Chapter 6

Differentiation

It's time for calculus!

Definition 6.1. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$. For real number D, we say that f has *derivative* D at the point a if the following two conditions hold:

1. The point a is an accumulation point of the domain of f.

2. If S is an open interval containing D, then there is an open interval T containing a such that if $t \in T$, $t \neq a$, and t is in the domain of f, then
$$\frac{f(t) - f(a)}{t - a} \in S.$$

In this case, we say that f is *differentiable* at a. If f does indeed have a derivative at some points in its domain, then the derivative of f is the function denoted by f', such that for each number x at which f is differentiable, $f'(x)$ is the derivative of f at x.

Note that the definition of derivative automatically excludes the kind of behavior we saw with continuous functions, where a function defined only at a single point was continuous.

Exercise 6.2. Explain why any function defined only on $\mathbb{Z}$ cannot have a derivative.

Exercise 6.3. Find and prove a formula for the derivative of $f(x) = 3$.

Problem 6.4. Find and prove a formula for the derivative of $g(x) = 2x - 5$.

The following problem provides an alternative definition for the derivative.

Problem 6.5. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$. Prove that f has derivative D at the point a if and only if the following two conditions hold:

1. The point a is an accumulation point of the domain of f.

2. If $\epsilon > 0$, then there exists $\delta > 0$ such that if t is in the domain of f and $|t - a| < \delta$, then
$$\left|\frac{f(t) - f(a)}{t - a} - D\right| < \epsilon.$$

Problem 6.6. Find the derivative of $h(x) = x^2 - x + 1$ at $x = 2$.

Problem 6.7. Find the derivative of $h(x) = x^2 + ax + b$ for any $a, b \in \mathbb{R}$.

Problem 6.8. If f is differentiable at x and $c \in \mathbb{R}$, show that the function cf also has a derivative at x and $(cf)'(x) = cf'(x)$.

Problem 6.9. If f and g are differentiable at x, show that the function $f + g$ also has a derivative at x and $(f + g)'(x) = f'(x) + g'(x)$.

We now pause our regularly scheduled program for a short discussion of limits. Many of you have had a burning desire to utilize the limits that you are familiar with from calculus and have felt like you were working with one hand tied behind your back. Partly to satisfy your desires and partly to speed things up in light of our transition to remote instruction, lets dig into limits for a bit.

Definition 6.10. Let $f : A \to \mathbb{R}$ be a function. Then the *limit* of f as x approaches a is L if the following two conditions hold:

1. The point a is an accumulation point of A, and
2. For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Notationally, we write this as
$$\lim_{x \to a} f(x) = L.$$

Exercise 6.11. Why do we require $0 < |x - a|$ in Definition 6.10?

Example 6.12. It should come as no surprise to you that $\lim_{x \to 5}(3x + 2) = 17$. Let's prove this using Definition 6.10. First, notice that the default domain of $f(x) = 3x + 2$ is the set of real numbers. So, any x-value we choose will be in the domain of the function. Now, let $\epsilon > 0$. Choose $\delta = \epsilon/3$. You'll see in a moment why this is a good choose for δ. Suppose $x \in \mathbb{R}$ such that $0 < |x - 5| < \delta$. We see that
$$|(3x + 2) - 17| = |3x - 15| = 3 \cdot |x - 5| < 3 \cdot \delta = 3 \cdot \epsilon/3 = \epsilon.$$
This proves the desired result.

Example 6.13. Let's try something a little more difficult. Let's prove that $\lim_{x \to 3} x^2 = 9$. As in the previous example, the default domain of our function is the set of real numbers. Our goal is to prove that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in \mathbb{R}$ such that $0 < |x - 3| < \delta$, then $|x^2 - 9| < \epsilon$. Let $\epsilon > 0$. We need to figure out what δ needs to be. Notice that
$$|x^2 - 9| = |x + 3| \cdot |x - 3|.$$
The quantity $|x - 3|$ is something we can control with δ, but the quantity $|x + 3|$ seems to be problematic.

To get a handle on what's going on, let's temporarily assume that $\delta = 1$ and suppose that $0 < |x-3| < 1$. This means that x is within 1 unit of 3. In other words, $2 < x < 4$. But this implies that $5 < x+3 < 7$, which in turn implies that $|x+3|$ is bounded above by 7. That is, $|x+3| < 7$ when $0 < |x-3| < 1$. It's easy to see that we still have $|x+3| < 7$ even if we choose δ smaller than 1. That is, we have $|x+3| < 7$ when $0 < |x-3| < \delta \leq 1$. Putting this altogether, if we suppose that $0 < |x-3| < \delta \leq 1$, then we can conclude that

$$|x^2 - 9| = |x+3| \cdot |x-3| < 7 \cdot |x-3|.$$

This work informs our choice of δ, but remember our scratch work above hinged on knowing that $\delta \leq 1$. If $\epsilon/7 \leq 1$, we should choose $\delta = \epsilon/7$. However, if $\epsilon/7 > 1$, the easiest thing to do is to just let $\delta = 1$. Let's button it all up.

