Final Exam

Your Name:

Names of Any Collaborators:

Instructions

This exam is worth a total of 26 points and worth 20% of your overall grade. The exam is due by 9AM on **Friday, May 8**. When you have finished your exam, **please email me a single PDF**. I'd like you to include the cover page, but if you are unable to do this, recreate a suitable replacement that includes your signature. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems/problems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Problem 2.32, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

 Written by D.C. Ernst

The following problems are mostly related to content from Chapters 4–7 of our course notes. Some of the problems are directly from the notes. You may freely use any result that appears in any chapter as long as the result appears *before* the problem in question. Moreover, you may use any result that appears below as long as it appears *before* the problem in question.

1. (4 points, Problem 7.38: Fundamental Theorem of Calculus, Part 1) Suppose f is continuous on $[a,b]$ and define $F:[a,b]\to\mathbb{R}$ via
$$F(x)=\int_a^x f.$$
Prove that for each $c\in[a,b]$, F is differentiable at c and $F'(c)=f(c)$.

2. (4 points, Problem 7.39: Fundamental Theorem of Calculus, Part 2) Suppose f is a function on $[a,b]$ such that f is differentiable at each point of $[a,b]$, and the function f' is continuous at each point in $[a,b]$. Then show that
$$\int_a^b f'=f(b)-f(a).$$

3. (4 points) Complete **two** of the following.

   (a) Let $f:A\to\mathbb{R}$ be a differentiable function such that $[a,b]\subseteq A$. Prove that f' attains every value between $f'(a)$ and $f'(b)$. This says that f' satisfies an intermediate value property.

   (b) Suppose f and g are integrable over $[a,b]$. Prove that if $f(x)\leq g(x)$ for all $x\in[a,b]$, then
$$\int_a^b f\leq\int_a^b g.$$

   (c) If f is integrable on $[a,b]$, then so is $|f|$ (where $|f|$ is given by $|f|(x):=|f(x)|$) and
$$\left|\int_a^b f\right|\leq\int_a^b |f|.$$

   (d) Suppose f is integrable on $[a,b]$ and define $g:[a,b]\to\mathbb{R}$ via
$$g(x)=\int_a^x f.$$
Prove that g is continuous on $[a,b]$.

   (e) Define $f:[0,1]\to\mathbb{R}$ via
$$f(x)=\begin{cases}\sin(1/x), & \text{if } x\neq 0\\ 0, & \text{if } x=0.\end{cases}$$
Prove that f is integrable over $[0,1]$.[1]

4. (2 points each) Each of the following statements is *false*. For **three** of the statements, provide a counterexample.

   (a) Suppose that f is integrable on $[a,b]$ and $f(x)\geq 0$ for all $x\in[a,b]$. If $\int_a^b f=0$, then f is the zero function on $[a,b]$.

   (b) If $f:A\to\mathbb{R}$ is a differentiable function such that f is one-to-one, then either $f'(x)>0$ or $f'(x)<0$ for all $x\in A$.

[1] *Hint:* Try to cleverly split the interval into two pieces and utilize Problem 7.31. You should be able to use Problem 7.34 on one part. Then try to make use of Problem 7.24.

(c) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|u - v| < \delta$, then $|f(u) - f(v)| < \epsilon$.

(d) Suppose f is continuous on $[a, b]$ and define $g : [a, b] \to \mathbb{R}$ via

$$g(x) = \int_a^x f.$$

If f is strictly increasing on $[a, b]$, then g is nondecreasing on $[a, b]$.

5. (2 points) A reoccurring theme this semester has been the importance of productive struggle in the learning process. Identify a problem or topic in this course that you found difficult initially, but that you eventually overcame. Describe how your struggle and perseverance were key to conquering this obstacle.

6. (2 points) Describe how your perceptions of learning, especially mathematics, have changed.

