Most of what we believe, we believe because it was told to us by someone we trusted. What I would like to suggest, however, is that if we rely too much on that kind of education, we could find in the end that we have never really learned anything.

Paul Wallace, physicist & theologian

Chapter 9

Integration

9.1 Introduction to Integration

Unlike with differentiation, we will need a number of auxiliary definitions for beginning integration.

Definition 9.1. A set of points $P = \{t_0, t_1, \dots, t_n\}$ is a **partition** of the closed interval $[a, b]$ if $a = t_0 < t_1 < \cdots t_{n-1} < t_n = b$. If $t_i - t_{i-1} = \frac{b-a}{n}$ for all i, we say that the partition is a **regular partition** of $[a, b]$. In this case, we may use the notation $\boxed{\Delta t := t_i - t_{i-1}}$.

Problem 9.2. Give some partitions, regular and not regular, of $[0, 1]$, $[2, 4]$, and $[-1, 0]$.

Definition 9.3. We say that a real function is **bounded** if it has bounded image set.

Important! For the next four definitions, we assume that f is a bounded real function with domain equal to some closed interval $[a, b]$.

Definition 9.4. Let f be a bounded real function with domain $[a, b]$ and let $\{t_0, t_1, \dots, t_n\}$ be a partition of $[a, b]$. We say that any sum S of the form

$$\boxed{S = \sum_{i=1}^{n} f(x_i)(t_i - t_{i-1}),}$$

where $x_i \in [t_{i-1}, t_i]$ is a **Riemann sum** for f on $[a, b]$.

Definition 9.5. Let f be a bounded real function with domain $[a, b]$ and let $P = \{t_0, t_1, \dots, t_n\}$ be a partition of $[a, b]$. For each $i \in \{1, 2, \dots, n\}$, define $\boxed{M_i := \sup\{f(x) \mid x \in [t_{i-1}, t_i]\}}$. We say that the sum

$$\boxed{U_P(f) := \sum_{i=1}^{n} M_i(t_i - t_{i-1}),}$$

is the **upper Riemann sum** for f with partition P.

Definition 9.6. Let f be a bounded real function with domain $[a,b]$ and let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a,b]$. For each $i \in \{1,2,\ldots,n\}$, define $\boxed{m_i := \inf\{f(x) \mid x \in [t_{i-1}, t_i]\}}$. We say that the sum

$$\boxed{L_P(f) := \sum_{i=1}^{n} m_i(t_i - t_{i-1}),}$$

is the **lower Riemann sum** for f with partition P.

Problem 9.7. Draw pictures that capture the concepts of upper and lower Riemann sums.

Contrary to the name, upper and lower Riemann sums are not always Riemann sums.

Problem 9.8. Give an example of an interval $[a,b]$, partition P, and bounded real function f such that $U_p(f)$ is not a Riemann sum.

Problem 9.9. Define $f : [0,1] \to \mathbb{R}$ via

$$f(x) = \begin{cases} 0, & x \in (0,1] \\ 1, & x = 0. \end{cases}$$

(a) Show that $U_P(f) > 0$ for all partitions of $[0,1]$.

(b) Show that for any positive number ε there is a partition P_ε such that $U_{P_\varepsilon}(f) < \varepsilon$.

(c) Fully describe all lower sums of f on $[0,1]$.

For the next problem, it will be useful to recall that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Problem 9.10. Define $f : [0,1] \to \mathbb{R}$ via $f(x) = x$. For each $n \in \mathbb{N}$, let P_n be the regular partition of $[0,1]$ given by $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$.

(a) Compute $U_{P_5}(f)$.

(b) Give a formula for $U_{P_n}(f)$.

(c) Compute $L_{P_5}(f)$.

(d) Give a formula for $L_{P_n}(f)$.

Problem 9.11. Suppose that f is a bounded real function on $[a,b]$ with lower bound m and upper bound M. Show that for any partition P of $[a,b]$, $U_P(f) \leq M(b-a)$ and $L_P(f) \geq m(b-a)$.

