Homework 9

Combinatorics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on our Discord server. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

New! At the top of each problem, I would like you to list the students that you discussed the problem with. If you worked with the same peers on every problem, then you can simply indicate that once at the top of your assignment.

1. Let $F(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$ and $G(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!}$. Prove that

$$F(t)G(t) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k} f_k g_{n-k}\right)\frac{t^n}{n!}.$$

2. Given a finite set U, let F be rule for producing a collection of labeled combinatorial structures on U, called F**-structures**. I recognize that the phrase "combinatorial structures" is sufficiently vague here, but for the purposes of this problem (and potentially some future problems), we will rely on our intuitive understanding. Some examples of labeled combinatorial structures are sets, subsets, permutations, graphs, planar binary trees, partitions, etc. Let $F[U]$ denote the collection of F-structures on the set U. We will assume that the number of F-structures on U is uniquely determined by $|U|$. We define the exponential generating function for the number of F-structures on $[n]$ via

$$F(t) := \sum_{n\geq 0} |F[n]| \frac{t^n}{n!}.$$

On the previous homework, we encountered $E[n]$ and $S[n]$, which are specific examples of the kind of thing defined above. There is one E-structures in $E[n]$, namely the set $[n]$, while there are $n!$ many S-structures in $S[n]$. Now, let F and G be two rules for producing combinatorial structures on any finite set. We define a new rule, denoted $F \cdot G$, for producing combinatorial structures as follows:

$$(F \cdot G)[U] := \bigsqcup_{A \subseteq U} F[A] \times G[U \setminus A],$$

where $\sqcup$ denotes "disjoint union". Informally, an $(F \cdot G)$-structure is an ordered pair formed by an F-structure and a G-structure over complementary disjoint sets. That is, we put an F-structure on a subset of U and then a G-structure on that subset's compliment. Prove that

$$(F \cdot G)(t) = F(t)G(t).$$

3. A **bicolored set** is a set in which every element is colored one of two colors (say red or blue). For example, $\{a, b, c\}$ and $\{a, b, c\}$ are two different bicolored sets.

   (a) Explain why every collection of bicolored sets with n elements is in bijection with the collection of $(E \cdot E)$-structures on $[n]$, where E is the rule given in Problem 7 on Homework 8. It follows that the corresponding exponential generating function is $(E \cdot E)(t) = E(t)E(t) = e^t e^t = e^{2t}$.

   (b) To really drive the point home, use the righthand side of the equation in Problem 1 above to compute $E(t)E(t)$.

   (c) Let $P[n]$ denote the collection of subsets we can make out of $[n]$, and let $|P[0]| = 1$ (i.e., the empty set is the only subset of a set of size 0). That is, a P-structure on $[n]$ is a subset of $[n]$. Find a closed form for the exponential generating function for the number of P-structures on $[n]$.

4. A **planar rooted tree** is similar to a planar binary tree, except that a node can have any number of subnodes; not just two (and we won't include the "trunk" of the tree). Let $\mathrm{PRT}(n)$ denote the collection of planar rooted trees with n edges. Determine $|\mathrm{PRT}(n)|$.

5. If possible, provide an example each of the following. If no such example exists, explain why.

   (a) A lattice L that is not a ranked poset.

   (b) A ranked poset P that is not a lattice.

   (c) A lattice L together with a subposet Q of L such that Q is not a lattice.

   (d) A ranked poset P together with a subposet Q of P such that Q is not ranked.

6. Suppose $(P, \leq_P)$ and $(Q, \leq_Q)$ are finite posets.

   (a) Define $\leq_{P\times Q}$ on $P \times Q$ via

   $$(x, y) \leq_{P\times Q} (x', y') \text{ if and only if } x \leq_P x' \text{ and } y \leq_Q y'.$$

   Prove that $(P \times Q, \leq_{P\times Q})$ is a poset.

   (b) Prove that if P and Q are ranked posets with rank function rk_P and rk_Q, respectively, then $P \times Q$ is a ranked poset with rank function

   $$\mathrm{rk}_{P\times Q}(x, y) := \mathrm{rk}_P(x) + \mathrm{rk}_Q(y).$$