Let $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon/7\}$ and suppose $0 < |x-3| < \delta$. We see that

$$|x^2 - 9| = |x+3| \cdot |x-3| < 7 \cdot |x-3| < 7 \cdot \delta \leq \epsilon$$

since

$$7 \cdot \delta = \begin{cases} 7, & \text{if } \epsilon > 7 \\ 7 \cdot \epsilon/7, & \text{if } \epsilon \leq 7. \end{cases}$$

Therefore, $\lim_{x \to 3} x^2 = 9$, as expected.

Problem 6.14. Prove that $\lim_{x \to 1}(17x - 42) = -25$ using Definition 6.10.

Problem 6.15. Prove that $\lim_{x \to 2} x^3 = 8$ using Definition 6.10.

Problem 6.16. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 17, & \text{if } x = 0. \end{cases}$$

Using Definition 6.10, prove that $\lim_{x \to 0} f(x) = 0$.

Problem 6.17. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ -1, & \text{if } x > 0. \end{cases}$$

Using Definition 6.10, prove that $\lim_{x \to 0} f(x)$ does not exist.

Problem 6.18. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Using Definition 6.10, prove that $\lim_{x \to a} f(x)$ does not exist for all $a \in \mathbb{R}$.

The definition of a function f being continuous at $x = a$ looks awfully similar to the definition of the limit of f as x approaches a. Let's explore this a bit.

Problem 6.19. Explain the similarities and differences between the definitions of continuity at $x = a$ versus the limit as x approaches a. State a theorem about continuity involving limits.

In order for limits to be a useful tool, we need to prove a few important facts.

Problem 6.20 (Limit Laws)**.** Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions. Prove each of the following using Definition 6.10.

(a) If $\lim_{x\to a} f(x)$ exists, then the limit is unique.

(b) If $c \in \mathbb{R}$, then $\lim_{x\to a} c = c$.

(c) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then

$$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x).$$

(d) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then

$$\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$$

(e) If $c \in \mathbb{R}$ and $\lim_{x\to a} f(x)$ exists, then

$$\lim_{x\to a}(c \cdot f(x)) = c \cdot \lim_{x\to a} f(x).$$

(f) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and $\lim_{x\to a} g(x) \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

(g) If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x) = f(\lim_{x\to a} g(x)) = f(b).$$

The next problem is extremely useful. It allows us to simplify our calculations when computing limits.

Problem 6.21. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions and let a be an accumulation point of A. If there exists an open interval S such that $f(x) = g(x)$ for all $x \in (S \cap A) \setminus \{a\}$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

provided one of the limits exists.

Let's return to derivatives.

Problem 6.22. Using Problem 6.5 and Definition 6.10, state a theorem for derivatives that involves limits.

We now return to our regularly scheduled program.

The next problem tells us that differentiability implies continuity.

Problem 6.23. Using Problem 6.22, show that if f has a derivative at $x = a$, then f is also continuous at $x = a$.

The next problems are the well-known Product and Quotient Rules for Derivatives. You will need to use Problem 6.23 in their proofs.

Problem 6.24. Suppose f and g are differentiable at x. Using Problem 6.22, prove each of the following:

(a) The function fg is differentiable at x. Moreover, its derivative function is given by

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

(b) The function f/g is differentiable at x provided $g'(x) \neq 0$. Moreover, its derivative function is given by

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Definition 6.25. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$. The non-vertical line L is *tangent* to the function f at the point $P = (a, b)$ means that:

1. a is an accumulation point of the domain of f,

2. P is a point of L, and

3. if A and B are non-vertical lines containing P with the line L between them (except at P), then there are two vertical lines H and K with P between them such that if Q is a point of f between H and K which is not P, then Q is between A and B.

If L is tangent to f at P, we say that L is a *tangent line* to f at $x = a$.

In the previous definition we write that we have three distinct lines, A, B, and L with L between A and B (except at P). By this we mean that for any point l on L (except P) there is a point α on A and a point β on B so that either α is below l which is below β or that β is below l which is below α.

Exercise 6.26. Try to draw a picture that captures the definition of tangent line. Your picture should include f, a, $f(a)$, P, L, A, B, H, K, Q, α, and β.

Problem 6.27. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$ such that f has a tangent line at $x = a$. Prove that f does not have two tangent lines at the point $(a, f(a))$.

Problem 6.28. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = |x|$.