Problem 9.12. Suppose that f is a bounded real function on $[a,b]$ and P is a partition of $[a,b]$. Show that $L_P(f) \leq U_P(f)$.

One consequence of Problem 9.11 is that the set of all upper, respectively lower, sums of f over $[a,b]$ is a bounded set. This implies that if f is a bounded real function on $[a,b]$, then the following supremum and infimum exist:

$$\inf\{U_P(f) \mid P \text{ is a partition of } [a,b]\}$$

$$\sup\{L_P(f) \mid P \text{ is a partition of } [a,b]\}$$

This leads to the following definition.

Definition 9.13. Let f be a bounded real function with domain $[a,b]$. The **upper integral** of f from a to b is defined via

$$\overline{\int_a^b} f := \inf\{U_P(f) \mid P \text{ is a partition of } [a,b]\}.$$

Similarly, the **lower integral** of f from a to b is defined via

$$\underline{\int_a^b} f := \sup\{L_P(f) \mid P \text{ is a partition of } [a,b]\}.$$

Problem 9.14. Compute the upper and lower integrals for the function in Problem 9.9.

Problem 9.15. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that $\underline{\int_0^1} f < \overline{\int_0^1} f$.

Definition 9.16. If P and Q are partitions of $[a,b]$ such that $P \subseteq Q$, then we we say that Q is a **refinement** of P, or that Q **refines** P.

Problem 9.17. Let f be a bounded real function with domain $[a,b]$. Prove that if P and Q are partitions of $[a,b]$ such that Q is a refinement of P, then $L_P(f) \leq L_Q(f)$ and $U_P(f) \geq U_Q(f)$.

Problem 9.18. Suppose f is a bounded real function on $[a,b]$. Use the previous problem to prove that

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Problem 9.19. Suppose f is continuous on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and that for some $z \in [a,b]$, $f(z) > 0$. Explain why $\int_a^b f$ exists and then show that $\int_a^b f > 0$.

Definition 9.20. Let f be a bounded real function with domain $[a,b]$. We say that f is (**Riemann**) **integrable** on $[a,b]$ if

$$\overline{\int_a^b} f = \underline{\int_a^b} f.$$

If f is integrable on $[a,b]$, then the common value of the upper and lower integrals is called the (**Riemann**) **integral** of f on $[a,b]$, which we denote via

$$\boxed{\int_a^b f} \quad \text{or} \quad \boxed{\int_a^b f(x)\,dx.}$$

Technically, we have defined the **Darboux integral**, with Riemann integrals coming from so-called Riemann sums. The two notions can be proved to be equivalent.

Problem 9.21. Give an example of a function f and an interval $[a,b]$ for which we know $\int_a^b f$ does not exist.

Problem 9.22. Is the function in Problem 9.9 integrable over $[0,1]$? If so, determine the value of the corresponding integral. If not, explain why.

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as poetry.
>
> Bertrand Russell, philosopher & mathematician

9.2 Properties of Integrals

There are so many facts about integrals, and unfortunately, we do not have time to prove them all! Nonetheless, we will hit some of the key results.

Problem 9.23. Prove that every constant real function is integrable over every interval $[a,b]$.

The following theorem is a useful characterization of when a function is integrable over a closed interval.

Problem 9.24. Suppose f is a bounded real function on $[a,b]$. Then f is (Riemann) integrable if and only if for every $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that $U_P(f) - L_P(f) < \varepsilon$.

It is important to recognize that the previous problem provides us with a technique for determining whether a function is integrable over a closed interval, but does not necessarily help us with determining the value of a particular integral.

Problem 9.25. Define $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x$. Using the tools we currently have at our disposal, prove that f is integrable on $[0,1]$ and compute the value of the integral.

The next set of theorems will vastly expand our repertoire of functions known to be integrable. First, we need a few definitions, which resemble the corresponding concepts we defined for sequences in Chapter 5.

