Homework 9

Abstract Algebra I

Complete the following problems. Note that you should only use results that we've discussed so far this semester.

Problem 1. Find all conjugacy classes and their sizes for the following groups.

(a) D_8

(b) Q_8

(c) A_4

Problem 2. If $[G : Z(G)] = n$, prove that every conjugacy class has at most n elements.

Problem 3. Assume G is a non-abelian group of order 15.

(a) Prove that the center of G is trivial.

(b) Use the fact that $\langle g \rangle \leq C_G(g)$ for all $g \in G$ to show that there is at most one possible class equation for G.

Hint: Use my favorite problem.

Problem 4. Prove that the center of S_n is trivial for all $n \geq 3$.

Problem 5. Find all finite groups that have exactly two conjugacy clases.

Problem 6. Prove that if n is odd, then the set of all n-cycles consists of two conjugacy classes of equal size in A_n.

Problem 7. A proper subgroup M of a group G is called *maximal* if whenever $M \leq H \leq G$, either $H = M$ or $H = G$.

(a) Prove that if M is a maximal subgroup of G, then either $N_G(M) = M$ or $N_G(M) = G$.

(b) Prove that if M is a maximal subgroup of G that is not normal in G, then the number of nonidentity elements of G that are contained in conjugates of M is at most $(|M|-1)[G : M]$.

Problem 8. Let H be a proper subgroup of a finite group G. Prove that $G \neq \cup_{g \in G} gHg^{-1}$. *Hint:* Use the previous problem.

Problem 9. Let $g_1, g_2, \ldots, g_r$ be representatives of the conjugacy classes of the finite group G and assume these elements pairwise commute. Prove that G is abelian.

Problem 10. Let p be a prime and let G be a group of order p^α. Prove that G has a subgroup of order p^β, for every β with $0 \leq \beta \leq \alpha$.