Homework 10

Combinatorics

You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in his or her own work. In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments with no questions asked**.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing assignments for this course, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, Discord, me, and your own intuition. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.** Please read NAU's Academic Integrity Policy.

Complete the following problems.

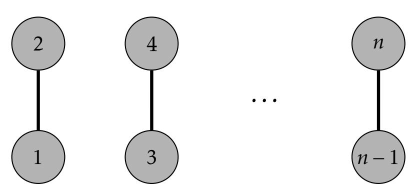
1. A *partition* of a positive integer n is a weakly decreasing sequence of nonnegative integers whose sum is n, i.e., $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition of n if $\lambda_1 \geq \cdots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$. We often draw partitions as a collection of n boxes that are upper- and left-justified, so that the number of boxes in row i is λ_i (we number rows top to bottom). Such a picture is called a *Young diagram*. Let $\mathcal{Y}$ be the set of all possible Young diagrams. The *Young lattice* is the poset $(\mathcal{Y}, \leq)$, where $\lambda \leq \mu$ if and only if the Young diagram μ contains the Young diagram λ with their northwest corners aligned. Note that the empty diagram (typically denoted by $\emptyset$) is the Young diagram for the integer 0. This is the unique minimal element in $\mathcal{Y}$.

   (a) Prove that $\mathcal{Y}$ is a ranked poset with rank function $\mathrm{rk}(\lambda) := |\lambda|$, where $|\lambda|$ is the number of one-by-one squares in λ.

   (b) Prove that $\mathcal{Y}$ is a lattice.

   (c) Find a "nice" expression for the rank generating function for $\mathcal{Y}$.

2. Find a reasonable formula for the number of linear extensions of the following poset on $[n]$ for n even.



3. Suppose P is an antichain on $[n]$ (i.e., there are no relations among $\{1, 2, \dots, n\}$).

   (a) Prove that $\Omega(P; k) = k^n$.

   (b) Prove that the order polynomial generating function for P is given by

$$H(P;t) = \frac{tS_n(t)}{(1-t)^{n+1}},$$

   where $S_n(t)$ is the nth Eulerian polynomial.

(c) Use parts (a) and (b) to derive the Carlitz identity.

4. Let $\Pi(n)$ denote the poset of all set partitions of $[n]$ ordered by reverse refinement.

   (a) Draw the Hasse diagrams for $\Pi(4)$, highlighting NC(4) as a sub-poset.

   (b) Prove that $\Pi(n)$ is a lattice.

   (c) Enumerate the number of maximal chains in $\Pi(n)$.