Definition 9.26. A real function f is (strictly) **increasing** if for each pair of points x and y in the domain of f satisfying $x < y$, we have $f(x) < f(y)$. The function is **nondecreasing** if under the same assumptions we have $f(x) \leq f(y)$. The notions of (strictly) **decreasing** and **nonincreasing** are defined analogously. We say that f is a **monotonic** function if f is either nondecreasing or nonincreasing.

Problem 9.27. Prove that if f is a bounded monotonic real function on $[a,b]$, then f is integrable on $[a,b]$.

Problem 9.28. Prove that each of the following exist. Do you know the value of any of these integrals knowing what we know now and perhaps some well-known area formulas?

(a) $\displaystyle\int_1^2 x^2\,dx$

(b) $\displaystyle\int_1^{17} e^{-x}\,dx$

(c) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

(d) $\displaystyle\int_0^1 \sqrt{1+x^4}\,dx$

The next problem tells us that the integral respects scalar multiplication and sums and differences of integrable functions.

Problem 9.29. Suppose f and g are integrable real functions on $[a,b]$ and let $c \in \mathbb{R}$. Prove each of the following:

(a) The function cf is integrable on $[a,b]$ and $\displaystyle\int_a^b cf = c\int_a^b f$.

(b) The function $f+g$ is integrable on $[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. This one is much harder than it looks!

(c) The function $f-g$ is integrable on $[a,b]$ and $\int_a^b (f-g) = \int_a^b f - \int_a^b g$. Consider using parts (a) and (b)

Unfortunately, products of integrable functions are not well behaved.

Problem 9.30. Find two real functions f and g that are integrable on $[0,1]$ such that fg is also integrable on $[0,1]$ but

$$\left(\int_0^1 f\right)\left(\int_0^1 g\right) \neq \int_0^1 fg.$$

Problem 9.31. Prove that if f is integrable on $[a,b]$, then there exists $m,M \in \mathbb{R}$ such that

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Problem 9.32. Assume that $[a,b]$ is a closed interval and suppose f is integrable on $[a,c]$ and $[c,b]$ for $c \in (a,b)$. Show that f is integrable on $[a,b]$ and that

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Problem 9.33. Suppose f is integrable on $[a,b]$. Prove that for every $c \in \mathbb{R}$, the function g defined via $g(x) = f(x-c)$ is integrable on $[a+c, b+c]$ and

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx.$$

Let's turn our attention to continuous functions. Consider using Problem 6.38 when approaching the next problem.

Problem 9.34. Suppose f is continuous on $[a,b]$. Prove that for every $\varepsilon > 0$, there exists a partition $P = \{t_0 = a, t_1, \ldots, t_{n-1}, t_n = b\}$ of $[a,b]$ such that for each $1 \leq i \leq n$, if $u,v \in [t_{i-1}, t_i]$, then $|f(u) - f(v)| < \varepsilon$.

Use the previous problem to tackle the next problem.

Problem 9.35. Prove that if f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Problem 9.36. Is the converse of the previous problem true? If so, prove it. Otherwise, provide a counterexample.

Problem 9.37. Suppose f is continuous on $[a,b]$. Prove that

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Definition 9.38. If f is integrable on $[a,b]$, then we define

$$\boxed{\int_b^a f = -\int_a^b f} \quad \text{and} \quad \boxed{\int_a^a f = 0.}$$

The next result is often referred to as the Mean Value Theorem for Integrals. Do you see why?

Problem 9.39 (Mean Value Theorem for Integrals)**.** Suppose f is continuous on $[a,b]$. Prove that there exists $c \in [a,b]$ such that

$$\int_a^b f = f(c)(b-a).$$

Can you draw a picture to capture the essence of this theorem?

Problem 9.40. Suppose f is integrable on $[a,b]$ and define $g : [a,b] \to \mathbb{R}$ via

$$g(x) = \int_a^x f.$$

Prove that g is continuous on $[a,b]$.

