Final Exam

Your Name:

Names of Any Collaborators:

Instructions

Answer each of the following questions. This exam is worth a total of 65 points. This exam is due by 5PM on **Thursday, December 15**. Your overall score on the Final Exam is worth 20% of your overall grade. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts. Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing the following problems, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. Further information:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using.
2. Unless you prove them, you cannot use any results from the course notes/book that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.** To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

1. (1 point each) For each statement below, determine whether it is TRUE or FALSE. Circle your answer. You do not need to justify your answer.

   (a) The number of outcomes for a race with n runners (with ties allowed) is $\displaystyle\sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}$.

   TRUE FALSE

   (b) For $0 \leq k \leq n-1$, $N_{n,k} \leq \left\langle {n \atop k} \right\rangle$.

   TRUE FALSE

   (c) For all $n \in \mathbb{N}$, $S_n(1) = C_n(1)$, where $S_n(t)$ is the nth Eulerian polynomial and $C_n(t)$ is the nth Narayana polynomial.

   TRUE FALSE

   (d) For all $n \in \mathbb{N}$, $\deg(S_n(t)) = \deg(C_n(t))$.

   TRUE FALSE

   (e) For all $n \in \mathbb{N}$, $w \in S_n(231)$ if and only if $w^{-1} \in S_n(231)$.

   TRUE FALSE

   (f) If $w \in S_n$, then $\mathrm{maj}(w) = \mathrm{inv}(w)$.

   TRUE FALSE

   (g) The number of permutations in S_n with major index k is equal to the coefficient on q^k in $[n]_q!$.

   TRUE FALSE

   (h) The number of subsets of $[n]$ of size k is equal to the coefficient on q^k in the expansion of $(1+q)^n$.

   TRUE FALSE

   (i) If $i <_P j$ in some labeled poset P, then every linear extension w of P satisfies $w(i) <_{\mathbb{N}} w(j)$.

   TRUE FALSE

   (j) The maximum adjacent sorting length of a permutation $w \in S_n$ is $\binom{n}{2}$.

   TRUE FALSE

   (k) If $u, v \in S_n$, then there exists a permutation w such that $\mathrm{Inv}(w) = \mathrm{Inv}(u) \cup \mathrm{Inv}(v)$.

   TRUE FALSE

   (l) If P is a finite poset having unique maximal and unique minimal elements, then P is a lattice.

   TRUE FALSE

   (m) For $n \geq 1$, $\displaystyle\sum_{p \in \mathrm{Dyck}(n)} q^{\mathrm{maj}(p)} = \sum_{p \in \mathrm{Dyck}(n)} q^{\mathrm{pk}(p)}$.

   TRUE FALSE

   (n) For $1 \leq k \leq n$, $\displaystyle\sum_{p \in L(k,n-k)} q^{\mathrm{maj}(p)} = \sum_{w \in S_n} q^{\mathrm{maj}(w)}$.

   TRUE FALSE

   (o) For all $n \in \mathbb{N}$, $\displaystyle\sum_{p \in \mathrm{Dyck}(n)} q^{\mathrm{maj}(p)} = \sum_{w \in S_n(231)} q^{\mathrm{maj}(w)}$.

   TRUE FALSE

   (p) For all $n \in \mathbb{N}$, $\displaystyle\sum_{p \in L(n,n)} q^{\mathrm{area}(p)} = \begin{bmatrix} 2n \\ n \end{bmatrix}_q$.

   TRUE FALSE

(q) If $(P, \leq)$ is a lattice and Q is a subposet of P with a unique minimum value and unique maximum value, then Q is also a lattice.

TRUE FALSE

(r) A *clique* of a graph G is defined as a subset of vertices of G such that every pair of vertices in the subset is adjacent in G. If the maximum cardinality of a clique in G is n, then $\chi(G) \geq n$.

TRUE FALSE

(s) If G is a graph consisting of connected components G_1 and G_2, then $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$.

TRUE FALSE

(t) If G_1 and G_2 are non-isomorphic graphs, then $P(G_1;t) \neq P(G_2;t)$.

TRUE FALSE

(u) If G is a graph consisting of connected components G_1 and G_2, then $P(G;t) = P(G_1;t)P(G_2;t)$.

TRUE FALSE

(v) If a graph G cannot be properly colored using k colors, then $t-k$ is a factor of $P(G;t)$.

TRUE FALSE

(w) If $G = (V, E)$ is a graph, then the number of orientations of G is $2^{|E|}$.

