Problem 6.21).

We can naturally form direct products of any finite collection of groups. For example, if G, H, and K are groups, then we can form the direct product $G \times H \times K$, where we use the respective operation in each component (see Theorem 6.5).

1. (4 points) Prove **one** of the following theorems. Neither of these theorems have anything to do with the discussion above! *Note:* The first theorem below is actually Theorem 4.44 from the book. If you choose to prove this theorem, you may only utilize results that occur before it in the book.

   Theorem 1 (Theorem 4.44)**.** If G is a finite cyclic group with generator g such that $|G| = n$, then for all $m \in \mathbb{Z}$, $|g^m| = \dfrac{n}{\gcd(n,m)}$.

   Theorem 2. If G is a cyclic group of order n and k is a positive integer such that $\gcd(n,k) = 1$ (i.e., n and k are relatively prime), then the function $\varphi : G \to G$ defined via $\varphi(x) = x^k$ for all $x \in G$ is an isomorphism.

2. (3 points each) Complete **five** of the following.

   (a) Determine whether $\mathbb{Z}_4 \times \mathbb{Z}_2$ is isomorphic to any of D_4, $\mathbb{Z}_8$, Q_8, and L_3. Justify your answer.

   (b) Determine whether $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to any of D_4, $\mathbb{Z}_8$, Q_8, and L_3. Justify your answer.

   (c) Determine whether $\mathbb{Z}_3 \times \mathbb{Z}_2$ is isomorphic to either of $\mathbb{Z}_6$ or D_3. Justify your answer.

   (d) Find an example that illustrates that not every subgroup of a direct product is the direct product of subgroups of the factors.

(e) Argue that all of the subgroups of Q_8 are normal in Q_8.

(f) Consider $\langle s\rangle = \{e, s\}$ and $\langle r^2, sr^2\rangle = \{e, r^2, sr^2, s\}$. It is clear that $\langle s\rangle \leq \langle r^2, sr^2\rangle \leq D_4$. Show that $\langle s\rangle \trianglelefteq \langle r^2, sr^2\rangle$ and $\langle r^2, sr^2\rangle \trianglelefteq D_4$, but $\langle s\rangle \ntrianglelefteq D_4$. This shows that normality is not transitive.

(g) Suppose G is a group and let $H \leq G$ such that $[G : H] = 2$. Explain why we must have $H \trianglelefteq G$.

It is important to point out that the claim in Part (g) is a sufficient condition for normality but not a necessary condition. For example, see Part (e).

3. (4 points each) Prove **two** of the following theorems.

Theorem 3. Suppose G and H are groups and let $(g, h) \in G \times H$. If $|g|$ and $|h|$ are finite, then $|(g, h)| = \operatorname{lcm}(|g|, |h|)$.

Theorem 4. The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if m and n are relatively prime.

Theorem 5. Suppose G is a group and let $H \leq G$. If $gHg^{-1} \subseteq H$ for all $g \in G$, then $H \trianglelefteq G$, where $gHg^{-1} := \{ghg^{-1} \mid h \in H\}$.

Note that Theorem 3 generalizes as one would expect (see Theorem 6.12 in the textbook).

It turns out that the converse of Theorem 5 is also true (see Theorem 5.35 in book). That is, $H \trianglelefteq G$ if and only if $gHg^{-1} \subseteq H$ for all $g \in G$. The set gHg^{-1} is often called the **conjugate of** H **by** g. Another way of thinking about normal subgroups is that they are "closed under conjugation." It's not too hard to show that if $gHg^{-1} \subseteq H$ for all $g \in G$, then we actually have $gHg^{-1} = H$ for all $g \in G$. This implies that $H \trianglelefteq G$ if and only if $gHg^{-1} = H$ for all $g \in G$. This seemingly stronger statement is sometimes used as the definition of normal subgroup.