> Mathematics has beauty and romance. It's not a boring place to be, the mathematical world. It's an extraordinary place; it's worth spending time there.
>
> Marcus du Sautoy, mathematician

9.3 Fundamental Theorem of Calculus

The next two problems are the crowning achievement of calculus and of this course. Collectively, these two problems are known as the Fundamental Theorem of Calculus.

Problem 9.41 (Fundamental Theorem of Calculus, Part 1)**.** Suppose f is continuous on $[a,b]$ and define $F : [a,b] \to \mathbb{R}$ via

$$F(x) = \int_a^x f.$$

Prove that for each $c \in (a,b)$, F is differentiable at c and $F'(c) = f(c)$.

It follows from Problem 9.40 that the function F in the previous theorem is continuous, as well.

Problem 9.42 (Fundamental Theorem of Calculus, Part 2)**.** Suppose f is a real function whose domain includes $[a,b]$ such that f is differentiable at each point of $[a,b]$, and the function f' is continuous at each point in $[a,b]$. Prove that

$$\int_a^b f' = f(b) - f(a).$$

These are the keys that made the efforts of Newton and Leibniz into the something one could calculate with. It is worth noting that the previous theorem is true even if f' is just integrable. Truly a great theorem!

It is important to point out that the function we are integrating in Problem 9.42 needs to be continuous. Moreover, this function must be some other function's derivative. Given f' in Problem 9.42, there is an entire family of functions that have the same derivative as f, each differing by a constant, according to Problem 8.22. Each of the functions in this family is referred to as an **antiderivative** of f' and any one of them can be used to compute $\int_a^b f'$ using the Fundamental Theorem of Calculus.

The crux of using the Fundamental Theorem of Calculus boils down to finding an antiderivative of the function you are integrating. Some functions do not have nice antiderivatives! For example, in part (d) of Problem 9.28, we argued that the function given by $f(x) = \sqrt{1+x^4}$ is integrable on $[0,1]$. However, this function does not have an antiderivative that you would recognize. Try asking WolframAlpha for the antiderivative of $f(x) = \sqrt{1+x^4}$ and see what you get.

Most functions you are familiar with are called elementary functions. Loosely speaking, a function is an **elementary function** if it is equal to a sum, product, and/or composition of finitely many polynomials, rational functions, trigonometric functions, exponential functions, and their inverse functions. These are the functions you typically encounter in high school, precalculus, and calculus. However, many functions are not elementary. For example, the function given in Problem 9.15 is not elementary. To complicate matters, many elementary functions do not have elementary antiderivatives. In fact, some rather innocent looking elementary functions do not have elementary antiderivatives. The function from part (d) of Problem 9.28 is such an example. Here are a few more elementary functions that do not have elementary antiderivatives:

- $\sqrt{1-x^4}$
- $\dfrac{1}{\ln(x)}$
- $\sin(x^2)$ and $\cos(x^2)$
- $\dfrac{\sin(x)}{x}$
- $\dfrac{e^x}{x}$
- e^{e^x}

Determining which elementary functions have elementary antiderivatives is not an easy task. The upshot is that utilizing the Fundamental Theorem of Calculus to compute an integral may be difficult for seemingly innocent looking functions.

Problem 9.43. Using Problem 9.42 and your knowledge of antiderivatives from first semester calculus, compute the integrals in parts (a) and (b) of Problem 9.28.

Problem 9.44. According to WolframAlpha,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

Explain why the techniques of this chapter cannot be used to verify this. How one might go about computing this integral? What definitions are needed?

> In the broad light of day mathematicians check their equations and their proofs, leaving no stone unturned in their search for rigour. But, at night, under the full moon, they dream, they float among the stars and wonder at the miracle of the heavens. They are inspired. Without dreams there is no art, no mathematics, no life.
>
> Michael Atiyah, mathematician