TRUE FALSE

(x) Every graph has at least one acyclic orientation.

TRUE FALSE

(y) If G is a graph, then $a(G)$ equals the number of NBC sets of G.

TRUE FALSE

2. (8 points) Complete **one** of the following problems.

(a) Find an explicit bijection between triangulations of a convex $(n+2)$-gon and $\mathrm{PB}(n)$.

(b) A *standard Young tableau* is a two dimensional array of numbers (from 1 to the number of entries in the array) that increases across rows and down columns. Let $\mathrm{SYT}(2,n)$ denote the collection of standard Young tableaux in a $2 \times n$ rectangular array. Enumerate $|\mathrm{SYT}(2,n)|$.

(c) A *non-nesting partition* of $[n]$ is a set partition $\pi = \{B_1, \ldots, B_k\}$ such that if $\{a,d\} \subseteq B_i$ and $\{b,c\} \subseteq B_j$ with $a < b < c < d$, then $B_i = B_j$. For example, $\{\{1,3\},\{2,4\}\}$ is non-nesting while $\{\{1,4\},\{2,3\}\}$ is not. Enumerate the number of non-nesting partitions of $[n]$.

3. (8 points) Complete **one** of the following problems.

(a) Recall the definition of parking function given on Part 2 of Exam 2. A parking function $(a_1, \ldots, a_n)$ is called *increasing* if $a_i \leq a_{i+1}$ for $1 \leq i \leq n-1$. Count the number of increasing parking functions of length n.

(b) Let $C_{\text{odd}}^{>1}(n)$ denote the collection of compositions of $[n]$ such that each part is odd and greater than one. Find a recurrence (including necessary initial conditions) for $|C_{\text{odd}}^{>1}(n)|$.

(c) Show that every $w \in S_n(123)$ is the "interweaving" (i.e., shuffle) of two decreasing sequences $a_1, a_2, \ldots, a_k$ ($a_m > a_{m+1}$) and $b_1, b_2, \ldots, b_{n-k}$ ($b_m > b_{m+1}$) (where we allow one of the sequences to be empty).

(d) Recall the definition of derangements and the corresponding sequence d_n given on Part 1 of Exam 1. Find a closed form for the exponential generating function for d_n:

$$D(z) := \sum_{n \geq 0} d_n \frac{z^n}{n!}.$$

4. (8 points each) Complete **three** of the following problems.

   (a) Let $M = \{\{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}\}$ be a multiset on $[m]$, where we write k^{n_k} to represent n_k copies of $k \in [m]$. Define inversions, descents, and the major index for permutations of M exactly the way we did for permutations without repetition. Let $P(M)$ denote the set of permutations of M. Prove that
   $$\sum_{w \in P(M)} q^{\mathrm{inv}(w)} = \sum_{w \in P(M)} q^{\mathrm{maj}(w)}.$$

   (b) For $n \geq 1$, prove that
   $$(1+t)(1+qt)(1+q^2t)\cdots(1+q^{n-1}t) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k.$$

   (c) Prove that
   $$\frac{1}{(1-t)(1-qt)(1-q^2t)\cdots(1-q^{n-1}t)} = \sum_{k\geq 0} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q t^k.$$

   (d) Determine the chromatic polynomial of the cycle graph C_n.

   (e) For a graph $G = (V, E)$, a subset of vertices $A \subseteq V$ is called *independent* if no two vertices in A are adjacent. Let $p_k(G)$ be the number of partitions of the vertices of G into k independent sets. Prove that
   $$P(G;t) = \sum_{k=1}^{|V|} p_k(G) \binom{t}{k} k! = \sum_{k=1}^{|V|} p_k(G) t(t-1)\cdots(t-k+1).$$

   (f) Prove that for a fixed edge e in a graph G, $a(G) = a(G \setminus e) + a(G/e)$.

   (g) Prove that if P and Q are locally finite posets with unique minimal elements $\hat{0}_P$ and $\hat{0}_Q$, respectively, then for all $p \in P$ and $q \in Q$, we have $\mu_{P\times Q}(p,q) = \mu_P(p)\mu_Q(q)$.

   (h) Let B_n denote the poset $(\mathcal{P}([n]), \subseteq)$. Compute the characteristic polynomial of B_n.

   (i) If P and Q are finite ranked posets, prove that the characteristic polynomial of $P \times Q$ satisfies $\chi(P \times Q) = \chi(P)\chi(Q)$.