Maximal and Prime Ideals

This section of notes roughly follows Section 7.4 in Dummit and Foote.

In this section of notes, we will study two important classes of ideals, namely **maximal** and **prime** ideals, and study the relationship between them.

Definition 36. An ideal M in a ring R is called a **maximal ideal** if $M \neq R$ and the only ideals containing M are M and R.

Example 37. Here are a few examples. Checking the details is left as an exercise.

(1) In $\mathbb{Z}$, all the ideals are of the form $n\mathbb{Z}$ for $n \in \mathbb{Z}^+$. The maximal ideals correspond to the ideals $p\mathbb{Z}$, where p is prime.

(2) Consider the integral domain $\mathbb{Z}[x]$. The ideals (x) (i.e., the subring containing polynomials with 0 constant term) and (2) (i.e, the set of polynomials with even coefficients) are not maximal since both are contained in the proper ideal $(2,x)$. However, as we shall see soon, $(2,x)$ is maximal in $\mathbb{Z}[x]$.

(3) The zero ring has no maximal ideals.

(4) Consider the abelian group $\mathbb{Q}$ under addition. We can turn $\mathbb{Q}$ into a trivial ring by defining $ab = 0$ for all $a, b \in \mathbb{Q}$. In this case, the ideals are exactly the additive subgroups of $\mathbb{Q}$. However, $\mathbb{Q}$ has no maximal subgroups, and so $\mathbb{Q}$ has no maximal ideals.

The next result states that rings with an identity $1 \neq 0$ always have maximal ideals. It turns out that we won't need this result going forward, so we'll skip its proof. However, it is worth noting that all known proofs make use of Zorn's Lemma (equivalent to the Axiom of Choice), which is also true for the proofs that a finitely generated group has maximal subgroups or that every vector spaces has a basis.

Theorem 38. In a ring with 1, every proper ideal is contained in a maximal ideal.

For commutative rings, there is a very nice characterization about maximal ideals in terms of the structure of their quotient rings.

Theorem 39 (Student Presentation 9)**.** Assume R is commutative. The ideal M is maximal iff R/M is a field.

Example 40. We can use the previous theorem to verify whether an ideal is maximal.

(1) Recall that $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ and that $\mathbb{Z}_n$ is a field iff n is prime. We can conclude that $n\mathbb{Z}$ is a maximal ideal precisely when n is prime.

(2) Define $\phi : \mathbb{Z}[x] \to \mathbb{Z}$ via $\phi(p(x)) = p(0)$. Then ϕ is surjective and $\ker(\phi) = (x)$. By the First Isomorphism Theorem for Rings, we see that $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$. However, $\mathbb{Z}$ is not a field. Hence (x) is not maximal in $\mathbb{Z}[x]$. Now, define $\psi : \mathbb{Z} \to \mathbb{Z}_2$ via $\psi(x) = x \mod 2$ and consider the composite homomorphism $\psi \circ \phi : \mathbb{Z} \to \mathbb{Z}_2$. It is clear that $\psi \circ \phi$ is onto

and the kernel of $\psi \circ \phi$ is given by $\{p(x) \in \mathbb{Z}[x] \mid p(0) \in 2\mathbb{Z}\} = (2, x)$. Again by the First Isomorphism Theorem for Rings, $\mathbb{Z}[x]/(2, x) \cong \mathbb{Z}_2$. Since $\mathbb{Z}_2$ is a field, $(2, x)$ is a maximal ideal.

Definition 41. Assume R is commutative. An ideal P is called a **prime ideal** if $P \neq R$ and whenever the product $ab \in P$ for $a, b \in R$, then at least one of a or b is in P.

Example 42. In any integral domain, the 0 ideal (0) is a prime ideal. What if the ring is not an integral domain?

Note 43. The notion of a prime ideal is a generalization of "prime" in $\mathbb{Z}$. Suppose $n \in \mathbb{Z}^+ \setminus \{1\}$ such that n divides ab. In this case, n is guaranteed to divide either a or b exactly when n is prime. Now, let $n\mathbb{Z}$ be a proper ideal in $\mathbb{Z}$ with $n > 1$ and suppose $ab \in \mathbb{Z}$ for $a, b \in \mathbb{Z}$. In order for $n\mathbb{Z}$ to be a prime ideal, it must be true that n divides either a or b. However, this is only guaranteed to be true for all $a, b \in \mathbb{Z}$ when p is prime. That is, the nonzero prime ideals of $\mathbb{Z}$ are of the form $p\mathbb{Z}$, where p is prime. Note that in the case of the integers, the maximal and nonzero prime ideals are the same.

Theorem 44. Assume R is a commutative ring. Then the ideal P is a prime ideal in R iff the quotient ring R/P is an integral domain.

Corollary 45. Assume R is a commutative ring. Every maximal ideal of R is a prime ideal.

Example 46. Recall that $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$. Since $\mathbb{Z}$ is an integral domain, it must be the case that (x) is a prime ideal in $\mathbb{Z}[x]$. However, as we saw in an earlier example, (x) is not maximal in $\mathbb{Z}[x]$ since $\mathbb{Z}$ is not a field. This shows that the converse of the previous corollary is not true.