(a) Prove that f is continuous on all at all points in its domain.

(b) Prove that f has a (non-vertical) tangent line at all points in its domain except $x = 0$.

Problem 6.29. Use the definition of tangent to show that if f is a function whose domain includes $(-1,1)$, and for each number $x \in (-1,1)$, $-x^2 \leq f(x) \leq x^2$, then the x-axis is tangent to f at the point $(0,0)$.

Problem 6.30. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$. Prove that f has a derivative at $x = a$ if and only if f has a non-vertical tangent line at the point $(a, f(a))$.

The upshot of Problems 6.27 and 6.30 is that derivatives are unique when they exist.

Problem 6.31. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$ and suppose f has a derivative at $x = a$. Explain why $f'(a)$ is the slope of the line tangent to f at the point $(a, f(a))$.

In light of Problem 6.30, if a function f does not have a tangent line or has a vertical tangent line at $x = a$, then f is not differentiable at $x = a$. Note that Problem 6.28 shows us that a function f that is is continuous at $x = a$ may or may not be differentiable at $x = a$. This problem also illustrates that a function f and its derivative f' might not have the same domain.

We probably should have done the next two problems sooner, but now is as good a time as any.

Problem 6.32. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = c$ for some constant $c \in \mathbb{R}$. Prove that f is differentiable on $\mathbb{R}$ and $f'(x) = 0$ for all $x \in \mathbb{R}$.

Problem 6.33. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = mx + b$ for some constants $m, b \in \mathbb{R}$. Prove that f is differentiable and $f'(x) = m$ for all $x \in \mathbb{R}$.

In the previous two problems, note that if we restrict the domain of the functions to a closed interval $[a, b]$, then we can conclude that we get the expected derivatives for all $x \in (a, b)$.

Problem 6.34. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is continuous at $x = 0$, but not differentiable at $x = 0$.

The next problem is sure to make your head hurt.

Problem 6.35. Define $g : \mathbb{R} \to \mathbb{R}$ via

$$g(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{otherwise.} \end{cases}$$

Now, define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2 g(x)$. Determine where f is differentiable.

The next result tells us that if a differentiable function attains a maximum value at some point in an open interval contained in the domain of the function, then the derivative is zero at that point. In a calculus class, we would said that differentiable functions attain local maximums at critical numbers.

Problem 6.36. Let $f : A \to \mathbb{R}$ be a function such that $[a,b] \subseteq A$, $f'(c)$ exists for some $c \in (a,b)$, and $f(c) \geq f(x)$ for all $x \in (a,b)$. Prove that $f'(c) = 0$.

Problem 6.37. Let $f : A \to \mathbb{R}$ be a function such that $f'(c) = 0$ for some $c \in A$. Does this imply that there exists an open interval (a,b) such that either $f(x) \geq f(c)$ or $f(x) \leq f(c)$ for all $x \in (a,b)$? If so, prove it. Otherwise, provide a counterexample.

The next problem asks you to prove a result called Rolle's Theorem.

Problem 6.38 (Rolle's Theorem)**.** Let $f : A \to \mathbb{R}$ be a function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, then prove that there exists a point $c \in (a,b)$ such that $f'(c) = 0$.[1]

We can use Rolle's Theorem to prove the next result, which is the well-known Mean Value Theorem.

Problem 6.39 (Mean Value Theorem)**.** Let $f : A \to \mathbb{R}$ be a function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$ and differentiable on (a,b), then prove that there exists a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$ [2]

Problem 6.40. Let $f : A \to \mathbb{R}$ be a function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$ and differentiable on (a,b) such that $f'(x) = 0$ for all $x \in (a,b)$, then prove that f is constant over $[a,b]$.[3]

Problem 6.41. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ such that $[a,b] \subseteq A$. Prove that if $f'(x) = g'(x)$ for all $x \in (a,b)$, then there exists $C \in \mathbb{R}$ such that $f(x) = g(x) + C$.

Problem 6.42. Is the converse of the previous problem true? If so, prove it. Otherwise, provide a counterexample.

[1] *Hint:* First, apply the Extreme Value Theorem to f and $-f$ to conclude that f attains both a maximum and minimum on $[a,b]$. If both the maximum and minimum are attained at the end points of $[a,b]$, then the maximum and minimum are the same and thus the function is constant. What does Problem 6.32 tell us in this case? But what if f is not constant over $[a,b]$? Try using Problem 6.36.

[2] *Hint:* Cleverly define the function $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. Is g continuous on $[a,b]$? Is g differentiable on (a,b)? Can we apply Rolle's Theorem to g using the interval $[a,b]$? What can you conclude? Magic!

[3] *Hint:* Try applying the Mean Value Theorem to $[a,t]$ for every $t \in (a,b)$.